Accounting for IBPs

Ops. having identical constituents & containing derivatives, can be related by IBP.

e.g. $O_1 = (\partial^4 \varphi)\, \varphi^{r-1}$, $O_2 = (\partial^3 \varphi)(\partial \varphi)\, \varphi^{r-2}$

$$\partial[(\partial^3 \varphi)\, \varphi^{r-1}]$$
$$= (\partial^4 \varphi)\, \varphi^{r-1} + (\partial^3 \varphi)(\partial \varphi)\, \varphi^{r-2}$$

So, O_1 & O_2 are same upto a total deriv.

thus, they are not independent.

* Start w/ an operator $\varphi^r \partial^k$ $(k \geq r)$

* Find the # of ways in which 'k' can be partitioned into 'r' parts

$\Rightarrow p(k; r)$

e.g. $\varphi^5 \partial^4 \rightarrow (\partial^4 \varphi)\, \varphi^4$, $(\partial^3 \varphi)(\partial \varphi)$, $(\partial^2 \varphi)(\partial^2 \varphi)$

So, $p = [4, 0]$; $[3, 1]$; $[2, 2]$

* Relations among $p(k; r)$ ops. through IBP are obtained by applying a total deriv. on $p(k-1; r)$

e.g.

$$O_1 = (\partial^4 \varphi)\, \varphi^{r-1} \equiv p(4, r-1)$$

$$O_2 = (\partial^3 \varphi)\, \varphi^{r-1} \equiv p(3, r-1)$$

$$\partial\left[p(3, r-1)\right] = p(4, r-1) + \sum_j O_j$$

# of independent ops

$$= p(k; r) - p(k; r-1)$$

* Performing the task at all orders we find the contribution of the following form

$$\sum_{k=0}^{d} (-1)^k D^k \, \mathrm{Tr}_{\square}(\wedge^k g)$$

representation

exterior product

$\Rightarrow$ k-form representation is obtained by the k^{th} exterior (anti-symm) product of the vector ($\square$) representation.

* Conformal character

$$\chi^{(d)}_{[\Delta;\ell]}(q;x)$$

$SO(d+2,\mathbb{C})$ is parametrized the $(r+1)$-dim torus, $r=\lfloor d/2 \rfloor$, by variables q (scaling dim) and $\vec{x}=(x_1,\ldots,x_r)$ for the params of $SO(d)$ torus $\Rightarrow q=e^{i\theta_q}$ $\theta_q \in \mathbb{C}$, $\theta_i \in \mathbb{R}$

$$= \sum_{n=0}^{\infty} q^{\Delta+n} \chi^{(d)}_{\mathrm{sym}^n(\Box)}(x)\, \chi^{(d)}_{\ell}(x)$$

$$= q^{\Delta} \chi^{(d)}_{\ell}(x)\, P^{(d)}(q;x)$$

$\chi^{(d)}_{\ell}(x) \Rightarrow$ character of spin-ℓ repⁿ of $SO(d)$

$$P^{(d)}(q,x) = \sum_{n=0}^{\infty} q^{n} \chi^{(d)}_{\mathrm{sym}^n(\Box)}(x)$$

$\Rightarrow$ momentum generating fⁿ.

$q^{\Delta}\chi^{(d)}_{\ell}(x) \Rightarrow$ contribution from primary block.

$P^{(d)} \Rightarrow$ generates contributions from all the descendents.

In even-dimension $\Rightarrow d = 2r$,

a group element $h \in SO(2r)$ in the vector repⁿ has eigenvalues

$$h_{\square}^{(2r)} \mapsto \mathrm{diag}\,(x_1, x_1^{-1}, \ldots, x_r, x_r^{-1})$$

The repⁿ $\mathrm{sym}^{(n)}(\square)$ is formed by the n^{th} fully symm. products of vector comp.
Here, each distinct degree-n monomial formed by above eigenvalues should show up precisely once in $\chi^{(2r)}_{\mathrm{sym}^{(n)}(\square)}$.

Thus,

$$\chi^{(2r)}_{\mathrm{sym}^{(n)}(\square)}(x) = \sum_{\sum_{i=1}^{r} a_i = n} (x_1)^{a_1} (x_1^{-1})^{\bar{a}_1} \cdots (x_r)^{a_r} (x_r^{-1})^{\bar{a}_r}$$

Now,

$$P^{(d)}(q;x) = \sum_{n=0}^{\infty} (-q)^n \chi^{(2r)}_{\text{anti-sym}^{(n)}(D)}(x)$$

$$= \prod_{i=1}^{r} \frac{1}{(1-qx_i)(1-q/x_i)}$$

$$= \left[\det_D \left(1-q\,h(x)\right)\right]^{-1}$$

$$SO(4,\mathbb{C}) \cong SU(2)_L \times SU(2)_R$$

$$q = D, \quad x_1 = \alpha\beta, \quad x_2 = \alpha/\beta$$

$$P^{(4)}(D;\alpha,\beta)$$

$$= 1 \Big/ \left[(1-D\alpha\beta)\left(1-\frac{D}{\alpha\beta}\right)\left(1-D\frac{\alpha}{\beta}\right)\left(1-D\frac{\beta}{\alpha}\right)\right]$$

$\left\{ \right.$ α & β are torus coordinates corresponding to $SU(2)_L$ & $SU(2)_R$.