Which can be represented in the basis of LDFR weight-tree, as:

$$(a_1, a_2, \ldots, a_{N-1})$$

$$= \lambda_1 (1, 0, \ldots, 0) + \lambda_2 (-1, 1, \ldots, 0)$$
$$+ \cdots + \lambda_{N-1} (0, 0, \ldots, -1, 1)$$
$$+ \lambda_N (0, \ldots, -1)$$

Solving this, we find

$$a_k = \lambda_k - \lambda_{k+1}, \quad k = 1, \ldots, N-1; \quad \lambda_N = 0$$

$$\lambda_N = 0,$$
$$\lambda_{N-1} = a_{N-1} = \left(\sum_{i=1}^{N-1} a_i\right) - \left(\sum_{j=1}^{N-2} a_j\right)$$

$$\lambda_k = a_k + a_{k+1} + \cdots + a_{N-1}$$
$$= \left(\sum_{i=1}^{N-1} a_i\right) - \left(\sum_{j=1}^{k-1} a_j\right)$$

Now, $\vec{r} = \vec{\lambda} + \vec{\rho}$, where $\rho_i = N - i$,

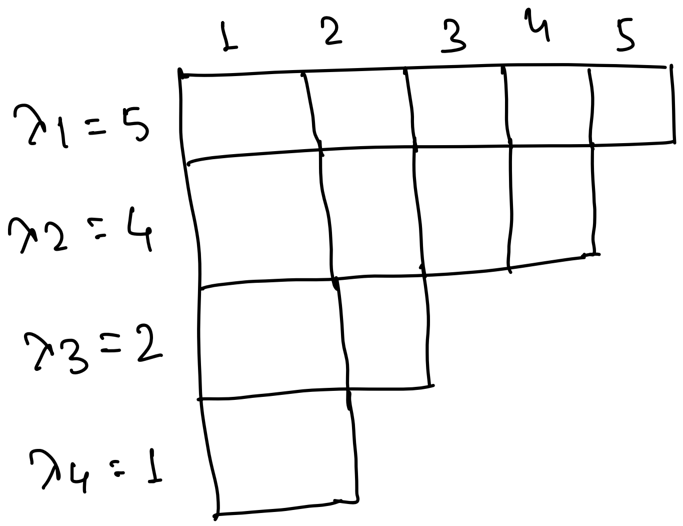
$i = 1, 2, \cdots, N$

Alternate interpretation of λ_i

Young diagram corresponding to R.

# of boxes in K_{th} row (from top) = λ_K

For SU(N), max^m # of vertical box is $(N-1) \Rightarrow \lambda_N = 0$



e.g.

SU(2): Rank 1.

$\epsilon_1 = z_1, \quad \epsilon_2 = z_1^{-1}$

$$\Delta(\epsilon) = \begin{vmatrix} \epsilon_1^1 & 1 \\ \epsilon_2^1 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & 1 \\ z_1^{-1} & 1 \end{vmatrix}$$

$$= (z_1 - z_1^{-1})$$

$$\Delta(\epsilon^{-1}) = (z_1^{-1} - z_1).$$

(1) Singlet $1 \equiv (0) \equiv \boxed{}$

$$\bar{\lambda} = (1, 1), \quad \bar{\rho} = (1, 0)$$

$$\bar{r} = \bar{\lambda} + \bar{\rho} = (2, 1)$$

$$\chi_1(\epsilon_1, \epsilon_2) = \frac{|\epsilon^2, \epsilon^1|}{\Delta(\epsilon)}$$

$$= \frac{1}{\Delta(\epsilon)} \begin{vmatrix} \epsilon_1^2 & \epsilon_1 \\ \epsilon_2^2 & \epsilon_2 \end{vmatrix} = \begin{vmatrix} z_1^2 & z_1 \\ z_1^{-2} & z_1^{-1} \end{vmatrix} / \Delta(\epsilon)$$

$$= (z_1 - z_1^{-1}) / (z_1 - z_1^{-1}) = 1$$

(2) $2 \equiv \bar{2} \equiv (1) \equiv \square$

$$\bar{\lambda} = (1, 0), \quad \bar{\rho} = (1, 0)$$

$$\bar{\gamma} = (2, 0)$$

$$x_2(\epsilon_1, \epsilon_2) = \frac{1}{\Delta(\epsilon)} |\epsilon^2, \epsilon^0| = \frac{|\epsilon^2, 1|}{\Delta(\epsilon)}$$

$$= \begin{vmatrix} \epsilon_1^2 & 1 \\ \epsilon_2^2 & 1 \end{vmatrix} / \Delta(\epsilon)$$

$$= \begin{vmatrix} z_1^2 & 1 \\ z_1^{-2} & 1 \end{vmatrix} / \Delta(\epsilon) = \frac{(z_1^2 - z_1^{-2})}{(z_1 - z_1^{-1})}$$

$$= (z_1 + z_1^{-1})$$

$$\boxed{x_2 = x_{\bar{2}}}$$

(3) Adjoint $3 \equiv (2) \equiv$ $\square\square$

$$\bar{\lambda} = (2, 0), \quad \bar{\rho} = (1, 0)$$

$$\bar{\gamma} = (3, 0)$$

$$\chi_3(\epsilon_1, \epsilon_2) = \frac{|\epsilon^3, \epsilon^0|}{\Delta(\epsilon)}$$

$$= \begin{vmatrix} z_1^3 & 1 \\ z_1^{-3} & 1 \end{vmatrix} / \Delta(\epsilon) = \frac{(z_1^3 - z_1^{-3})}{(z_1 - z_1^{-1})}$$

$$= \frac{(z_1 - z_1^{-1})(z_1^2 + z_1^{-2} + 1)}{(z_1 - z_1^{-1})}$$

$$= (z_1^2 + z_1^{-2} + 1)$$

<u>check</u>: $\boxed{\chi_2 \cdot \chi_2 = \chi_3 + \chi_1}$

$SU(3)$: Rank 2

HW of LDF $= (1, 0)$

Dynkin diagram $\Rightarrow$ 

Cartan Matrix $\Rightarrow \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$L_1 = (1, 0)$

$L_2 = L_1 - \alpha_1 = (1, 0) - (2 \; -1)$

$= (-1, 1)$

$L_3 = L_2 - \alpha_2 = (-1, 1) - (-1, 2)$

$= (0, -1)$

$\epsilon_1 = z_1, \quad \epsilon_2 = z_1^{-1} \cdot z_2, \quad \epsilon_3 = z_2^{-1}$

$\bar{\rho} = (N-1, N-2, \dots, 0)$

$= (2, 1, 0)$

$$\Delta(\epsilon) = |\epsilon^2, \epsilon^1, \epsilon^0| = \prod_{1 \le i < j \le 3} (\epsilon_i - \epsilon_j)$$

$$= \begin{vmatrix} \epsilon_1^2 & \epsilon_1 & 1 \\ \epsilon_2^2 & \epsilon_2 & 1 \\ \epsilon_3^2 & \epsilon_3 & 1 \end{vmatrix} = (\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)(\epsilon_1 - \epsilon_3)$$

$$= (z_1 - z_1^{-1} \cdot z_2)(z_1^{-1} z_2 - z_2^{-1})(-z_2^{-1} + z_1)$$

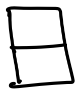
(1) $3 \equiv (1, 0) \equiv \square$

$\bar{\lambda} = (1, 0, 0)$, $\bar{\rho} = (2, 1, 0)$

$\bar{r} = (3, 1, 0)$

$$x_3 = |\epsilon^3, \epsilon^1, \epsilon^0| / \Delta(\epsilon)$$

$$= \begin{vmatrix} \epsilon_1^3 & \epsilon_1 & 1 \\ \epsilon_2^3 & \epsilon_2 & 1 \\ \epsilon_3^3 & \epsilon_3 & 1 \end{vmatrix} \Big/ \Delta(\epsilon) = z_1 + \frac{z_2}{z_1} + z_2^{-1}$$

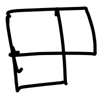
(2) $\bar{3} \equiv (0,1) \equiv$ 

$\bar{\lambda} = (1,1,0)$, $\bar{\rho} = (2,1,0)$

$\bar{r} = (3,2,0)$

$$x_{\bar{3}} = \frac{|\epsilon^3 \ \ \epsilon^2 \ \ \epsilon^0|}{\Delta(\epsilon)} = z_2 + \frac{z_1}{z_2} + z_1^{-1}$$

$$\equiv x_3 (z_1 \leftrightarrow z_2)$$

(3) $8 \equiv (1,1) \equiv$ 

$\bar{\lambda} = (2,1,0)$, $\bar{\rho} = (2,1,0)$

$\bar{r} = \bar{\lambda} + \bar{\rho} = (4,2,0)$

$$x_8 = |\epsilon^4 \ \ \epsilon^2 \ \ \epsilon^0| \,/\, \Delta(\epsilon)$$

$$= z_1 z_2 + \frac{1}{z_1 z_2} + 2 + \frac{z_1}{z_2^2} + \frac{z_1^2}{z_2} + \frac{z_2}{z_1^2} + \frac{z_2^2}{z_1}$$

$$= \left[(x_3 \cdot x_{\bar{3}}) - 1 \right] \text{ (check)}$$

Haar Measure

$$\int_{SU(N)} d\mu_{SU(N)} = \frac{1}{(2\pi i)^{N-1} N!}$$

$$\left\{ \oint \prod_{l=1}^{N-1} \frac{dz_l}{z_l} \Delta(\epsilon) \Delta(\epsilon^{-1}) \right\}$$

$\Delta(\epsilon) \Rightarrow$ Vandermonde determinant.

<u>SU(2):</u> $\Delta(\epsilon) = z_1 - z_1^{-1}$

$$\Delta(\epsilon^{-1}) = z_1^{-1} - z_1$$

$$\Delta(\epsilon)\Delta(\epsilon^{-1}) = \left(z_1 - \frac{1}{z_1}\right)\left(\frac{1}{z_1} - z_1\right)$$

$$= \left(1 + 1 - z_1^2 - \frac{1}{z_1^2}\right)$$

$$= \left(1 - \frac{1}{z_1^2}\right)\left(1 - z_1^2\right)$$

$$d\mu_{SU(2)} = \frac{1}{2!(2\pi i)} \frac{dz_1}{z_1} \Delta(t)\Delta(t^{-1})$$

$$= \frac{1}{2(2\pi i)} \frac{dz_1}{z_1} (1-z_1^2)\left(1-\frac{1}{z_1^2}\right)$$

SU(3):

$$d\mu_{SU(3)} = \frac{1}{3!(2\pi i)^2} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \Delta(t)\Delta(t^{-1})$$

$$\Delta(t) = (z_1 - z_1^{-1} z_2)(z_1^{-1} z_2 - z_2^{-1})$$

$$(-z_2^{-1} + z_1)$$

$$d\mu_{SU(3)} = \frac{1}{6} \frac{1}{(2\pi i)^2} \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

$$\Big[(1-z_1 z_2)\left(1-\frac{1}{z_1 z_2}\right)\left(1-\frac{z_1}{z_2^2}\right)\left(1-\frac{z_2^2}{z_1}\right)$$

$$\left(1-\frac{z_2}{z_1^2}\right)\left(1-\frac{z_1^2}{z_2}\right)$$ (check!)