Construction of Invariant Operators

Natural unit : $\hbar = c = 1$

$$M = L^{-1} = T^{-1}$$

In d-dim,

action, $S = \int d^dx \, \mathcal{L}$

$\hookrightarrow$ Lagrangian density

Measure ($\mathcal{M}$)

Consider $d = 3+1$

$M[\mathcal{M}] = 4$, $M[\mathcal{L}] = -4$ as $M[S] = 0$

(Light) DoF : Scalar (φ), Fermion (ψ), Gauge field (A_μ), field strength tensor ($F_{\mu\nu}$), Covariant derivative (D_μ)

* Our interest in computation of gauge invariant $\mathcal{L}$.

* Gauge field doesn't appear explicitly in $\mathcal{L} \Rightarrow$ Not counted as DoF.

* Covariant derivative is given the "status of a field"

Now,

$[\mathcal{L}] = 4, \; [D_\mu] = 1$

$\Rightarrow [\varphi] = [A_\mu] = 1, \; [\psi] = 3/2, \; [F_{\mu\nu}] = 2$

$\{\partial_\mu A_\nu - \partial_\nu A_\mu + \cdots\}$

Consider $O_i \Rightarrow$ composite op. of $M[O_i] = i$

$\alpha_i \Rightarrow$ coefficients of these O_i w/

$M[\alpha_i] = 4 - i$ $\quad \boxed{i \in \mathbb{Z}_n^+}$

st $M[\alpha_i O_i] = 4$.

$O_i \Rightarrow$ Symmetry invariant op.

internal — space-time (non-compact!)

internal: global — Local (gauge)

* $\mathcal{L} = \sum_{i=1}^{\infty} \alpha_i O_i$

- **Role of Symmetry**

$\phi \Rightarrow$ Real singlet scalar

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - k_1 \varphi - k_2 \varphi^2 - k_3 \varphi^3 - k_4 \varphi^4 \quad (\text{upto } i = 4).$$

- Impose $\varphi \to -\varphi$ symmetry ($\mathbb{Z}_2$)

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - k_2 \varphi^2 - k_4 \varphi^4$$

- φ is complex: $\varphi \to e^{i\theta}\varphi$, $U(1)$ global

$$\mathcal{L} = (\partial_\mu \varphi)^* (\partial^\mu \varphi) - k_2 (\varphi^\dagger \varphi) - k_4 (\varphi^\dagger \varphi)^2$$

We could write 2-more terms (up to $i=4$) $\Rightarrow$ $\underbrace{D^4, \; D^2 X}_{\text{excluded.}}$

Why?

We're working on a manifold w/out boundary $\Rightarrow$ No boundary ops.

$\rightsquigarrow$ total derivatives are excluded (No physical impact)

So, $\mathcal{L}' = \mathcal{L}'' +$ total derivative ops.

$$\Rightarrow \boxed{\mathcal{L}' \equiv \mathcal{L}''}$$

Category	Invariant Polynomial	Operators
Scalar potential	$\varphi^n \ (n \leq 4)$	$\varphi^\dagger\varphi$, $(\varphi^\dagger\varphi)^2$
Scalar kinetic term	$\varphi^2 D^2$	$(D_\mu\varphi)^\dagger (D^\mu\varphi)$
Fermion kinetic term	$\psi^2 D$	$i\bar{\psi} \not{D} \psi$
Gauge kinetic term	X^2	$X_{\mu\nu} X^{\mu\nu}$
Yukawa interaction	$\psi^2 \varphi$	$\bar{\psi}_i \varphi \psi_j$

Invariant Polynomial (internal Symm.)

$\varphi \Rightarrow$ complex scalar field

$\varphi \to e^{i\theta}\varphi, \quad \varphi^* \to \varphi^* e^{-i\theta}$

Invariant polynomial is of the form

$(\varphi^* \varphi)^n \quad \forall\, n = \mathbb{Z}^+_n, 0.$

Hilbert Series $\Rightarrow$ infinite series consisting of all invariants.

$$\mathcal{H}(\varphi, \varphi^*) = \sum_{n=0}^{\infty} C_n \, (\varphi^* \varphi)^n$$

$\hookrightarrow$ multiplicity of invariants for each 'n'

ex/ $\{\varphi_1, \varphi_2\} \Rightarrow$ real singlet scalars

then at $n=2$, we have $\{\varphi_1^2, \varphi_2^2, \varphi_1 \varphi_2\}$

so $C_2 = 3$ (assuming $\varphi_1 \varphi_2 = \varphi_2 \varphi_1$)

$\mathcal{H}(\varphi, \varphi^*) \Rightarrow$ geometric series $(1-\varphi^*\varphi)^{-1}$

$$\mathcal{H}(\varphi, \varphi^*) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-\varphi e^{i\theta})(1-\varphi^* e^{-i\theta})}$$

considering $z = e^{i\theta} \Rightarrow |z| = 1$

$$\mathcal{H}(\varphi, \varphi^*) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{1}{(1-\varphi z)(1-\varphi^* z^{-1})}$$

$$[(1-\varphi z)(1-\varphi^* z^{-1})]^{-1}$$

$$= [1 + (\varphi^*\varphi) + (\varphi^*\varphi)^2 + (\varphi^*\varphi)^3 + \dots]$$

"invariant"

$$+ z[\varphi + \varphi(\varphi^*\varphi) + \dots]$$

$$+ z^2[\varphi^2 + \varphi^2(\varphi^*\varphi) + \dots]$$

$$+ \frac{1}{z}[\varphi^* + \varphi^*(\varphi^*\varphi) + \dots]$$

$$+ \frac{1}{z^2}[\varphi^{*2} + \varphi^{*2}(\varphi^*\varphi) + \dots] + \dots$$

Again,

$$\left[(1-\varphi z)(1-\varphi^* z^{-1})\right]^{-1}$$

$$= \exp\left[-\log(1-\varphi z) - \log(1-\varphi^* z^{-1})\right]$$

$$= \exp\left[\sum_{r=1}^{\infty}\left\{\frac{(\varphi z)^r}{r} + \frac{(\varphi^* z^{-1})^r}{r}\right\}\right]$$

So,

$$\mathcal{H}(\varphi, \varphi^*) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} \exp\left[\sum_{r=1}^{\infty} \frac{(\varphi z)^r}{r} + \frac{(\varphi^* z^{-1})^r}{r}\right]$$

Integral Measure $\Rightarrow \dfrac{dz}{z}$

$\Rightarrow$ Haar Measure

$$\exp\left[\sum_{r=1}^{\infty} \frac{(\varphi z)^r + (\varphi^* z^{-1})^r}{r}\right] = \mathrm{PE}[\varphi, \varphi^*]$$

$\Rightarrow$ Plethystics Exponential

Invariant operator ⇒ taking tensor products of representations.

leading to construct a "Singlet" under all the "considered" symmetries

e.g./ SU(2) : $2 \otimes 2 = 3 \oplus 1$,

$3 \otimes 3 = 5 \oplus 3 \oplus 1$

SU(3) : $\bar{3} \otimes \bar{3} = \bar{6}_s \oplus 3_a$

$3 \otimes \bar{3} = 8 \oplus 1$

$8 \otimes 8 = 1_s \oplus 27_s \oplus 10_s \oplus \overline{10}_a \oplus 8_s \oplus 8_a$

⇒ Dictates which representation to choose to create a "Singlet"

⇒ Taking tensor products of these representations amount to character multiplication.

Compact Lie Groups and Torus

Torus (more specifically maximal torus) subgroups play crucial role in the representation theory of compact Lie groups (CLG)

* A torus in a CLG is a compact, connected abelian subgroups of CLG
  - maximal when maximal among subgroups.

* Rank of CLG ≡ dimension of maximal torus.
  - → maximal number of diagonal generators
  - → equals to the number of nodes in Dynkin diagram

* Unitary group $U(N)$ has subgroups of all diagonal matrices as maximal torus.

$$\mathbb{T}^N = \operatorname{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}\},$$

w/ $\theta_i \in \mathbb{R}\ \forall i$.

* Special Unitary group $SU(N)$ has maximal torus $\mathbb{T}^{N-1}$ as

$$\mathbb{T}^{N-1} = \operatorname{diag}\{e^{i\theta_1}, e^{i(\theta_2-\theta_1)}, \ldots, e^{-\theta_{N-1}}\}$$

Characters (SU(N))

Connected Compact Groups $\Leftrightarrow$ Maximal Torus

$SU(N)$

$$\mathbb{T}^{N-1} = \underbrace{U(1) \times \cdots \times U(1)}_{(N-1)}$$

Consider an irreducible repⁿ $M \in SU(N)$

character corresponding to M $[D(g)]$

$$\Rightarrow \boxed{\chi_D(g) = \mathrm{Tr}\, D(g)}$$

Weyl character is given as

$$\chi^{(M(\epsilon))}_{r_1, \ldots, r_{N-1}} = \frac{|\epsilon^{r_1}, \epsilon^{r_2}, \ldots, \epsilon^{r_{N-1}}, 1|}{|\epsilon^{N-1}, \epsilon^{N-2}, \ldots, \epsilon, 1|}$$

where,

$$M(\epsilon) = \mathrm{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_N)$$

w/ $\prod_{a=1}^{N} \epsilon_a = 1$,

Here, $(r_1, r_2, \ldots, r_{N-1})$ each are integers

st $r_1 > r_2 > \ldots > r_{N-1} > 0$

- $|\epsilon^{r_1}, \epsilon^{r_2}, \ldots, \epsilon^{r_{N-1}}, 1|$ (Numerator)

$$= \begin{vmatrix} \epsilon_1^{r_1} & \epsilon_1^{r_2} & \cdots & \epsilon_1^{r_{N-1}} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \epsilon_N^{r_1} & \epsilon_N^{r_2} & \cdots & \epsilon_N^{r_{N-1}} & 1 \end{vmatrix}$$

- $|\epsilon^{N-1}, \epsilon^{N-2}, \ldots, \epsilon, 1|$ (denominator)

$$= \begin{vmatrix} \epsilon_1^{N-1} & \epsilon_1^{N-2} & \cdots & \epsilon_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \epsilon_N^{N-1} & \epsilon_N^{N-2} & \cdots & \epsilon_N & 1 \end{vmatrix} = \prod_{1 \le a < b \le N} (\epsilon_a - \epsilon_b)$$

$\Rightarrow$ Vandermonde determinant.

In case of $SU(N)$, the rank is $(N-1)$.

Maximal torus $\mathbb{T}^{N-1}$ is given as

$(N-1)$ copies of $U(1)$

Each $U(1)$ is represented by a unimodular

complex variable $z = e^{i\theta}$

$$\mathbb{T}^{N-1} \equiv \underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{(N-1)}$$

* ϵ_a's are related to z_i's How?

Two bases : $\{\epsilon_a\}$ & $\{z_i\}$

$$\begin{cases} a = 1, \ldots, N; \\ i = 1, \ldots, (N-1) \end{cases}$$

$$\epsilon_a = \epsilon_a(\{z_i\})$$

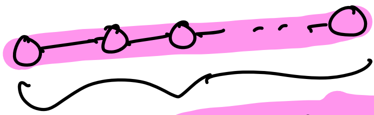
* Finding relation between ϵ_a & z_i

$$\mathbb{T}^{N-1} = \text{diag}\left\{e^{i\theta_1},\ e^{i(\theta_2-\theta_1)},\ e^{i(\theta_3-\theta_2)},\ \dots,\ e^{-i\theta_{N-1}}\right\}_{(N\times N)}$$

Here, θ_i's parametrizes the coordinates of torus.

Cartan matrix of $SU(N)$

Dynkin-diagram



$(N-1)$ # of blobs.

$$A_{SU(N)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} \Big|\quad\Big|$$

$$= \begin{pmatrix} 2 & -1 & 0 & \cdots & & \\ -1 & 2 & -1 & \cdots & & \\ & -1 & 2 & \ddots & 0 & \\ & 0 & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

* SU(N) has $(N-1)$ # of Fundamental Repⁿ (FR)

Dynkin label of FR is given as

$\boxed{(0^{a_1}\ 1^{a_2}\ 0^{a_3})}$, where $\boxed{a_1+a_2+a_3 = (N-1)}$

The Dynkin label of Lowest Dimensional FR (LDFR) is given as

$$(1, \underbrace{0, \ldots, 0}_{(N-2)})$$

<u>Define,</u>

$$L_1 = \underbrace{(1, 0, 0, \ldots 0)}_{(N-1)}$$

$$L_2 = L_1 - \alpha_1 = (-1, 1, 0, \ldots 0)$$

$$L_3 = L_2 - \alpha_2, \quad L_K = L_{K-1} - \alpha_{K-1}$$

$$L_N = L_{N-1} - \alpha_{N-1}$$

$$= (0, 0, \ldots, 0, -1)$$

$\left.\right\}$ $\boxed{\text{Weight tree}}$

generic defn of $(N-1)$ tuple L_i as

$$L_i = \left(l_i^{(1)}, l_i^{(2)}, l_i^{(3)}, \ldots, l_i^{(N-1)}\right)$$

then relatn betn ϵ_a and z_i is given as

$$\epsilon_i = z_1^{l_i^{(1)}} \times z_2^{l_i^{(2)}} \times z_3^{l_i^{(3)}} \cdots \times z_{N-1}^{l_i^{(N-1)}}$$

Simplifying we find,

$$\epsilon_1 = z_1^1 \times z_2^0 \cdots \times z_{N-1}^0 = z_1 \quad ,$$

$$\epsilon_2 = z_1^{-1} \times z_2^1 \times \cdots \times z_{N-1}^0 = z_1^{-1} z_2 \quad ,$$

$$\epsilon_k = z_1^0 \times \cdots \times z_{k-1}^{-1} \times z_k \times \cdots = z_{k-1}^{-1} z_k \, ,$$

$$\epsilon_N = z_{N-1}^{-1}$$

Computation of r_i's

Let's assume Dynkin label of a repn R is

$(a_1, a_2, \ldots, a_{N-1})$