# Lecture 1: From Classical and Quantum Chaos to Thermalization in Isolated Quantum Systems

Jed Pixley

ICTS Lecture Series:

Quantum Dynamics in the pre-fault tolerant era

6/3/2026

RUTGERS THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS CENTER FOR Materials Theory
FLATIRON INSTITUTE Center for Computational Quantum Physics
ICTS INTERNATIONAL CENTRE for THEORETICAL SCIENCES TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Office of Naval Research Science & Technology
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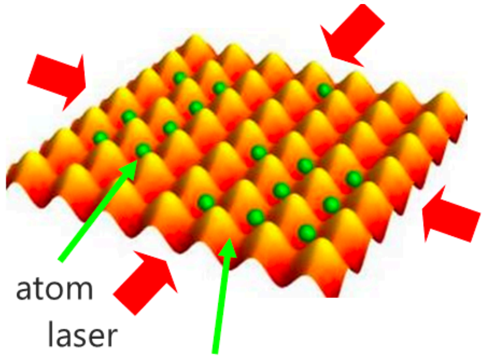
ALFRED P. SLOAN FOUNDATION Driven by the promise of great ideas

# Quantum Technology of Today

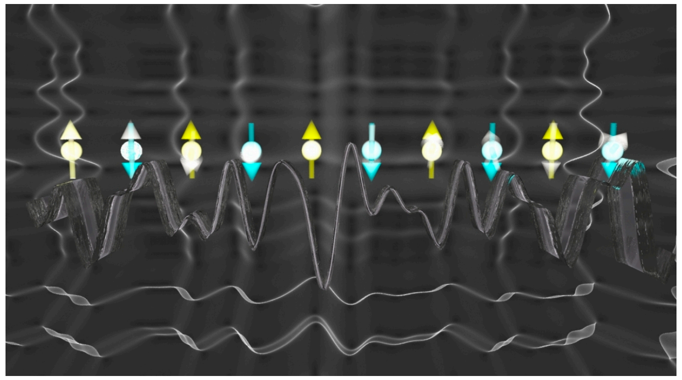
Huge advances in recent decades:

Cold atomic gases



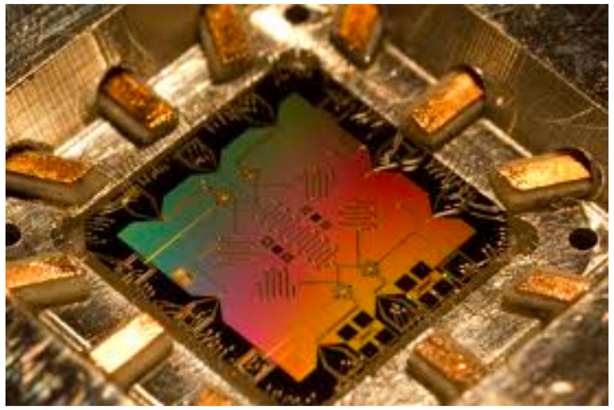

M. Kozuma

Trapped ions



C. Monroe

Superconducting qubits

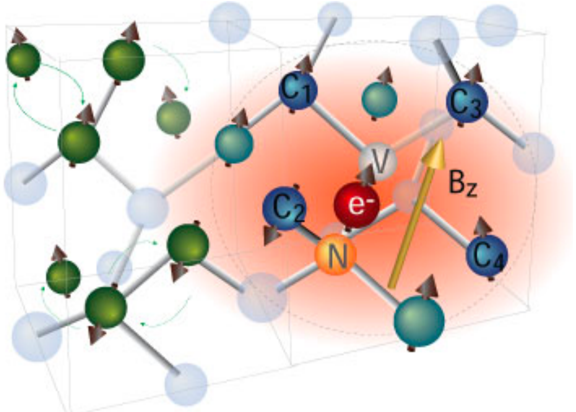


J. Martinis

Neutral atoms

M. Endres

Nitrogen-vacancy centers in diamond




P. Cappellaro

Highly coherent quantum systems with measurement capabilities

Applications: Computing, Simulation, Sensing…

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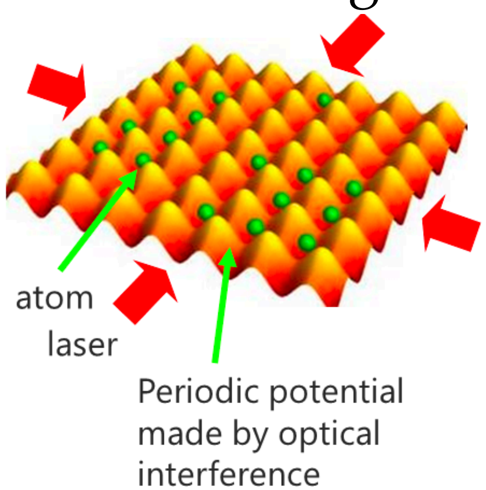


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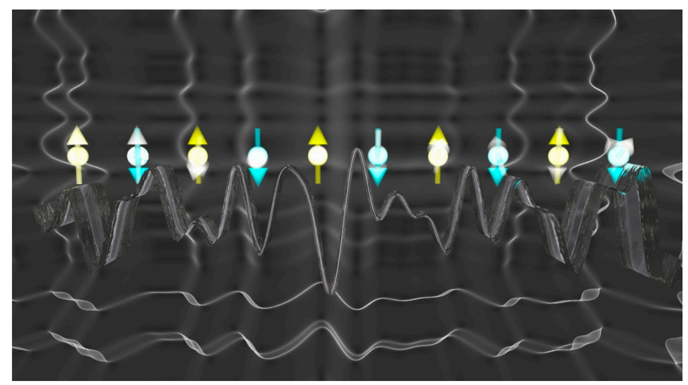

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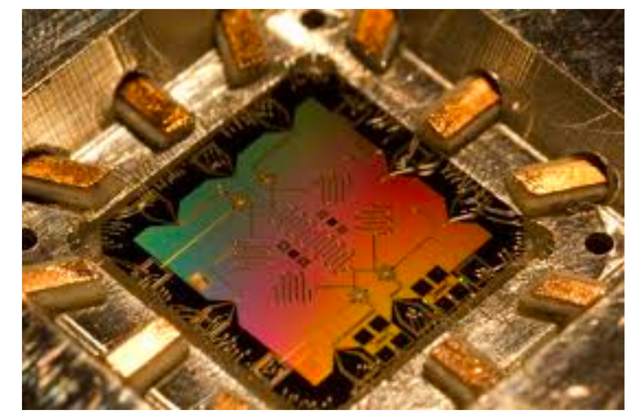

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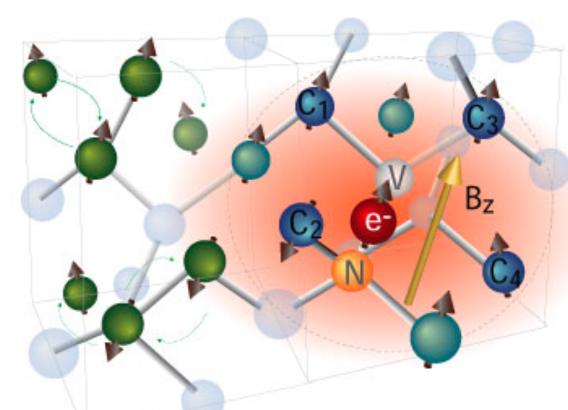



P. Cappellaro

Generic quantum dynamics is chaotic/thermal, to hard to control in general.

How can we control and manipulate quantum systems?

# Lecture Series Overview

Quantum dynamics in the pre fault tolerant era

**Lecture 1: From classical and quantum chaos to thermalization in isolated quantum systems**

Lecture 2: Analog and digital quantum simulators from unitary dynamics to midcircuit measurements.

Lecture 3: Monitored quantum dynamics in random quantum circuits

Lecture 4: Adaptive quantum circuits and control induced phase transitions

Lecture 5: Open quantum dynamics software tutorial

# Lecture Series, learning goals

I. **Lecture 1: Classical and Quantum Chaos, from single particle to many-body**

II. Lecture 2: Quantum platforms

III. Lecture 3: Monitored dynamics, interplay of unitary and projective evolution.

IV. Lecture 4: Adaptive dynamics, controlling quantum dynamics

V. Lecture 5: Numerical approaches to adaptive quantum dynamics

# Lecture 1: Goals

**Understanding generic quantum dynamics**

Classical dynamics, chaotic vs integrable

Quantum dynamics, chaos vs integrable

Thermalization in isolated quantum systems

Models for generators of the dynamics:
Hamiltonian, Floquet, and random quantum circuits

Breakdown of thermalization

# Outline

I. Lecture series layout
II. Classical chaos
III. Quantum chaos
IV. Many-body quantum chaos and thermalization
V. Evading thermalization

# Unpredictability of Chaos

Classical Chaos: unpredictable deterministic evolution of the classical equations of motion.

Weather

Google

Public transit


Feldman, *Chaos and Fractals an Elementary Introduction*, (2012)

# Unpredictability of Chaos

Classical Chaos: unpredictable deterministic evolution of the classical equations of motion.

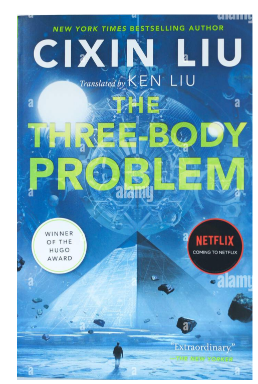
Weather

Google

3 gravitational bodies


Wikipedia

Public transit


Feldman, *Chaos and Fractals an Elementary Introduction*, (2012)

# Unpredictability of Chaos

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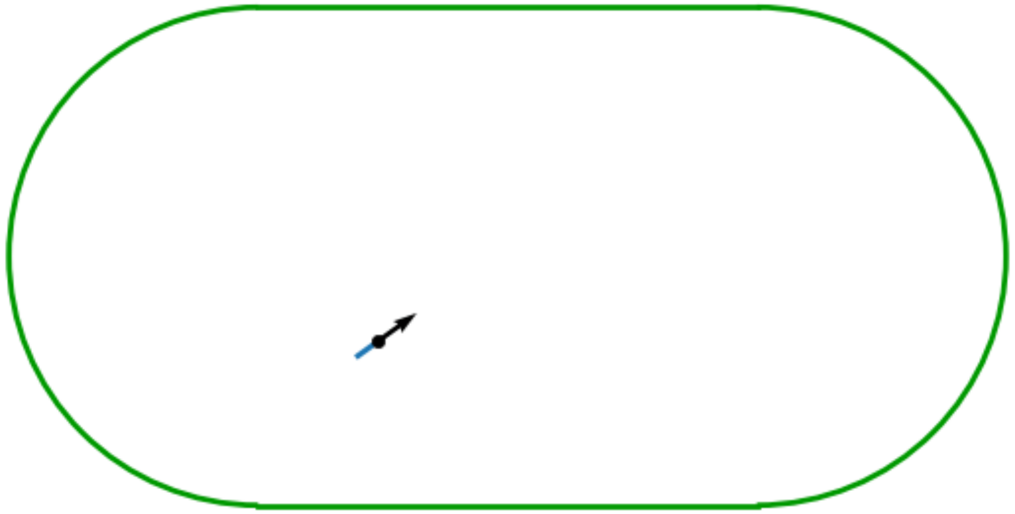
Billiards in a "stadium" geometry


Public transit

Wikipedia

# Unpredictability of Chaos

Classical Chaos: unpredictable deterministic evolution of the classical equations of motion.

Quantify chaos: Exponential sensitivity to initial conditions

$$\mathbf{x}'(t=0) = \mathbf{x}(0) + \delta \qquad \delta \ll 1$$

# Unpredictability of Chaos

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Quantify chaos: Exponential sensitivity to initial conditions

$$\mathbf{x}'(t=0) = \mathbf{x}(0) + \delta \qquad \delta \ll 1$$

$$||x(t) - x'(t)|| \sim e^{t\lambda_{\text{Lyapunov}}}$$

$\lambda_{\text{Lyapunov}} > 0$ Chaotic

$\lambda_{\text{Lyapunov}} < 0$ Stable

# Dynamical Systems

Continuous time
$$\frac{dx^1}{dt} = F_1(x^1, x^2, x^3, \ldots, x^N)$$
$$\frac{dx^2}{dt} = F_2(x^1, x^2, x^3, \ldots, x^N)$$
$$\vdots$$
$$\frac{dx^N}{dt} = F_N(x^1, x^2, x^3, \ldots, x^N)$$

$N$ first order order differential equations
$$\frac{d\mathbf{x}}{dt} = \mathbf{F}[\mathbf{x}(t)]$$

Discrete time
$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$$

# Hamiltonian Systems

Dynamics generated by a Hamiltonian depends on position, momentum, and time $H(\mathbf{q}, \mathbf{p}, t)$

$$\mathbf{q} = (q_1, q_2, \ldots, q_N)$$
$$\mathbf{p} = (p_1, p_2, \ldots, p_N)$$

Equations of motion

$$\mathrm{d}\mathbf{p}/\mathrm{d}t = -\partial H(\mathbf{p},\ \mathbf{q},\ t)/\partial \mathbf{q},$$
$$\mathrm{d}\mathbf{q}/\mathrm{d}t = \partial H(\mathbf{p},\ \mathbf{q},\ t)/\partial \mathbf{p}.$$

*Chaos in dynamical systems,* Ott

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Written in the general construction, we have

$$\mathrm{d}\tilde{\mathbf{x}}/\mathrm{d}t = \mathbf{F}(\tilde{\mathbf{x}},\ t) \qquad \tilde{\mathbf{x}} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$$

$$= \begin{pmatrix} d\mathbf{p}/dt \\ d\mathbf{q}/dt \end{pmatrix}$$

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$$\mathbf{F}(\tilde{\mathbf{x}},\ t) = \mathbf{S}_N \cdot \partial H/\partial \tilde{\mathbf{x}} \qquad = \begin{pmatrix} d\mathbf{p}/dt \\ d\mathbf{q}/dt \end{pmatrix}$$

$$\frac{\partial H}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \partial H/\partial \mathbf{p} \\ \partial H/\partial \mathbf{q} \end{bmatrix} \qquad \mathbf{S}_N = \begin{bmatrix} \mathbf{O}_N & -\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{O}_N \end{bmatrix}$$

NxN identity matrix — NxN zero matrix

*Chaos in dynamical systems*, Ott

# Iterative maps

Consider 1D maps $x_{n+1} = f(x_n)$

Initial condition $x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} \cdots \xrightarrow{f} x_n$

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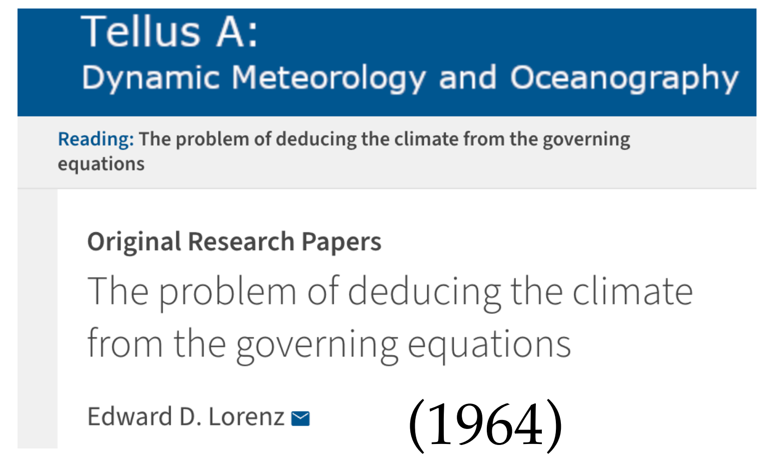
Weather forecasting


Tellus A:
Dynamic Meteorology and Oceanography

Reading: The problem of deducing the climate from the governing equations

Original Research Papers

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Edward D. Lorenz

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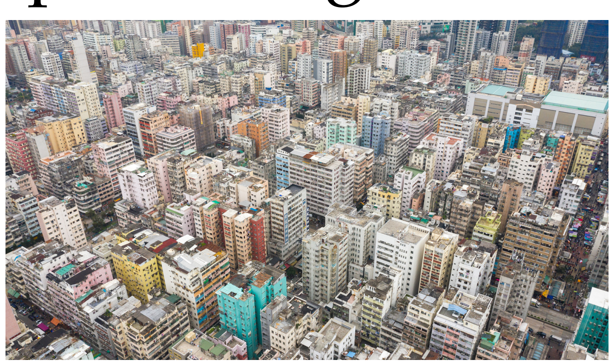

Nature Vol. 261 June 10 1976 459

**review article**

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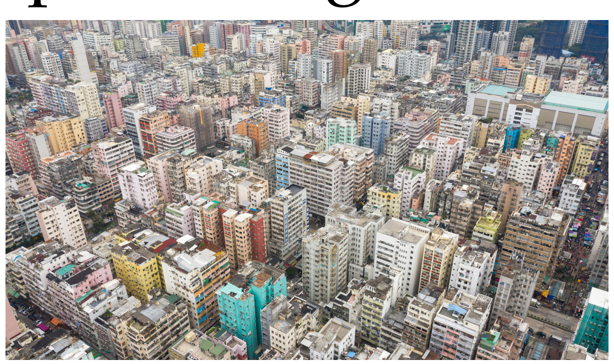

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**Simple mathematical models with very complicated dynamics**

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Logistic map $x_{n+1} = r x_n (1 - x_n)$

# From Periodic To Chaotic

Logistic map $$x_{n+1} = rx_n(1 - x_n)$$

Example: $r = 1.2$ **stable**

$$x_0 = 0.4$$

$$x'_0 = 0.4 + 10^{-9}$$

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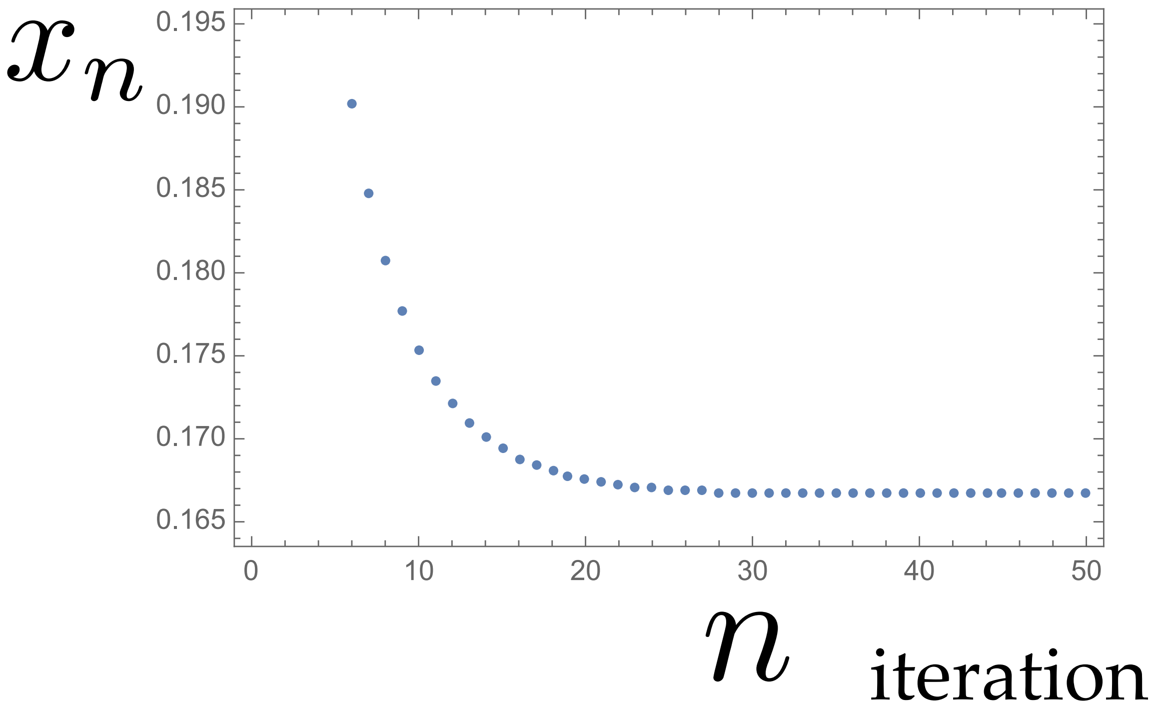

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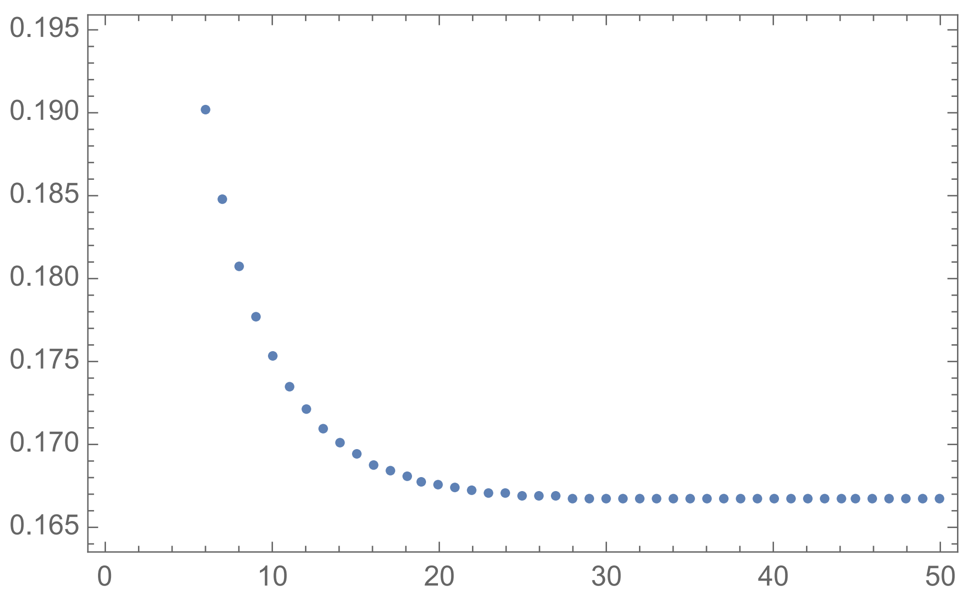
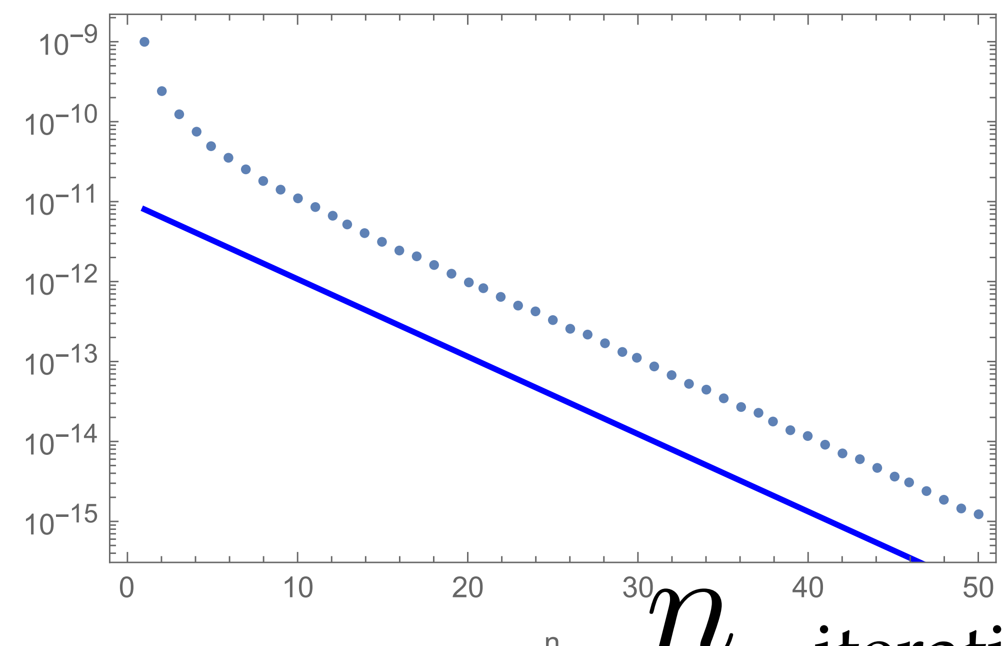
$x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} \cdots \xrightarrow{f} x_n$ $|x'_n - x_n|$

$x_n$




$n$ iteration

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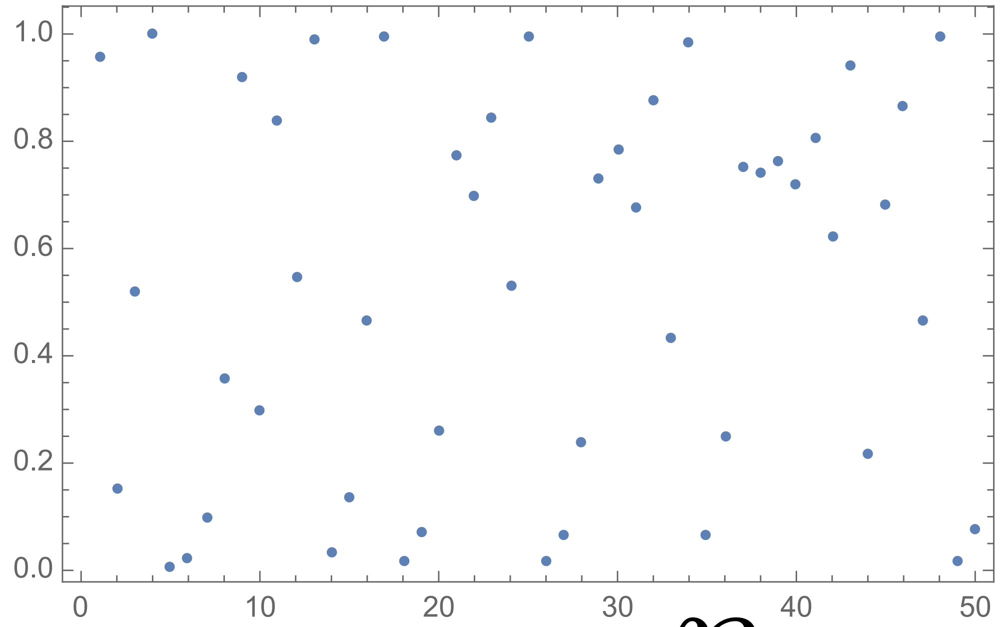


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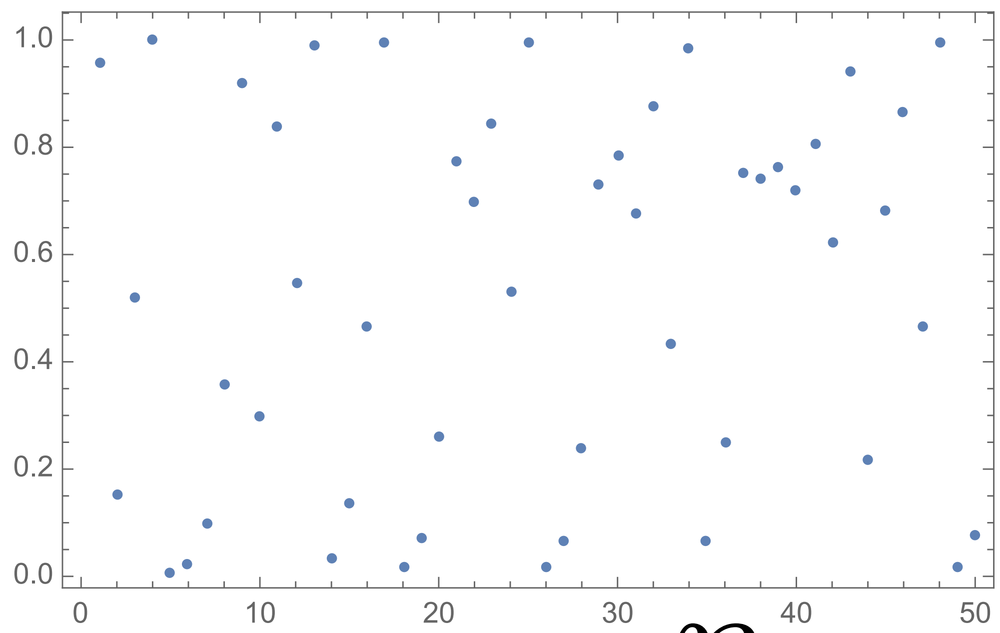
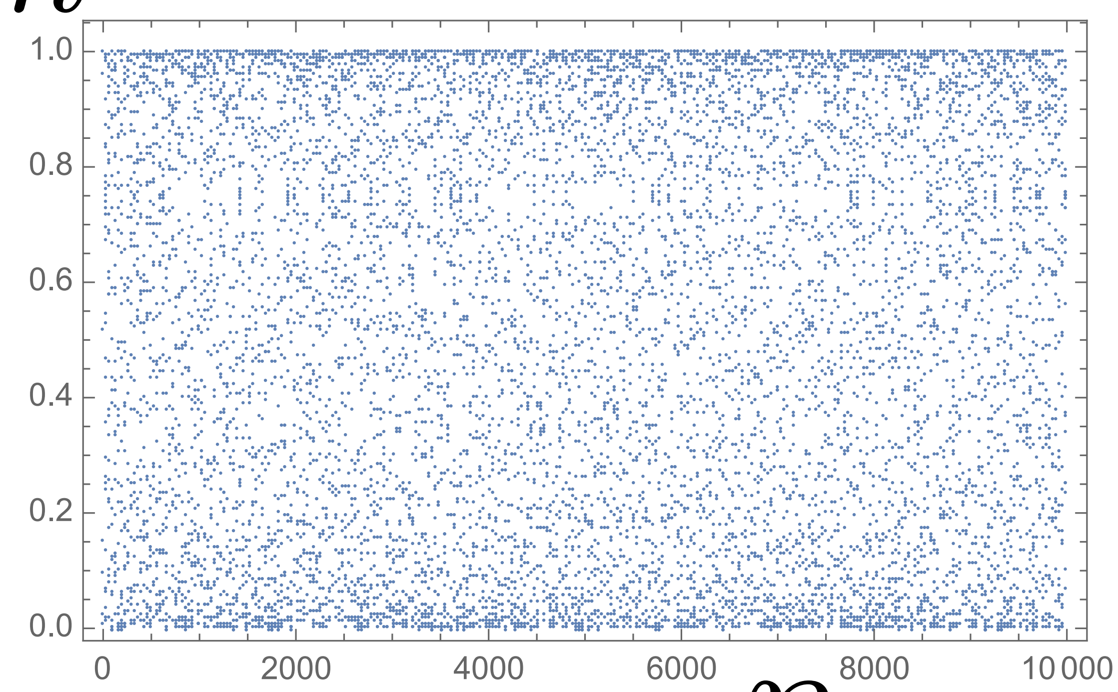



$n$ iteration

$x_n$ Large number of iterations!

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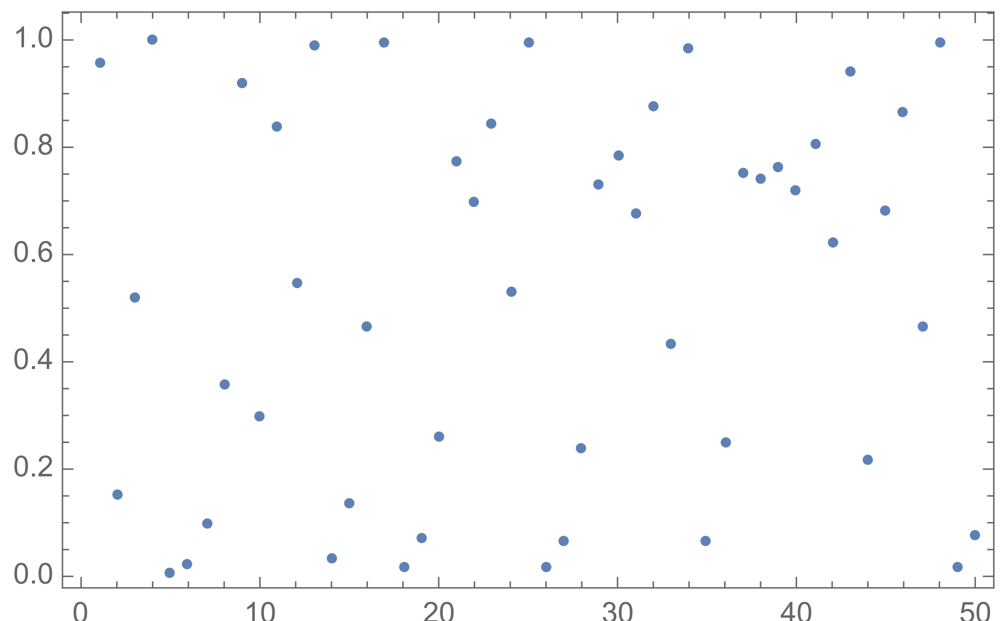
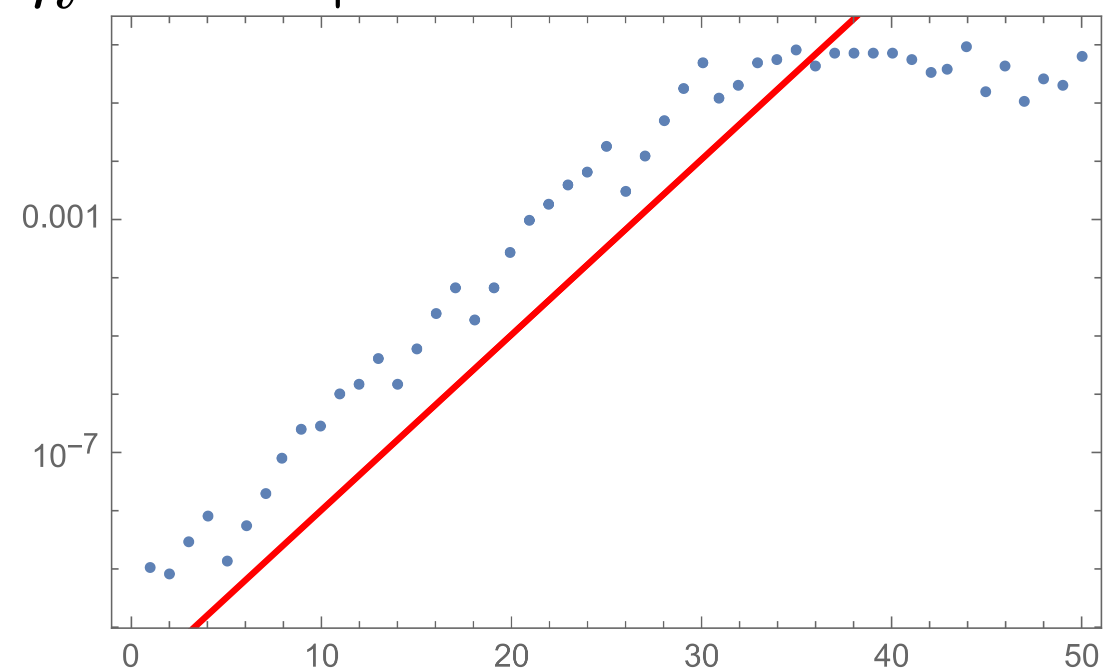



$n$ iteration

$|x'_n - x_n|$




$n$ iteration

# The Lyapunov Exponent

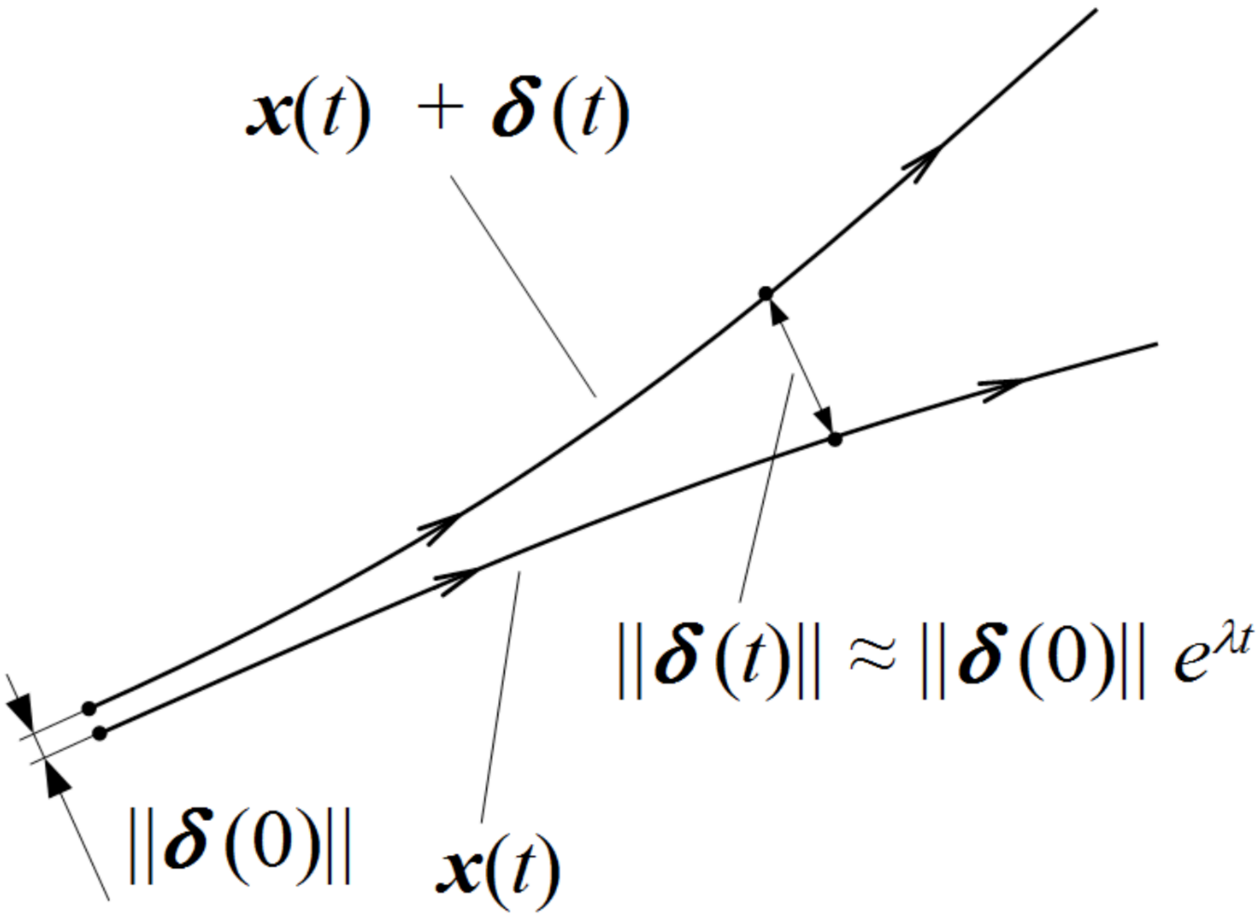
$$\mathbf{X}(t) = \mathbf{x}(t) + \delta(t)$$

Consider an infinitesimal perturbation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}[\mathbf{x}(t)]$$

Linearizing in $\delta$

$$\frac{d\delta}{dt} = \frac{d\mathbf{F}}{d\mathbf{x}}\delta \equiv \mathbf{F}'[\mathbf{x}(t)]\delta(t)$$



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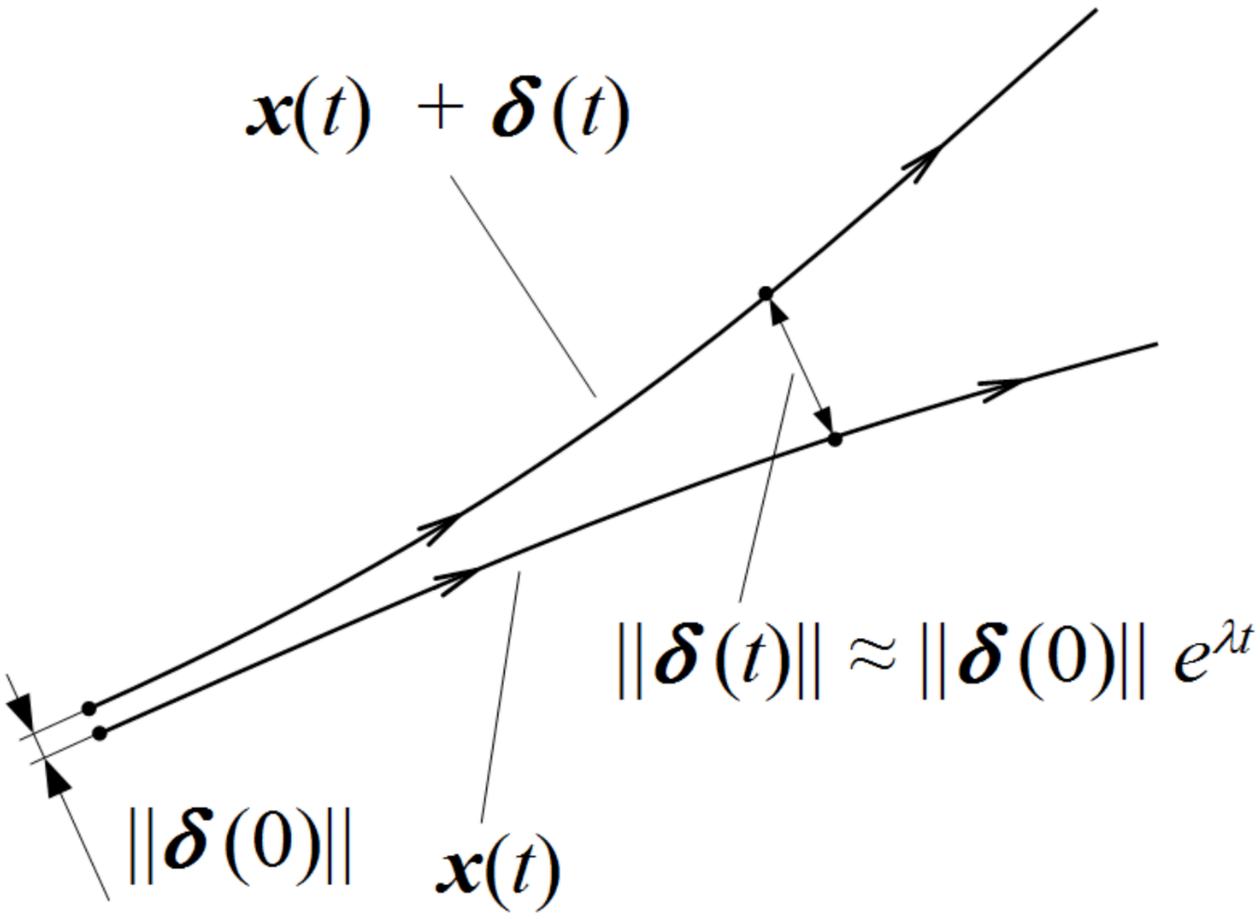
$$\delta(t) = \exp\left\{\int_0^t dt'\mathbf{F}'[\mathbf{x}(t')]\right\}\delta(0)$$



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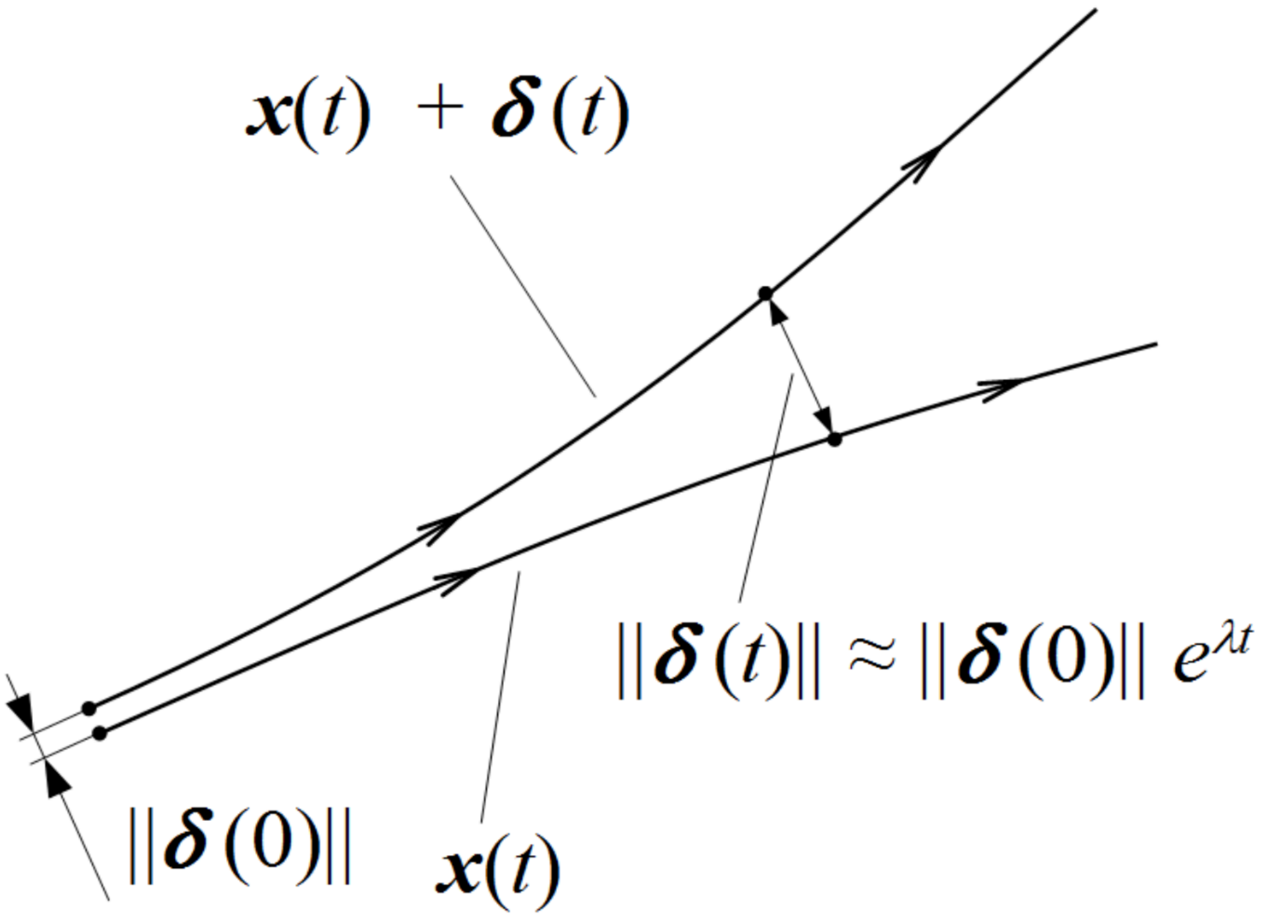

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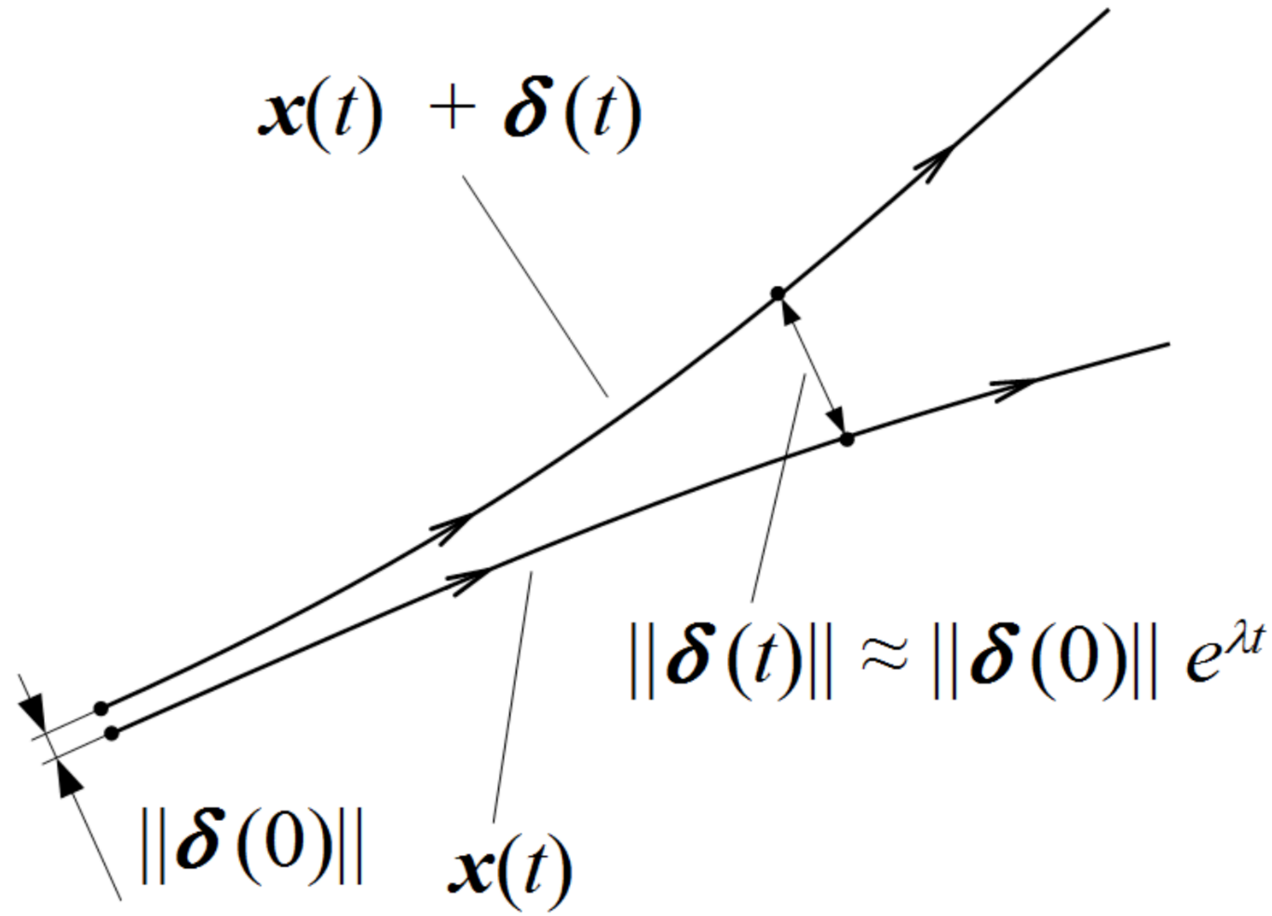



$$\delta_n = \frac{d\mathbf{f}}{d\mathbf{x}}\delta_{n-1} = \mathbf{f}'[\mathbf{x}_{n-1}]\delta_{n-1} = \mathbf{f}'[\mathbf{x}_{n-1}]\mathbf{f}'[\mathbf{x}_{n-2}]\delta_{n-2}$$

$$\delta_n = \mathbf{f}'[\mathbf{x}_{n-1}]\mathbf{f}'[\mathbf{x}_{n-2}]\ldots\mathbf{f}'[\mathbf{x}_0]\delta_0 = \prod_{k=0}^{n-1}\mathbf{f}'[\mathbf{x}_k]\delta_0$$

# The Lyapunov Exponent

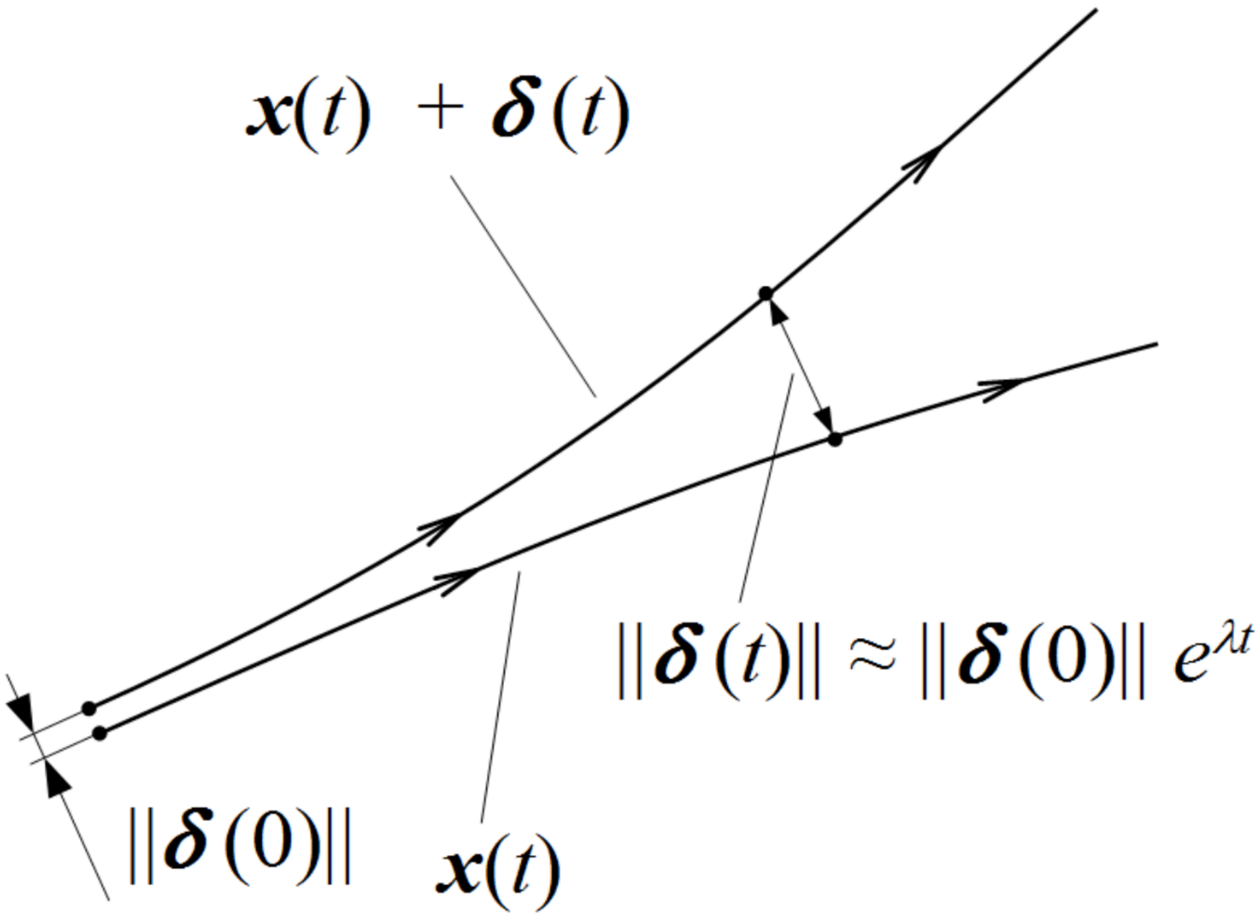
$$\mathbf{X}_n = \mathbf{x}_n + \delta_n$$

Consider an infinitesimal perturbation

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$$

$$\delta_n = \prod_{k=0}^{n-1} \mathbf{f}'[\mathbf{x}_k]\delta_0$$

$$\log|\delta_n| = \sum_{k=0}^{n-1} \log|\mathbf{f}'[\mathbf{x}_k]| + \log|\delta_0|$$



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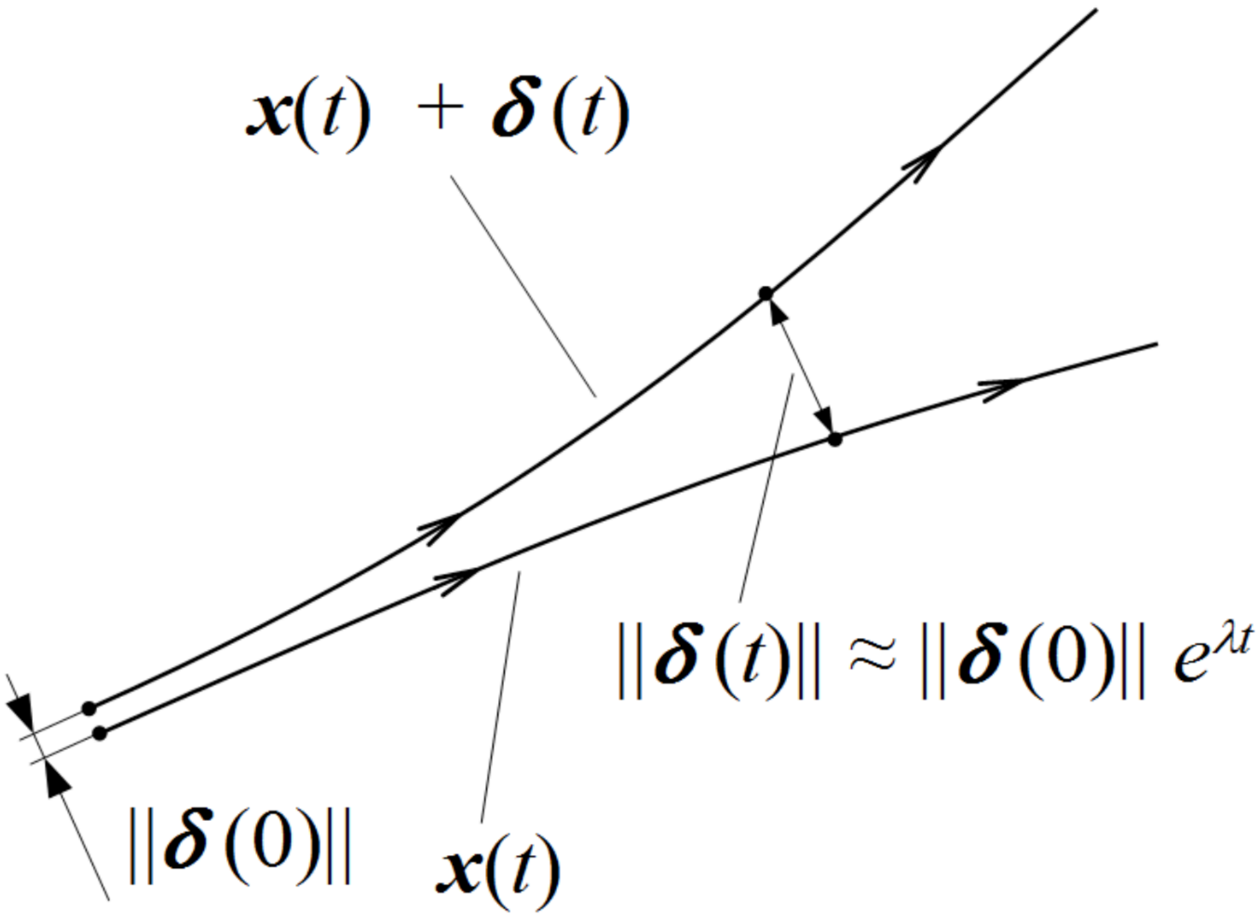
$$|\delta_n| = |\delta_0| \exp\left(\sum_{k=0}^{n-1} \log|\mathbf{f}'[\mathbf{x}_k]|\right)$$



# The Lyapunov Exponent

$$\delta(t) = \exp\left\{\int_0^t dt' \mathbf{F}'[\mathbf{x}(t')]\right\}\delta(0) \sim e^{\lambda t}$$

The Lyapunov Exponent
(continuous time)

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t dt' \mathbf{F}'[\mathbf{x}(t')]$$

$$|\delta_n| = |\delta_0| \exp\left(\sum_{k=0}^{n-1} \log|\mathbf{f}'[\mathbf{x}_k]|\right) \sim e^{\lambda n}$$

The Lyapunov Exponent
(discrete time)

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|\mathbf{f}'[\mathbf{x}_k]|$$

# The Lyapunov Exponent

Logistic map $$x_{n+1} = r x_n (1 - x_n)$$

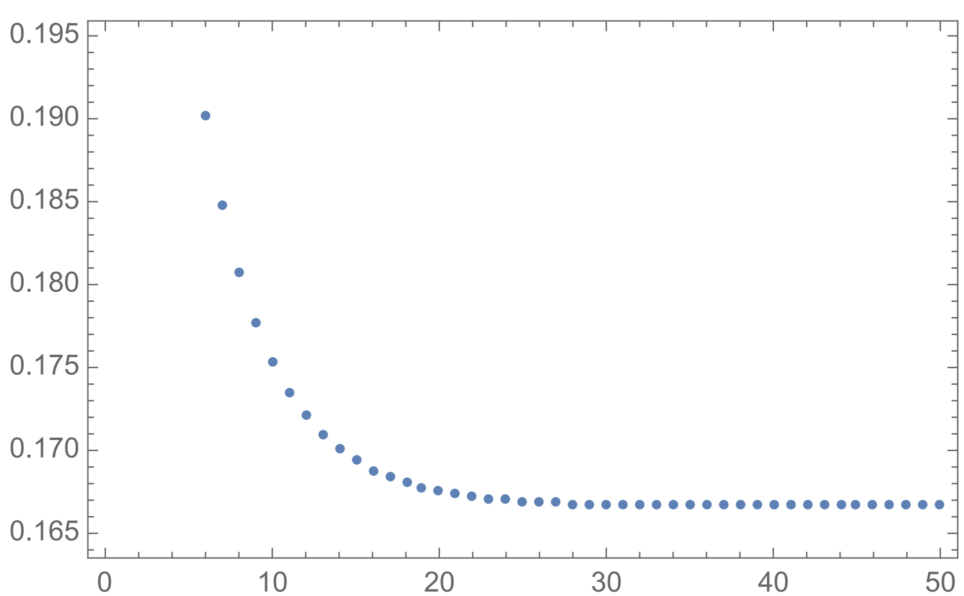
Example: $r = 1.2$ **stable**

$$x_0 = 0.4$$

$$x_0' = 0.4 + 10^{-9} \qquad x_0' \xrightarrow{f} x_1' \xrightarrow{f} x_2' \xrightarrow{f} \cdots \xrightarrow{f} x_n'$$

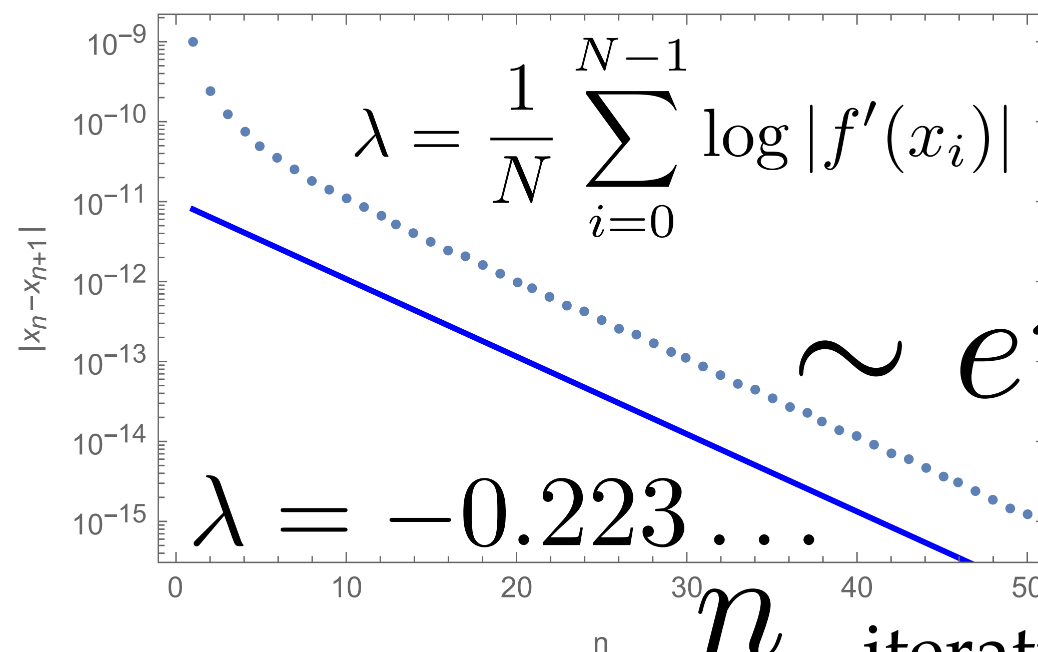
$$x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} \cdots \xrightarrow{f} x_n$$

$x_n$




$n$ iteration

$$|x_n' - x_n|$$




$$\lambda = \frac{1}{N} \sum_{i=0}^{N-1} \log |f'(x_i)|$$

$$\sim e^{\lambda t}$$

$$\lambda = -0.223\ldots$$

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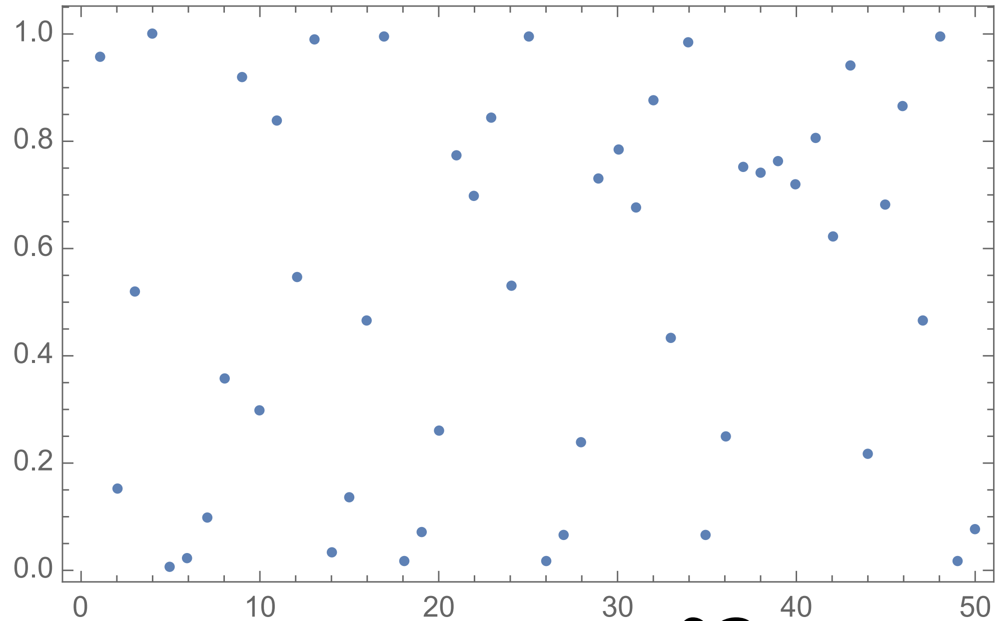
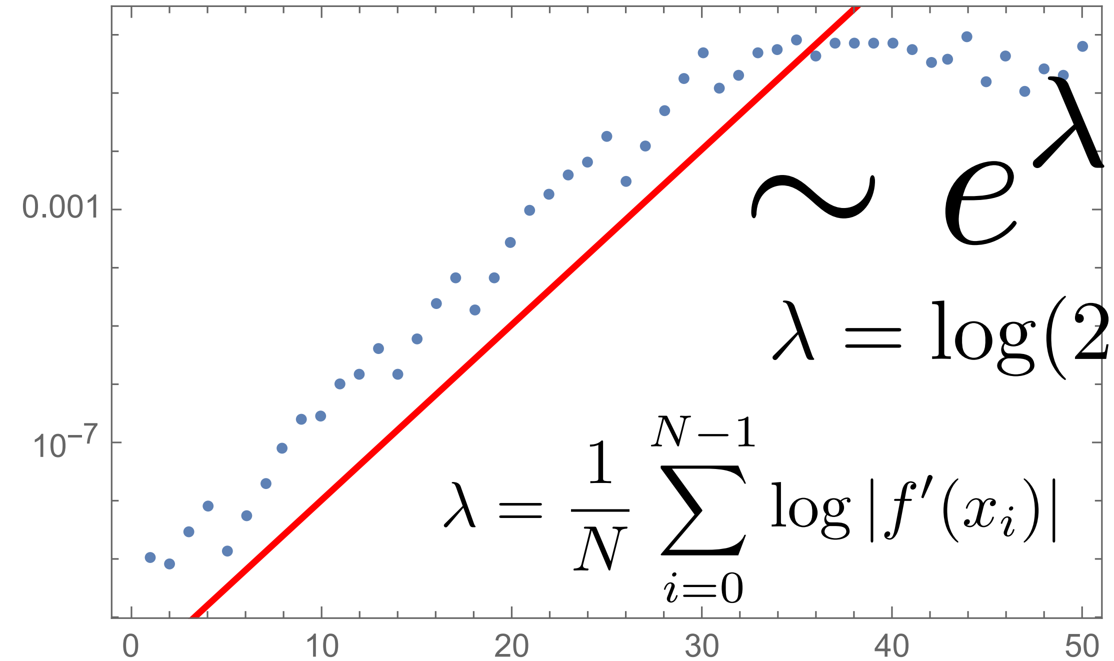
$$x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} \cdots \xrightarrow{f} x_n$$

$x_n$




$n$ iteration

$|x'_n - x_n|$




$\sim e^{\lambda t}$

$\lambda = \log(2)$

$$\lambda = \frac{1}{N} \sum_{i=0}^{N-1} \log |f'(x_i)|$$

$n$ iteration

# Route to chaos: Maps

Logistic map $$x_{n+1} = rx_n(1 - x_n)$$

Vary parameter, *r*

in the limit $n \to \infty$

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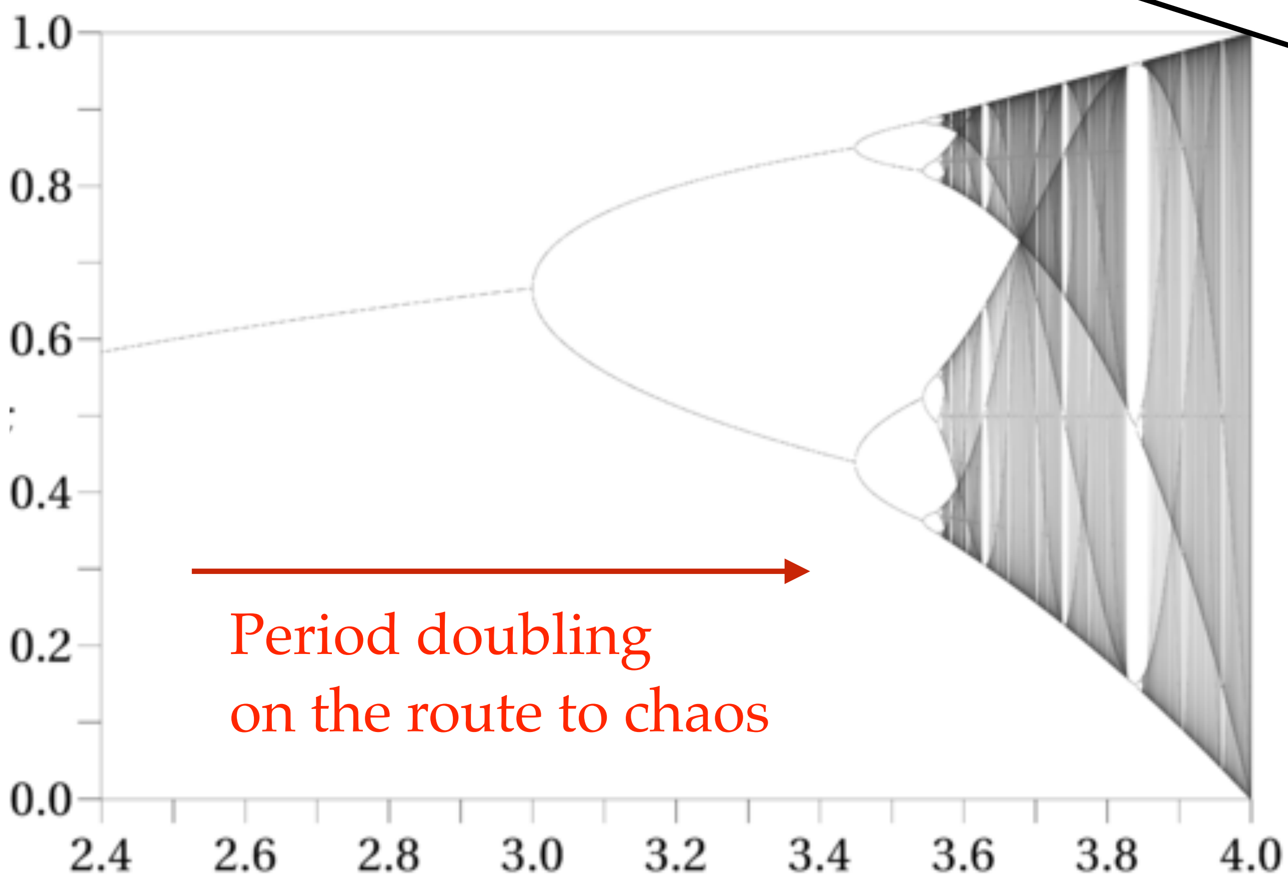



"Strength of chaos" $r$

Vary parameter, $r$

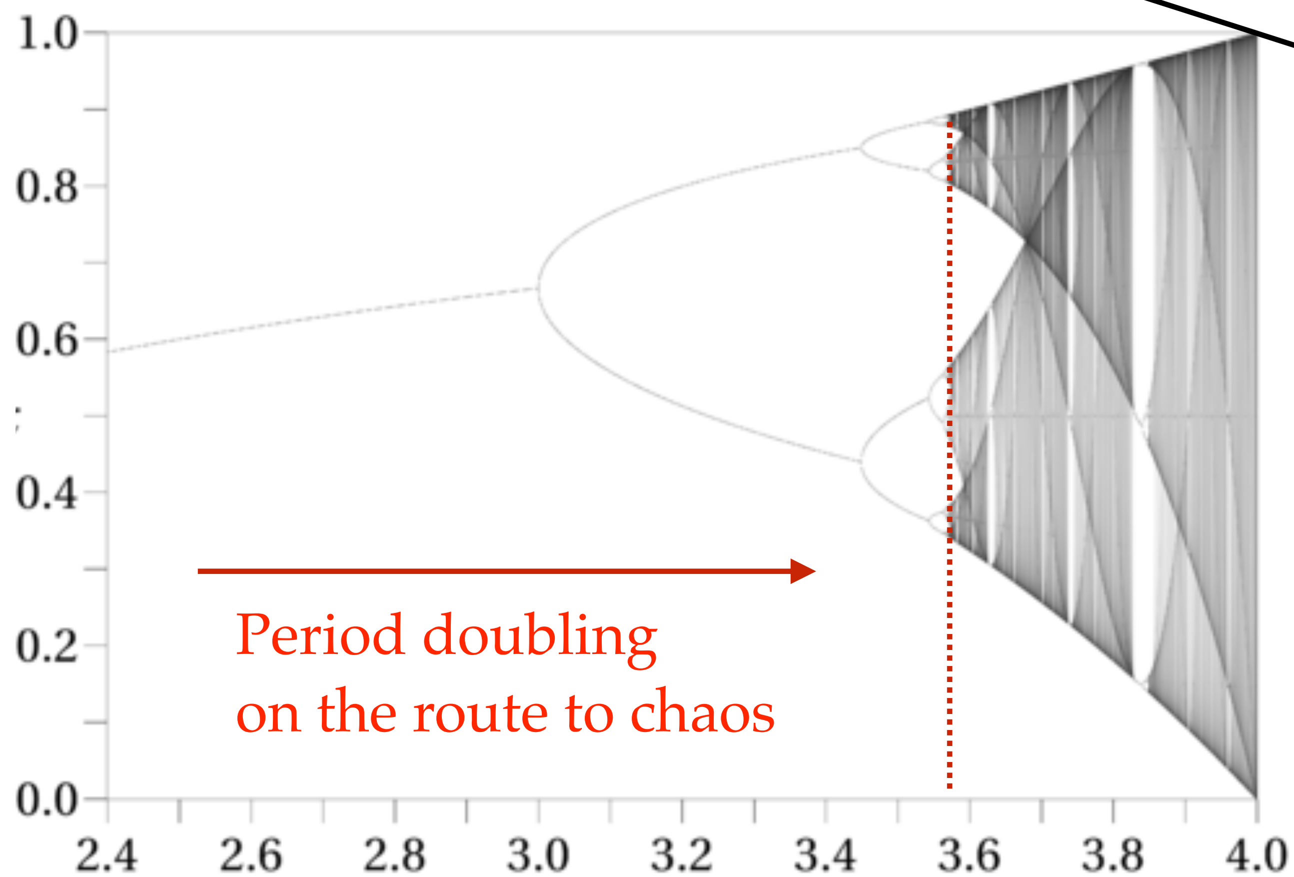
in the limit $n \to \infty$

# Route to chaos: Maps

Logistic map $x_{n+1} = r x_n (1 - x_n)$

Vary parameter, $r$

$\lim_{n \to \infty} x_n$

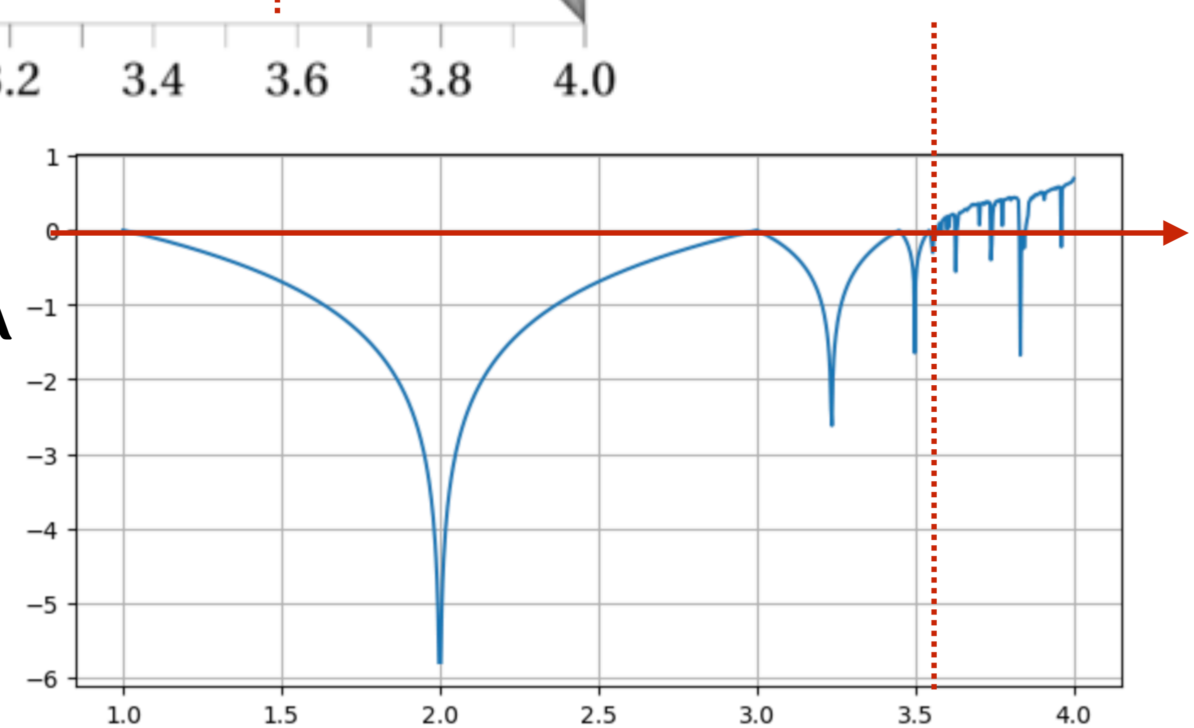



$r$

"Strength of chaos"

$\lambda$

Lyapunov exponent




$r$

# Route to chaos: Maps

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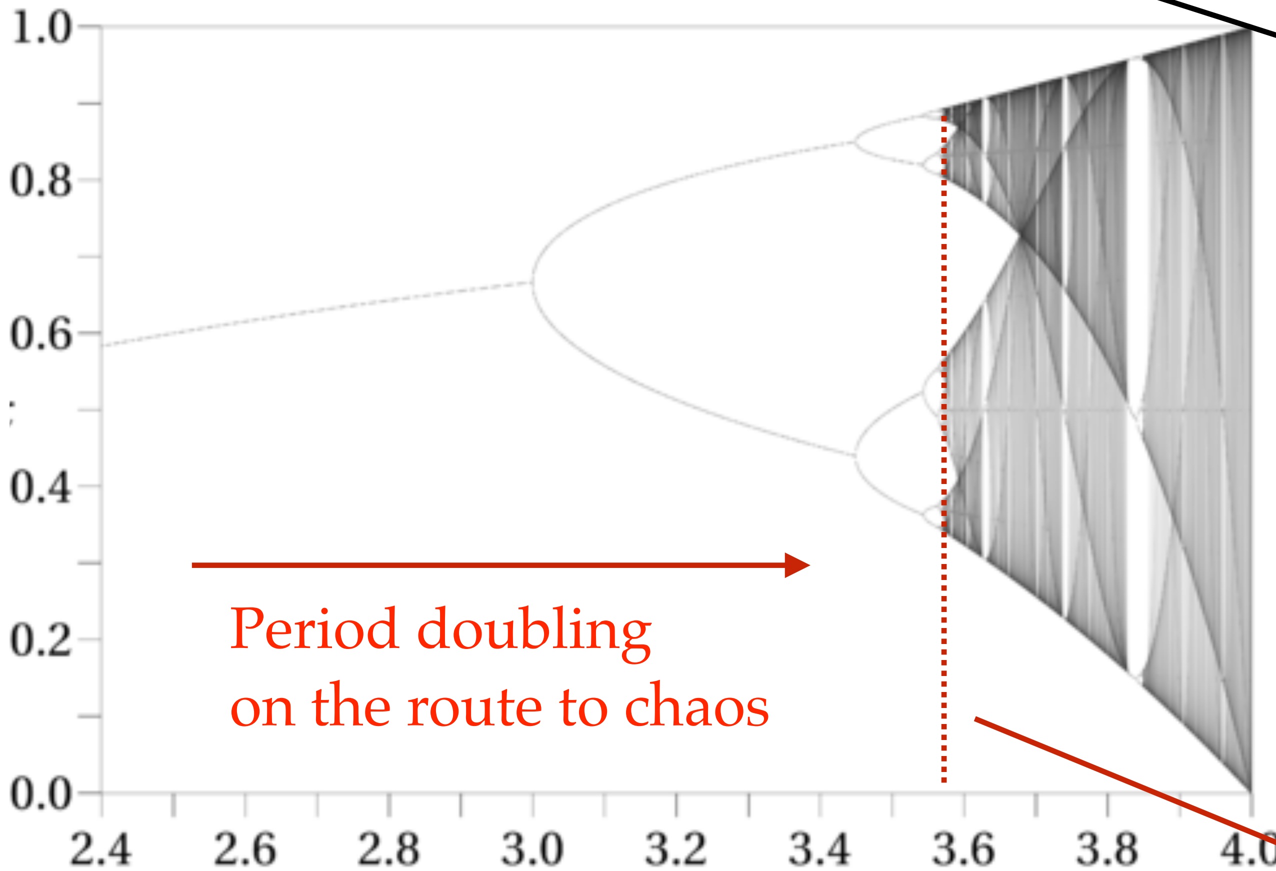



Vary parameter, $r$

Period doubling
on the route to chaos

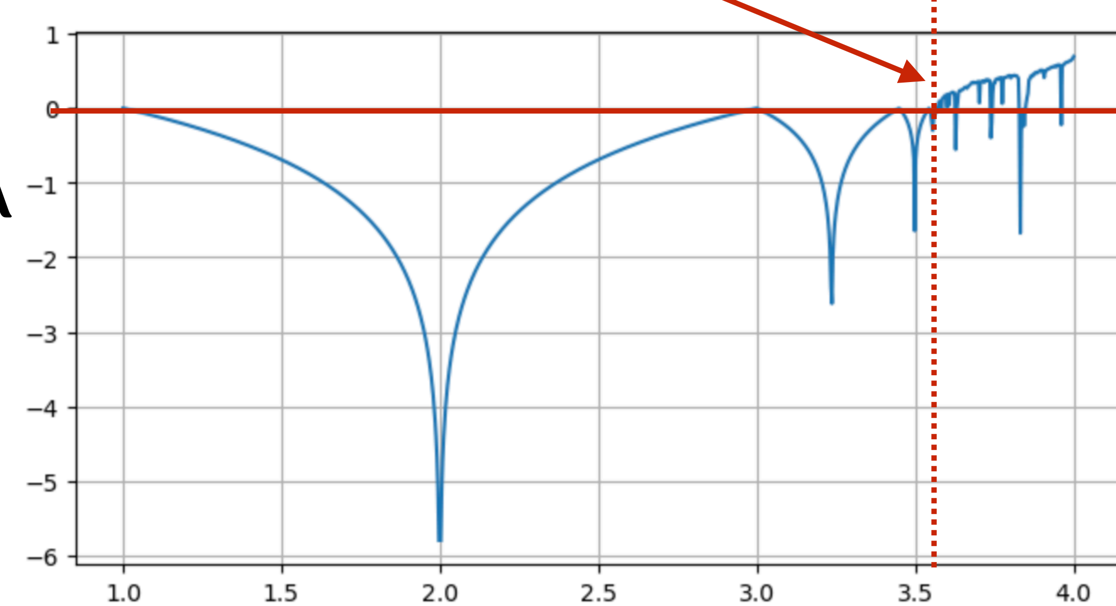
$r$

"Strength of chaos"

Onset of chaos

$\lambda$

Lyapunov
exponent




$r$

# A Simpler Chaotic Map

For *r=4*, where this is chaotic it can be reduced to the Bernoulli map

$$x_{n+1} = 4x_n(1 - x_n) \qquad x_n = \sin^2\left(\frac{\pi t_n}{2}\right)$$

$$\sin^2(\pi t_{n+1}/2) = 4\sin^2(\pi t_n/2)\cos^2(\pi t_n/2) = \sin^2(\pi t_n)$$

$$\longrightarrow \qquad \pi t_{n+1}/2 = \pi t_n \bmod \pi$$

Bernoulli map
(or doubling map)

$$t_{n+1} = 2t_n \mod 1$$

$$0 < t < 1$$

$$\boxed{\lambda_{\text{Lyapunov}} = \ln 2}$$

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binary bit strings on a classical computer

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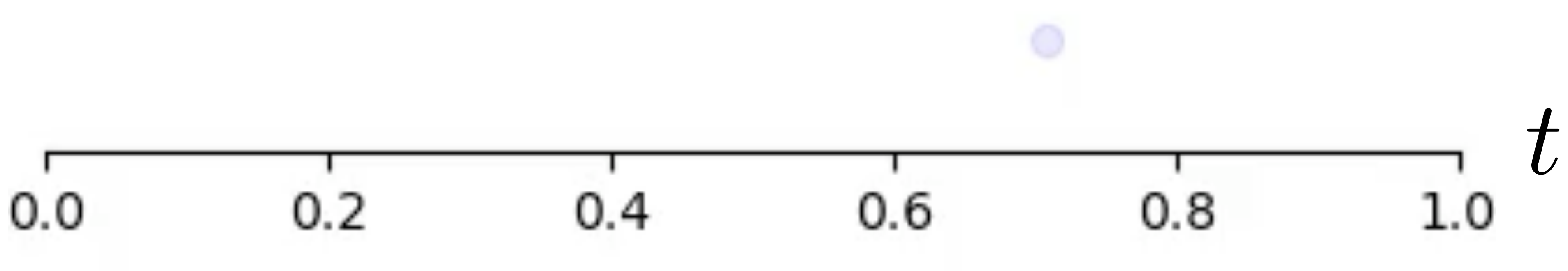
Starting from

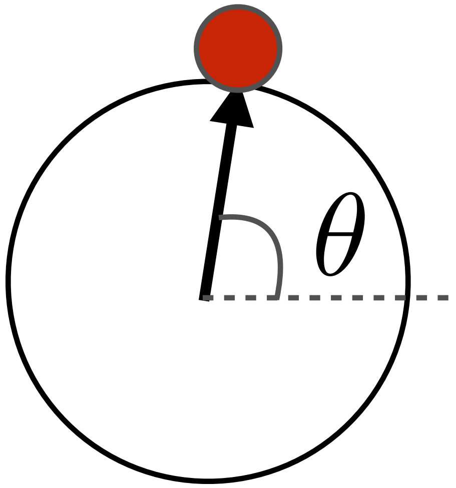
$$t_0 = \frac{1}{\sqrt{2}}$$



# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**

$$H_{\mathrm{KR}} = \frac{p^2}{2I} + k\cos(\theta) \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$




*Chaos in dynamical systems*, Ott

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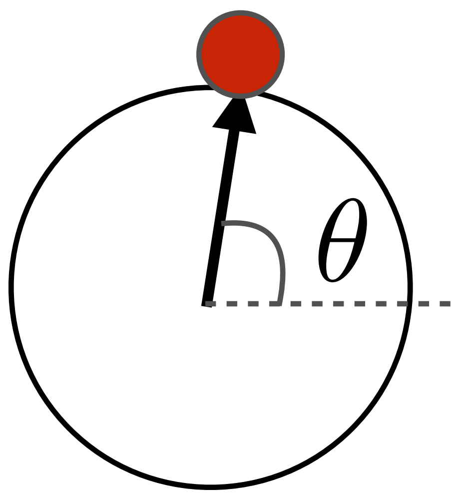


$$H_{\mathrm{KR}} = \frac{p^2}{2I} + k\cos(\theta) \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Equations of motion

$$\frac{dp}{dt} = k \sin\theta \sum_n \delta(t - nT)$$

$$\frac{d\theta}{dt} = \frac{p}{I}$$

Constant at times not equal to $nT$

Let $p_n, \theta_n$ be these constant at times $t = nT + 0^+$

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# Chaotic Hamiltonian Systems: Kicked Rotor

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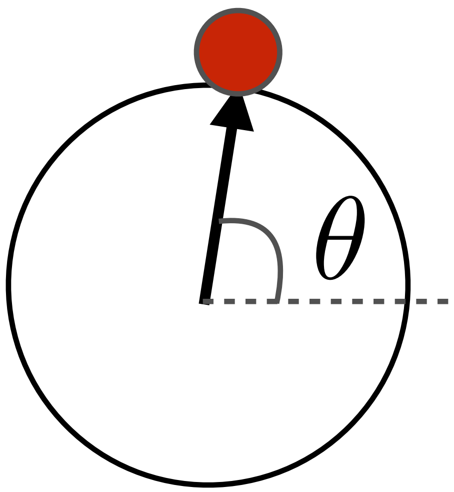


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$$\int_{mT}^{(m+1)T} dt(\ldots)$$

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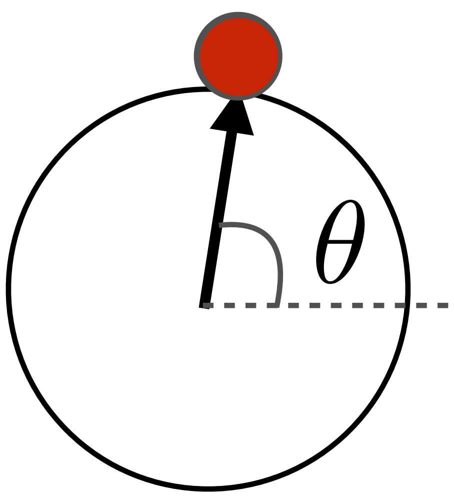


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Integrating the equations, across one period $\int_{mT}^{(m+1)T} dt(\dots)$

Let $p_n, \theta_n$ be these constant at times $t = nT + 0^+$

$$p((m+1)T) - p(mT) = p_{m+1} - p_m = k \sin\theta_{m+1}$$

$$\theta((m+1)T) - \theta(mT) = \theta_{m+1} - \theta_m = p_m \frac{T}{I}$$

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# Chaotic Hamiltonian Systems: Kicked Rotor

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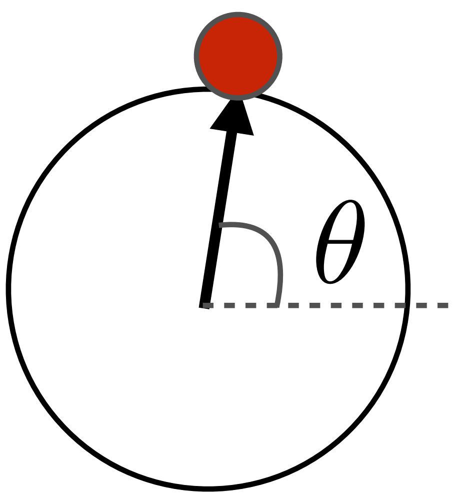



This is the "standard map"

$$\theta_{m+1} = (\theta_m + p_m \frac{T}{I}) \text{ modulo } 2\pi$$

Setting *T/I=1*

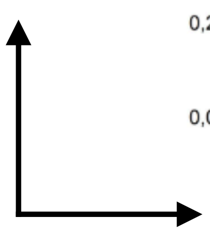
$$p_{m+1} = k \sin \theta_{m+1} + p_m$$

$k = 0$

Integrable

$p_{m+1} = p_m$

Increasing $k$



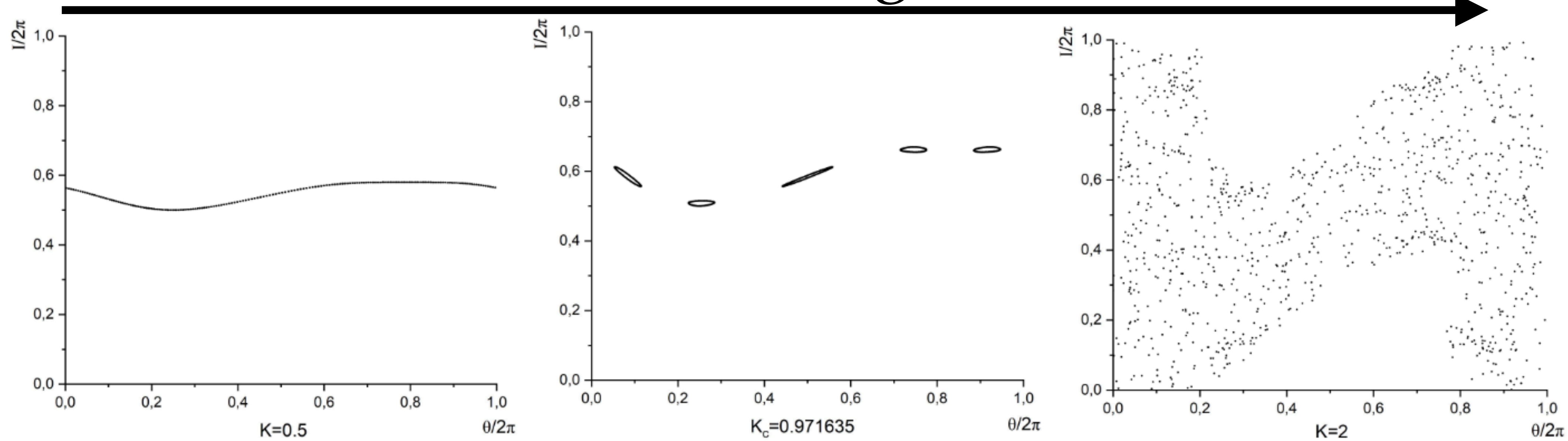




Figure 1: Single trajectory starting at point $(\pi/2, \pi)$ for different $K$

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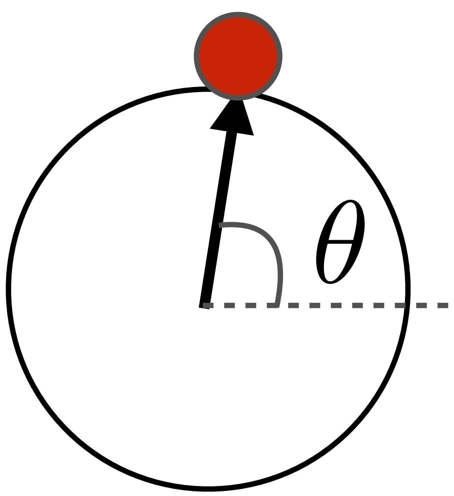
# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**

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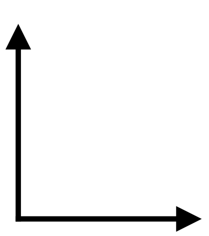


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Mixed phase space

$k < k_c$

Critical

$k_c = 0.971635$

Chaotic

$k > k_c$

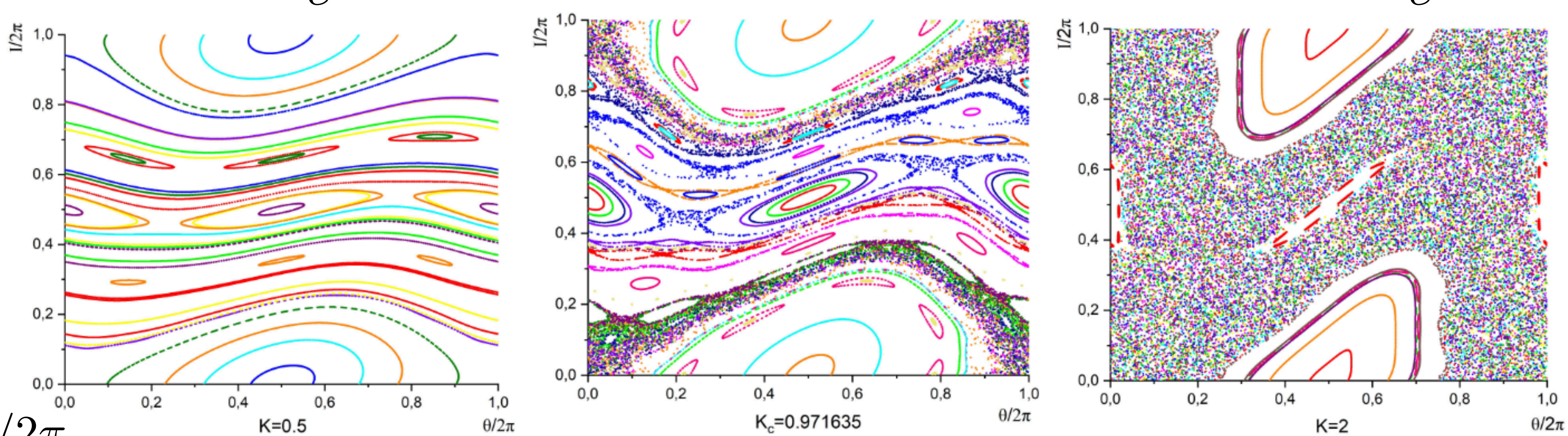
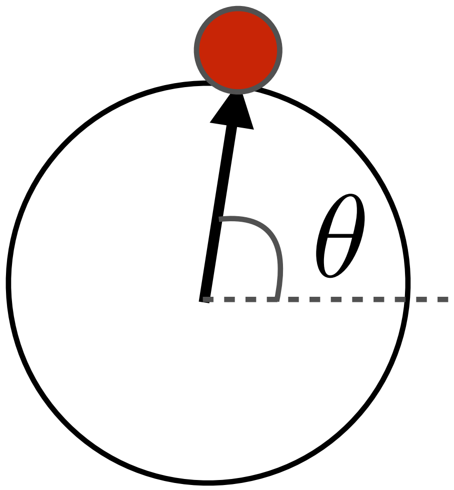



Figure 2: Poincaré surfaces for different $K$

# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**

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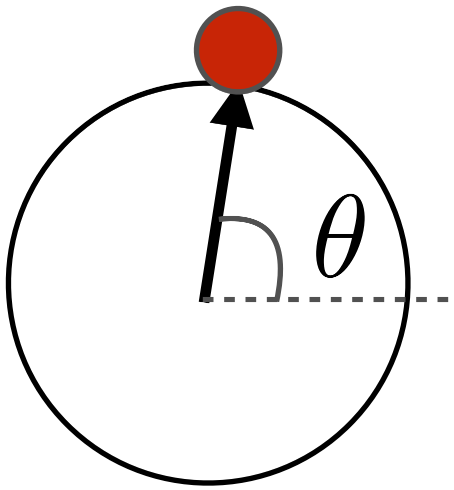


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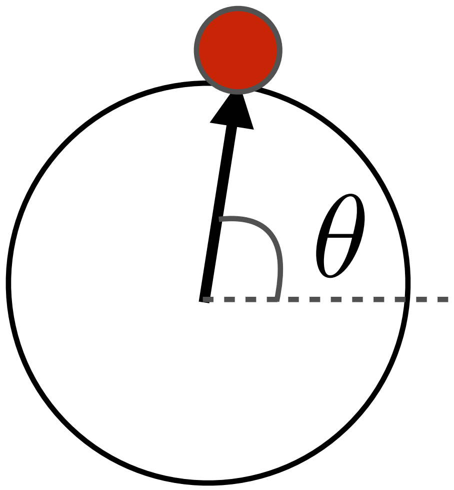
# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**

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$$p_{m+1} = \sum_{j=1}^{m+1} \sin(\theta_j) + p_0$$

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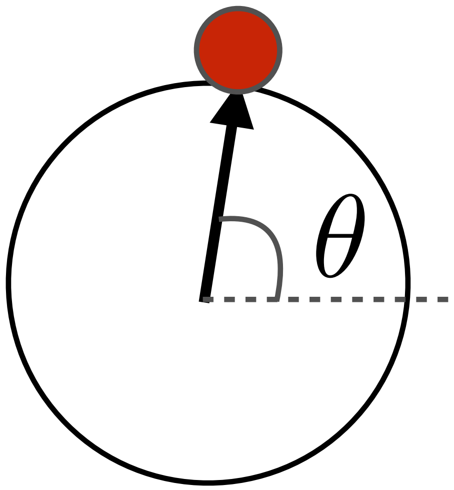
# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**




Diffusion in the kicked rotor

$$p_{m+1} = \sum_{j=1}^{m+1} \sin(\theta_j) + p_0$$

Average over kicks

$$\langle (p_m - p_0)^2 \rangle = k^2 \sum_{i,j}^{m} \langle \sin\theta_i \sin\theta_j \rangle$$

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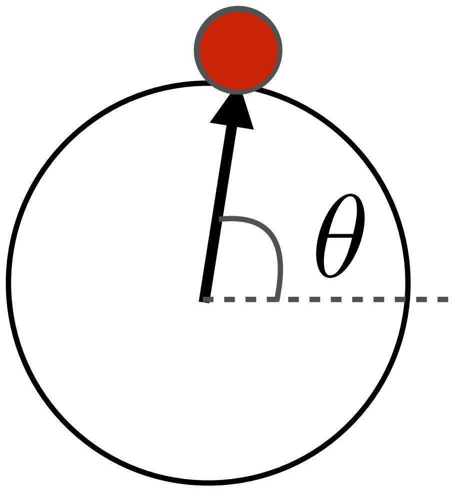


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In the limit *k>>1*, kicks are totally random and uncorrelated

$$\langle \sin\theta_i \sin\theta_j \rangle = \delta_{i,j} \langle \sin(\theta_i)^2 \rangle = \delta_{i,j} \frac{1}{2}$$

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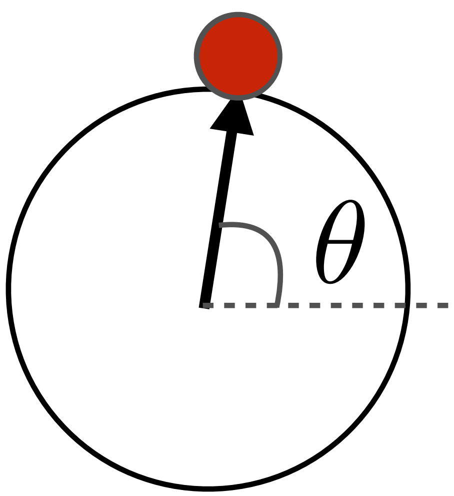


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$$\langle (p_m - p_0)^2 \rangle = k^2 m/2$$

The root-mean-square of a diffusion process

$$\langle (p_m - p_0)^2 \rangle_{\rm rms} = 2Dm$$

$$D \approx k^2/4$$

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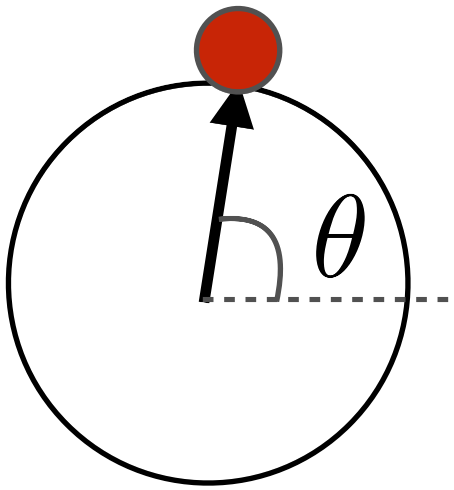
# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**

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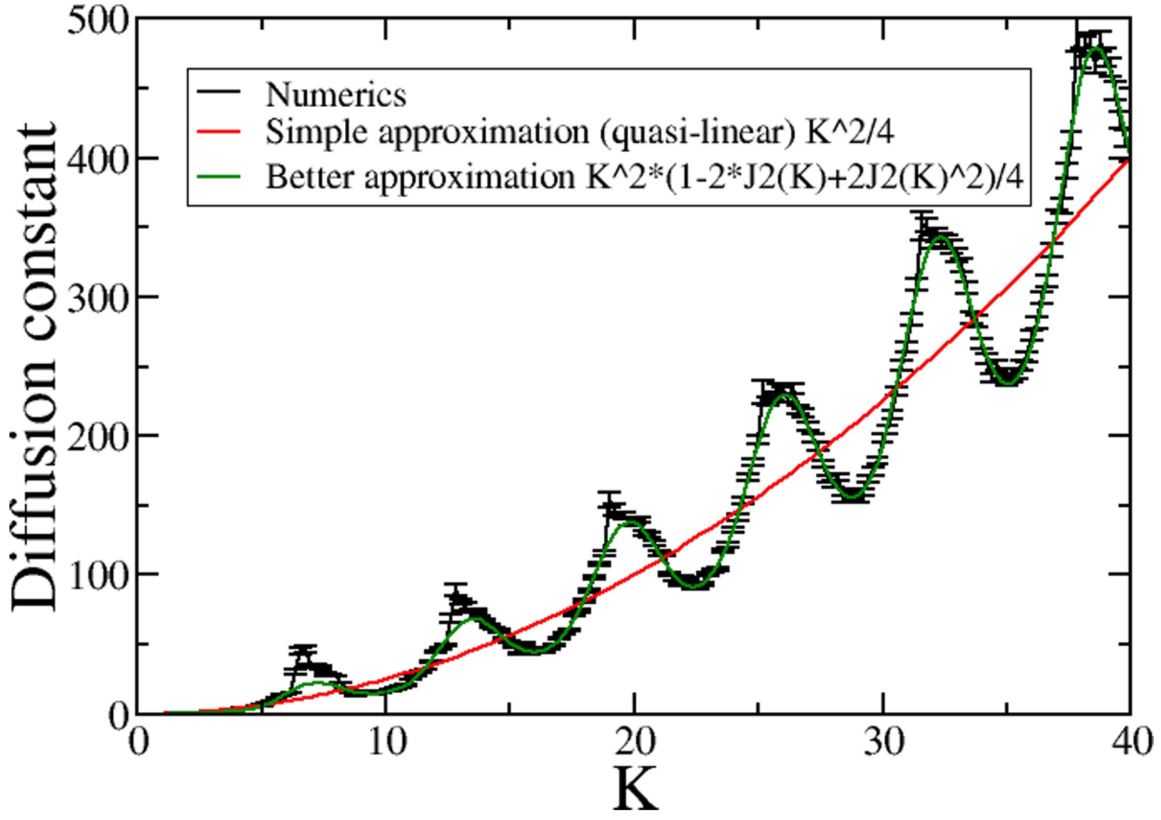


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# Chaotic Hamiltonian Systems: Kicked Rotor

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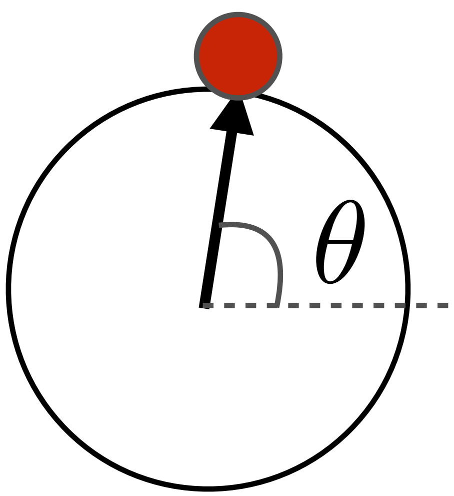


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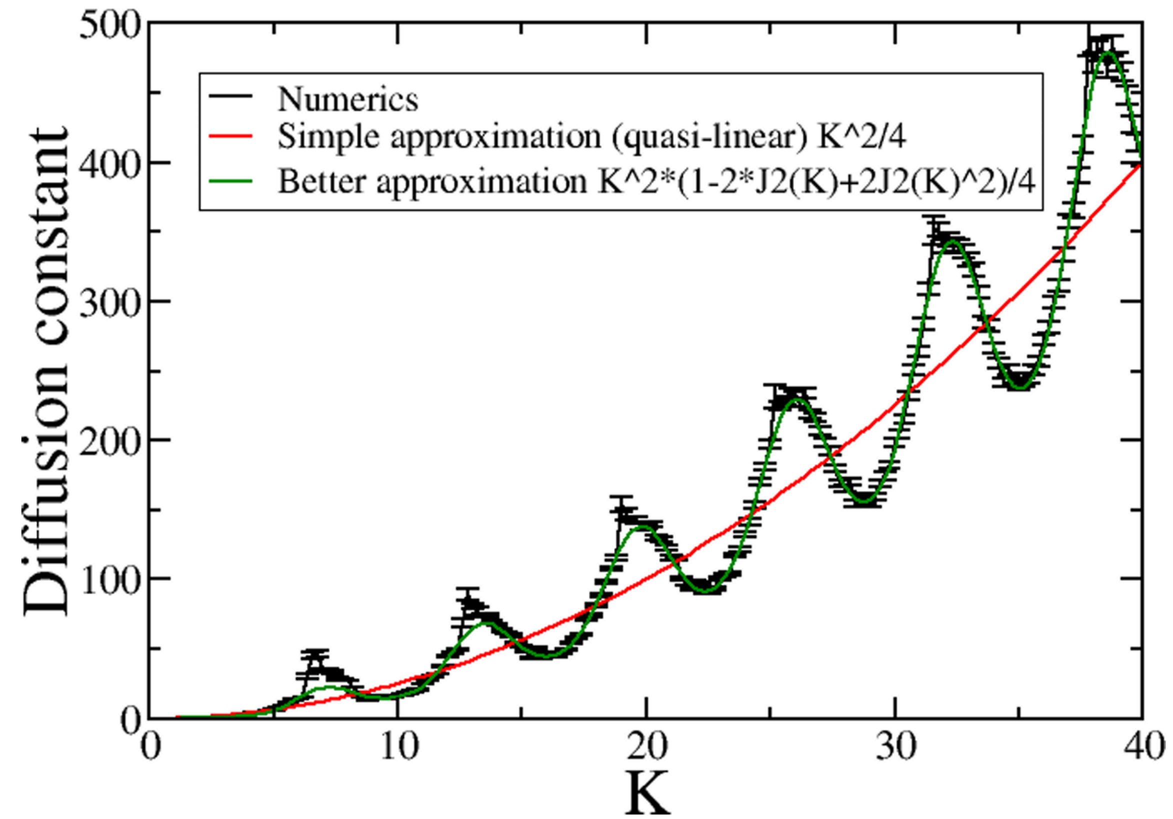



Corrections in the diffusion constant (Correlations between kicks)

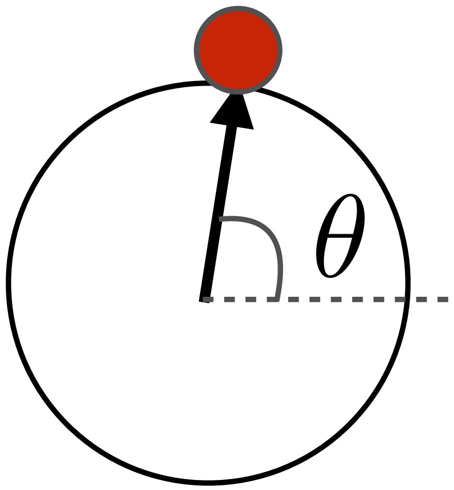
$$D \approx \frac{k^2}{4}(1 - 2J_2(k) + 2J_2^2(k))$$

Rechester and White, PRL (1980)

*Chaos in dynamical systems*, Ott

# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**




Lyapunov exponent

$$d(t) \approx d(0)e^{\lambda(x,p)t},$$

$$d(t) = \sqrt{[x'(t) - x(t)]^2 + [p'(t) - p(t)]^2}$$

$$\lambda = \langle\!\langle \lambda(x,p) \rangle\!\rangle = \left\langle\!\!\left\langle \lim_{t\to\infty} \lim_{d(0)\to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)} \right\rangle\!\!\right\rangle$$

*Chaos in dynamical systems*, Ott    Chirikov, Phys. Rep. (1979)    Rosenbaum, Ganeshan, Galitski, PRL (2017)

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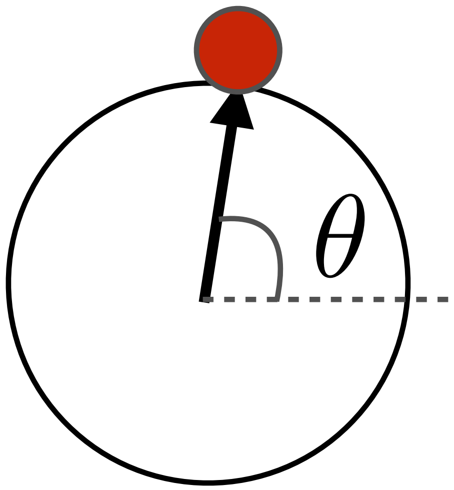


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2 nearby trajectories

$$\begin{cases} p_{n+1} = p_n + K \sin x_n \\ x_{n+1} \underset{\text{mod } 2\pi}{=} x_n + p_{n+1} \end{cases}$$

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$$\eta_n = p'_n - p_n$$

$$\xi_n = x'_n - x_n$$

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*Chaos in dynamical systems*, Ott

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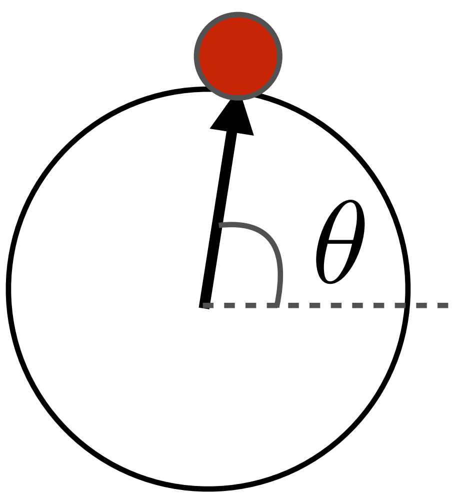



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# Chaotic Hamiltonian Systems: Kicked Rotor

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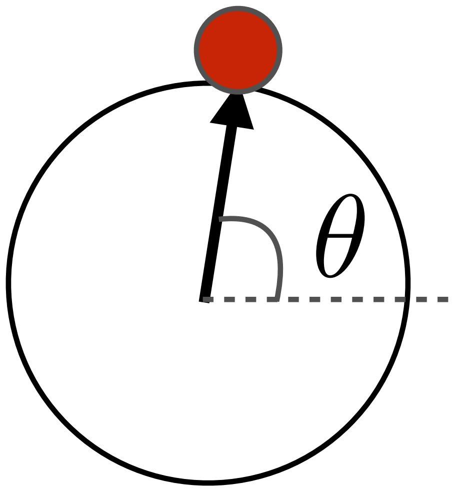


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Trig identity

$$\sin x'_n - \sin x_n = \sin(x_n + \xi_n) - \sin x_n$$
$$= \sin x_n (\cos \xi_n - 1) + \sin \xi_n \cos x_n.$$

To leading order in the deviations

$$\eta_{n+1} = \eta_n + (K \cos x_n)\xi_n$$
$$\xi_{n+1} = \xi_n + \eta_{n+1}$$

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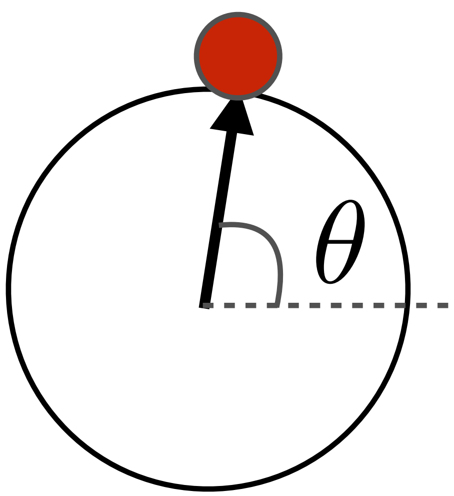
# Chaotic Hamiltonian Systems: Kicked Rotor

**Kicked rotor**




Lyapunov exponent

$$\eta_{n+1} = \eta_n + (K \cos x_n)\xi_n$$

$$\xi_{n+1} = \xi_n + \eta_n + (K \cos x_n)\xi_n$$

To leading order in the deviations

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & k\cos(x_n) \\ 1 & k\cos(x_n)+1 \end{pmatrix} \begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix}$$

Eigenvalues $\lambda_\pm = 1 + \frac{k\cos(x_n)}{2} \pm \sqrt{k\cos(x_n)\left(1 + \frac{k\cos(x_n)}{4}\right)}$

Focusing on $k > 0$

$$\lambda \approx \langle\langle l_+[k\cos(x_n)]\rangle\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln |l_+[k\cos(x)]|$$

*Chaos in dynamical systems*, Ott Chirikov, Phys. Rep. (1979) Rosenbaum, Ganeshan, Galitski, PRL (2017)

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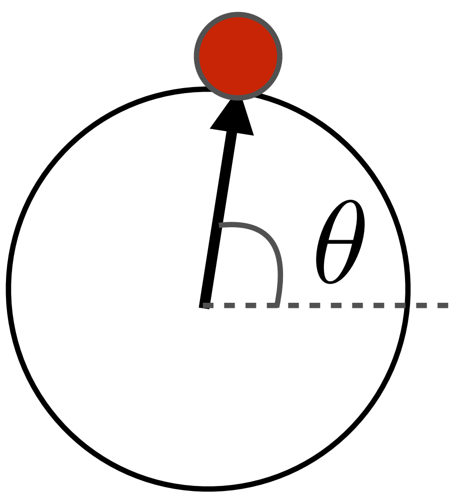



At large k

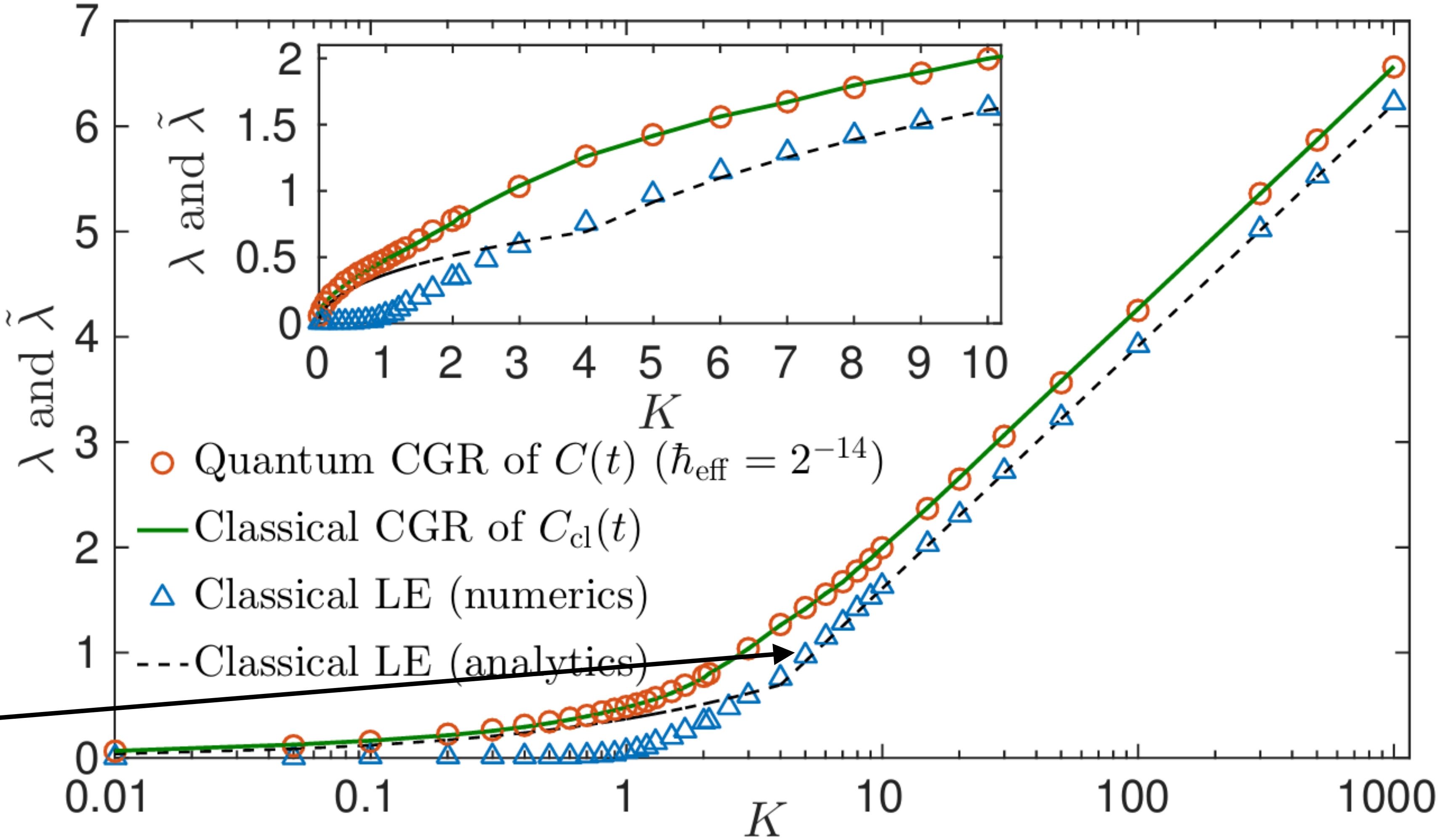
$$\lambda \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln(k|\cos(x)|)$$

$$= \log(k/2)$$

Lyapunov exponent

Eigenvalues $\lambda_\pm = 1 + \frac{k\cos(x_n)}{2} \pm \sqrt{k\cos(x_n)\left(1+\frac{k\cos(x_n)}{4}\right)}$

$k > 0$ $\lambda \approx \langle\langle l_+[k\cos(x_n)]\rangle\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln|l_+[k\cos(x)]|$

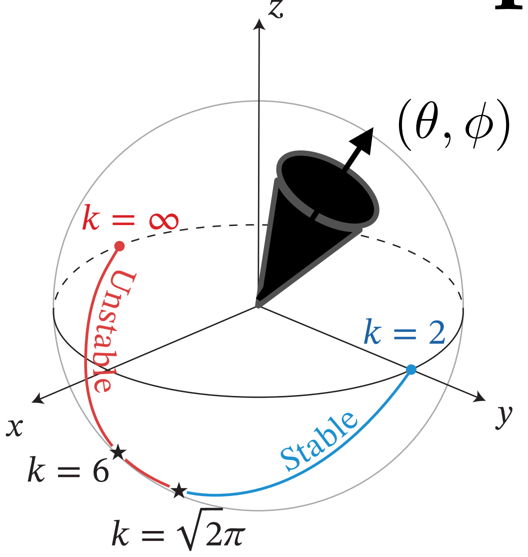



**Focus on** **blue data** **and on black dashed line**

*Chaos in dynamical systems*, Ott    Chirikov, Phys. Rep. (1979)    Rosenbaum, Ganeshan, Galitski, PRL (2017)

# Chaotic Hamiltonian Systems: Kicked Top

**Kicked top**




$$H_{\mathrm{KT}} = \alpha J_y + \frac{k J_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$\alpha = \pi/(2T)$.

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

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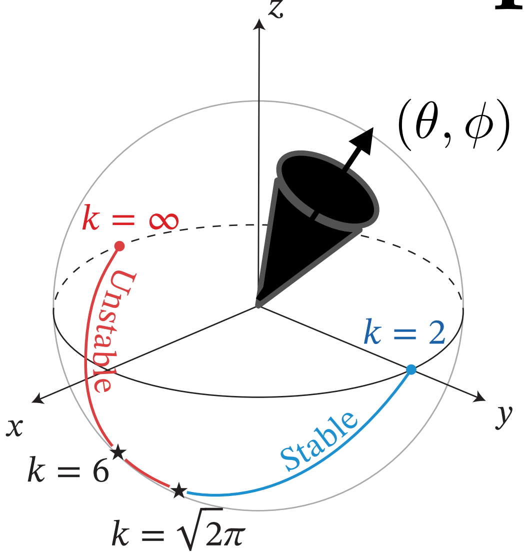


$$H_{\mathrm{KT}} = \alpha J_y + \frac{kJ_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$\alpha = \pi/(2T)$

$\{J_i, J_j\} = i\epsilon_{ijk} J_k$

Conserved angular momentum $\{\mathbf{J}^2, H\} = 0$

Equation of Motion $$\frac{d\mathbf{J}}{dt} = \{\mathbf{J}, H\} = -\mathbf{J} \times \frac{\partial H}{\partial \mathbf{J}}$$

Classical limit $S \to \infty$ $\quad x = J_x/S, y = J_y/S, z = J_z/S$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

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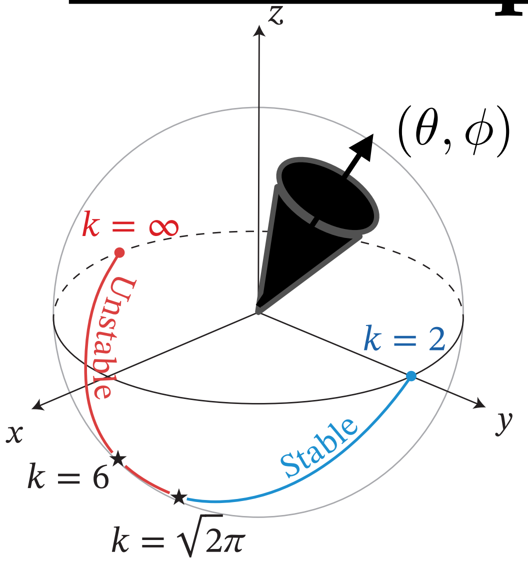


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Integrating the equations, across one period

$$\int_{mT}^{(m+1)T} dt(\dots)$$

$$x_{n+1} = z_n\cos(kx_n) + y_n\sin(kx_n)$$

$$y_{n+1} = -z_n\sin(kx_n) + y_n\cos(kx_n)$$

$$z_{n+1} = -x_n$$

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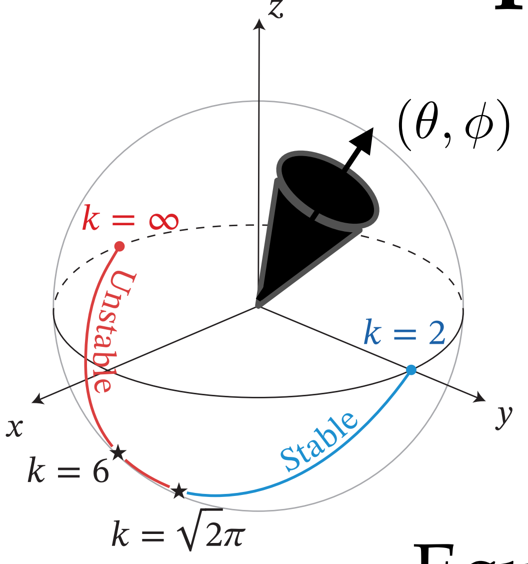


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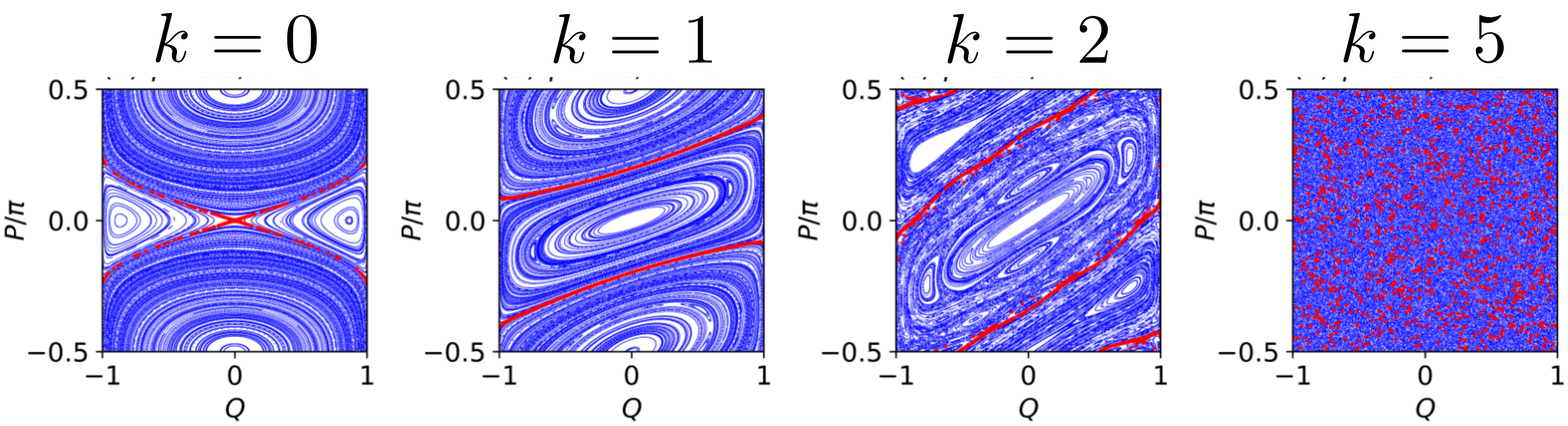
Phase space coordinates

$z = Q$

$x = \sqrt{1 - Q^2} \cos(2P)$

$y = \sqrt{1 - Q^2} \sin(2P)$




Passarelli, et al, Quantum (2025)

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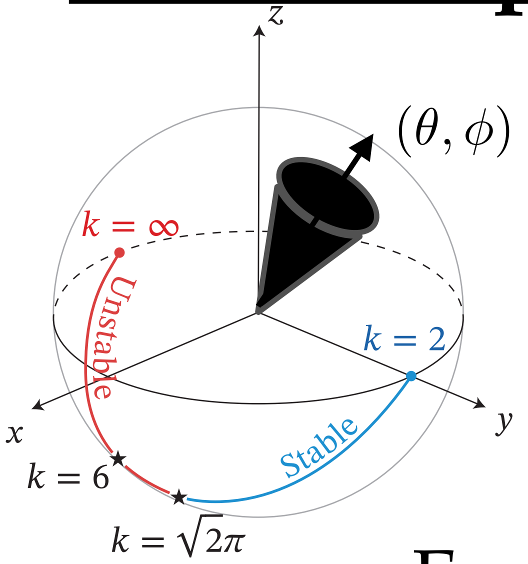


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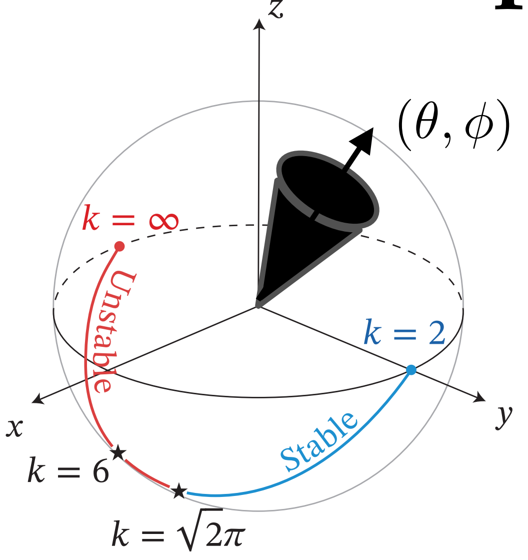
$$z_{n+1} = -x_n$$

Phase space is compact, NO notion of diffusion in the long time limit

But in the limit of short times and large kicking strengths it is diffusive

# Chaotic Hamiltonian Systems: Kicked Top

## Kicked top

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Phase space is compact, NO notion of diffusion in the long time limit

Diffusion in the kicked top, in the limit of large kicking strength (*k*>>*1*) and SHORT times!

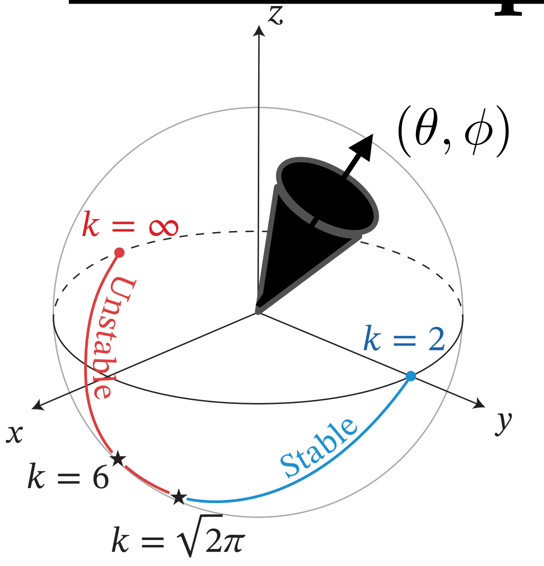
$$\langle [z_{n+1} - z_n]^2 \rangle = \langle [x_n + z_n]^2 \rangle$$

*k*>>*1*, all kicks are random and independent

$$\approx \langle x_n^2 \rangle + \langle z_n^2 \rangle = 2/3$$

# Chaotic Hamiltonian Systems: Kicked Top

**Kicked top**

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Diffusion in the kicked top, in the limit of large kicking strength (*k*>>*1*) and SHORT times!

$$\langle [z_{n+1} - z_n]^2 \rangle \approx \langle x(t)^2 \rangle + \langle z(t)^2 \rangle = 2/3$$

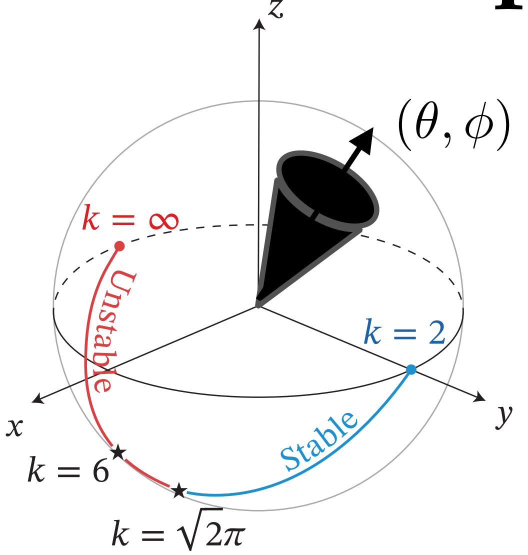
*k*>>*1*, all kicks are random and independent

$$\langle [z_{n+1} - z_n]^2 \rangle = 2D \qquad D \approx 1/3$$

*k* independent!
Consequence of compact phase space

# Chaotic Hamiltonian Systems: Kicked Top

**Kicked top**

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Stable fixed points of the map $x_{n+1} = x_n = x_0$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

Prasad, et al, Kulkarni, JHP, et al, arXiv (2026)

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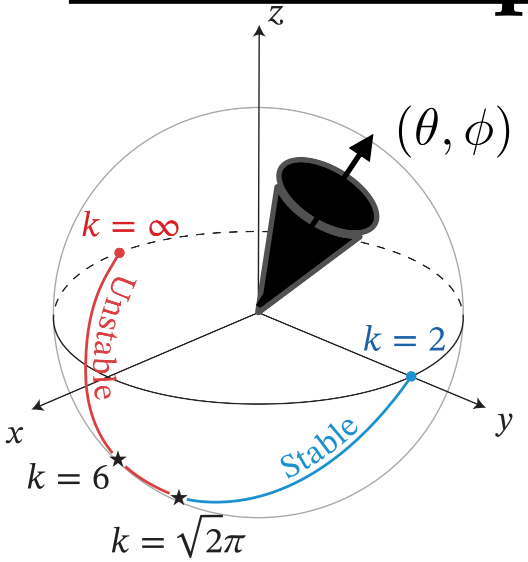


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Stable fixed points of the map $x_{n+1} = x_n = x_0$

$$z_0 = -x_0 \qquad y_0 = x_0 \sin(kx_0) + y_0 \cos(kx_0)$$

$$y_0/x_0 = \sin(kx_0)/(1 - \cos(kx_0)) = \cot(kx_0/2)$$

$$x_0^2 + y_0^2 + z_0^2 = 1 \qquad x_0^2 = \frac{\sin^2\left(\frac{kx_0}{2}\right)}{1 + \sin^2\left(\frac{kx_0}{2}\right)}$$

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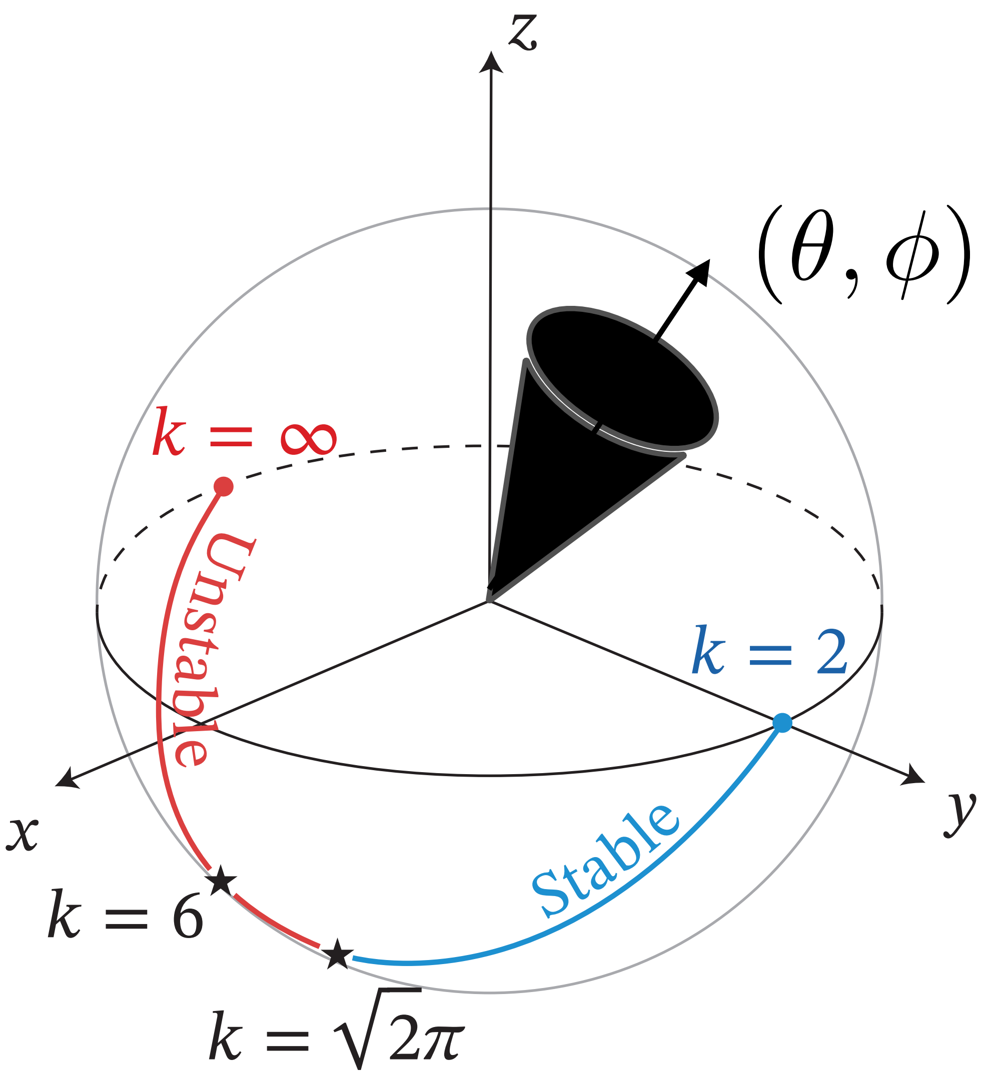
Stable fixed points of the map




$$x_0^2 = \frac{\sin^2\left(\frac{kx_0}{2}\right)}{1+\sin^2\left(\frac{kx_0}{2}\right)}$$

This must satisfy $\leq 1/2$

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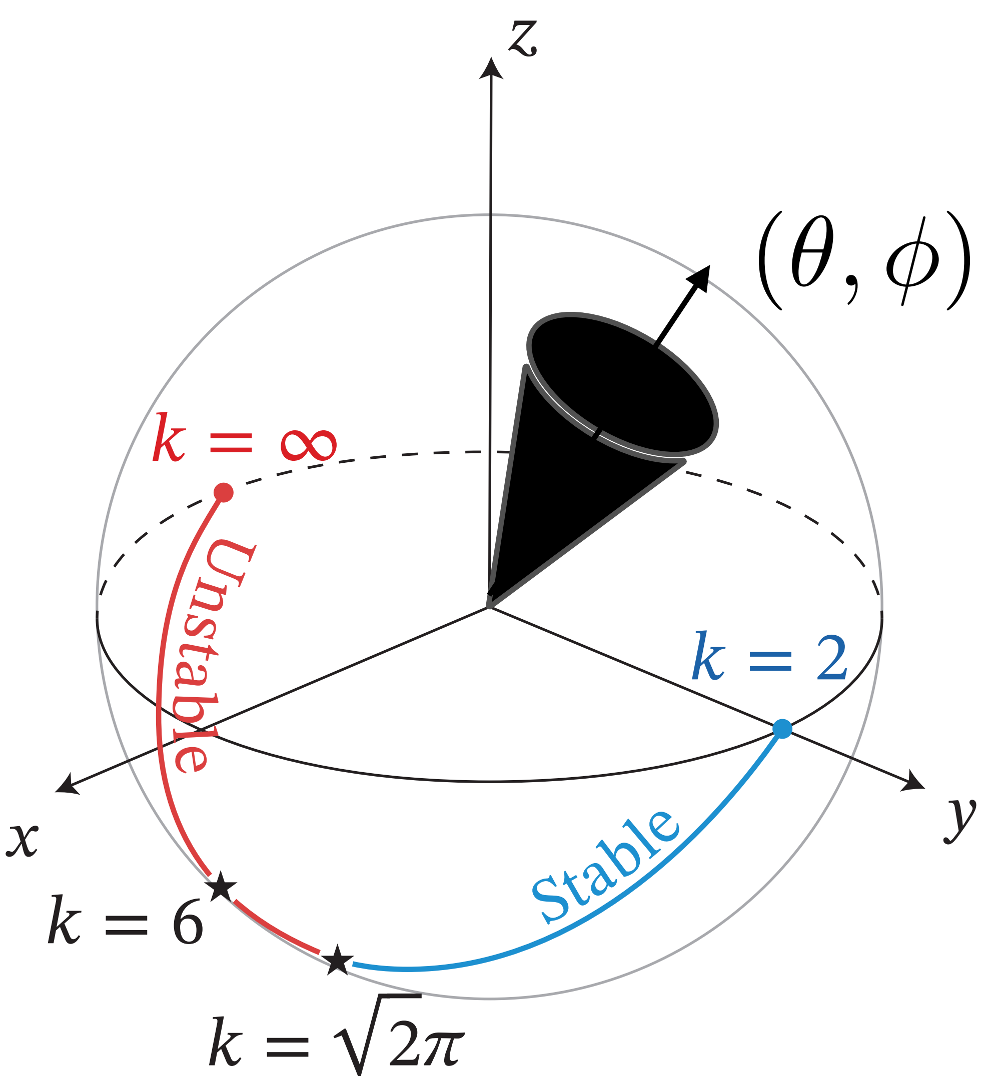


$$x_0^2 = \frac{\sin^2\left(\frac{kx_0}{2}\right)}{1 + \sin^2\left(\frac{kx_0}{2}\right)}$$

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For at least one solution to exist in the small $x_0$ limit

$$x_0^2 = (kx_0/2)^2 \left(1 - (kx_0/2)^2 + \ldots\right)$$

$k \geq 2$ Guarantees one solution exists with $x_0 > 0$

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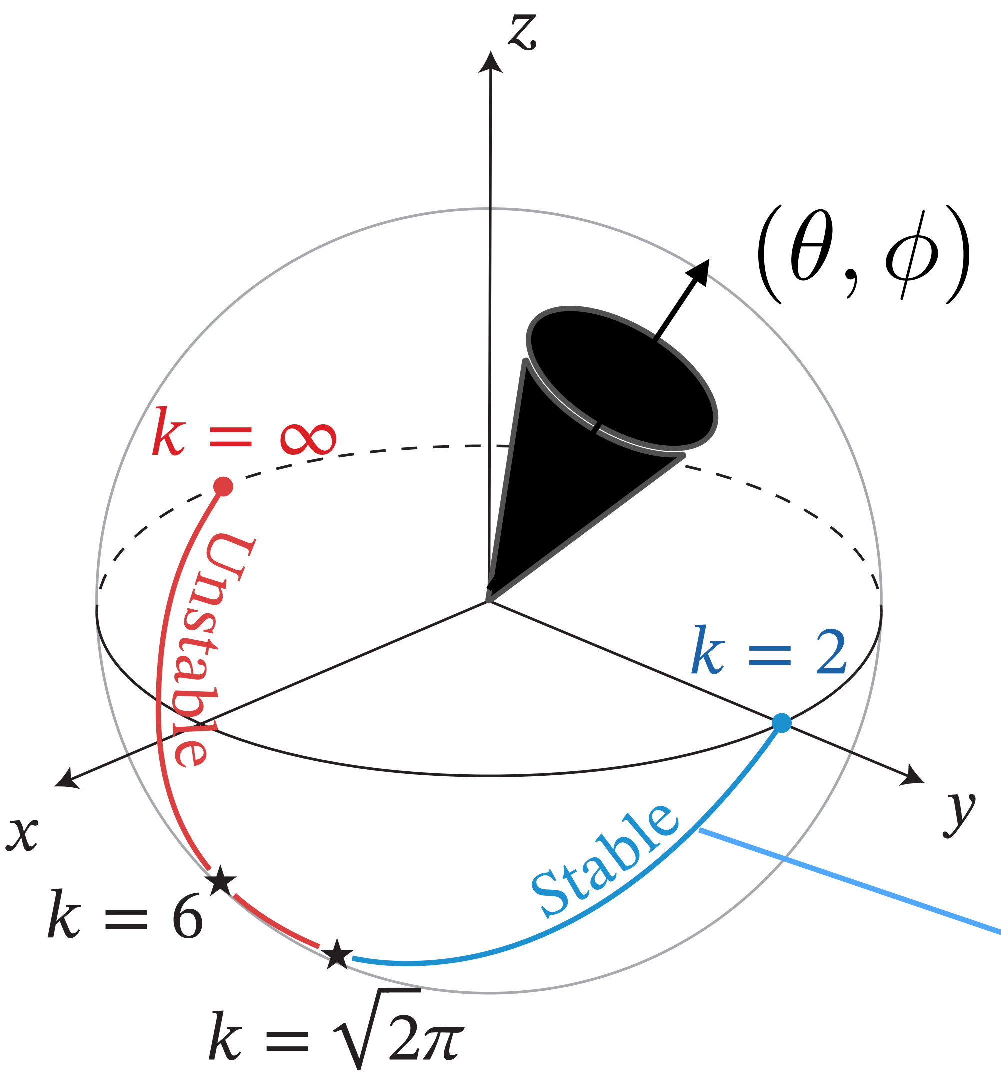


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$$x_0^2 = (kx_0/2)^2\left(1-(kx_0/2)^2+\dots\right)$$

$k \geq 2$ Guarantees one solution exists with $x_0 > 0$

$$x_0 = \frac{\sqrt{3}}{2}\sqrt{k-2} + O((k-2)^{3/2})$$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

Prasad, et al, Kulkarni, JHP, et al, arXiv (2026)

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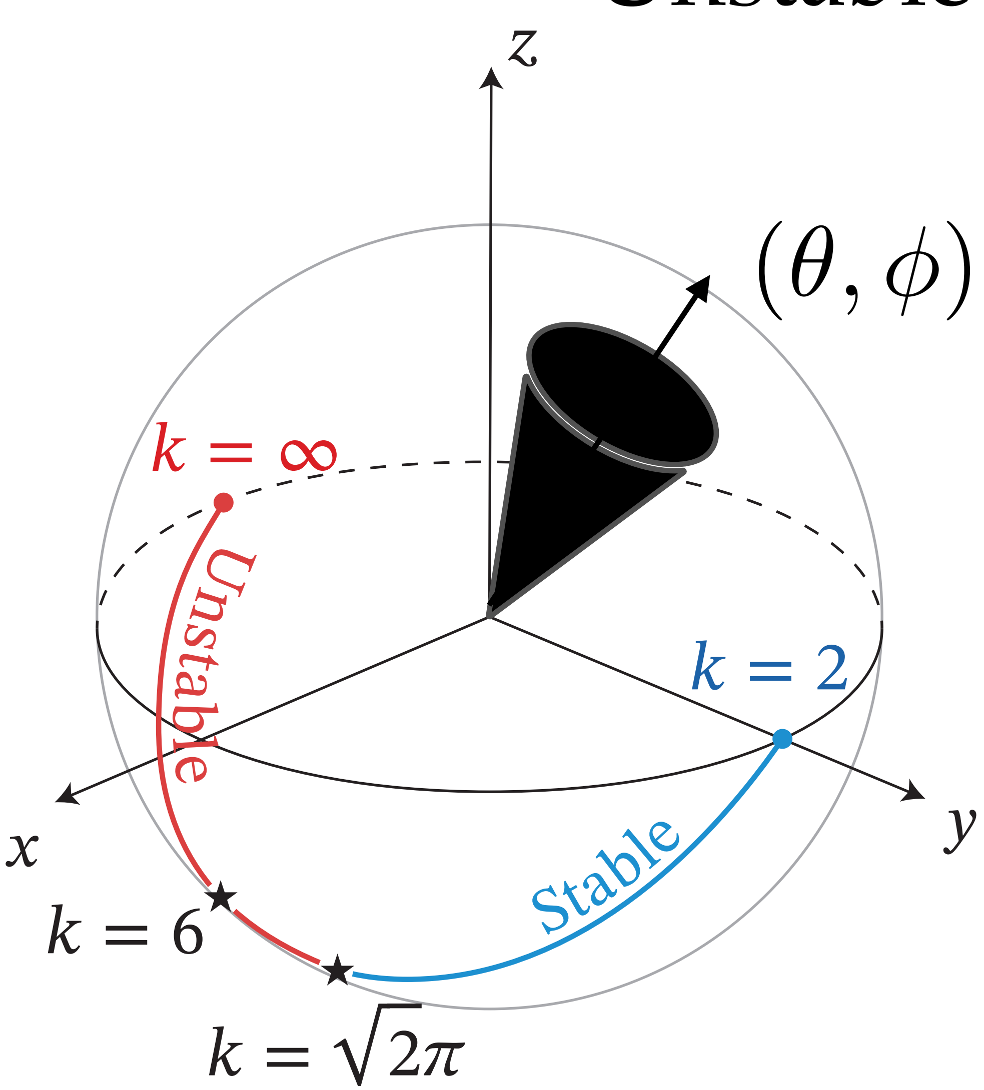
## Unstable fixed points of the map




Linearize about the fixed points $x_0^2 = \frac{\sin^2\left(\frac{kx_0}{2}\right)}{1+\sin^2\left(\frac{kx_0}{2}\right)}$

$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta z' \end{bmatrix} = \begin{bmatrix} kx_0 \cot(kx_0/2) & \sin(kx_0) & \cos(kx_0) \\ -kx_0 & \cos(kx_0) & -\sin(kx_0) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

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# Chaotic Hamiltonian Systems: Kicked Top

## Kicked top

$$H_{\rm KT} = \alpha J_y + \frac{k J_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

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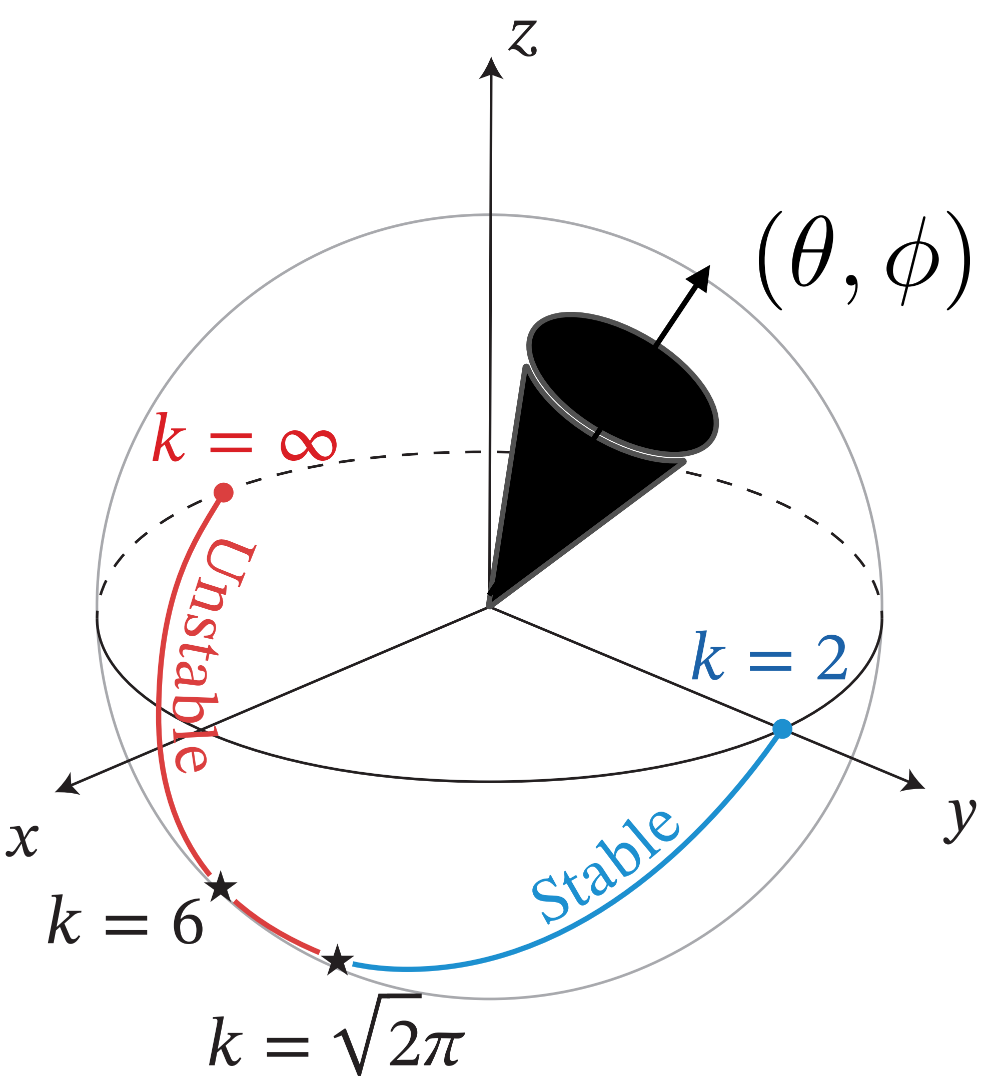
## Unstable fixed points of the map

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Eigenvalues $1$ & $\lambda_{\pm}(k) = -\left[h(k) \pm \sqrt{h(k)^2 - 1}\right]$

$$h(k) = \sin^2(kx_0/2) - \frac{kx_0}{2}\cot(kx_0/2)$$

$|h(k)| < 1$ For stability

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

Prasad, et al, Kulkarni, JHP, et al, arXiv (2026)

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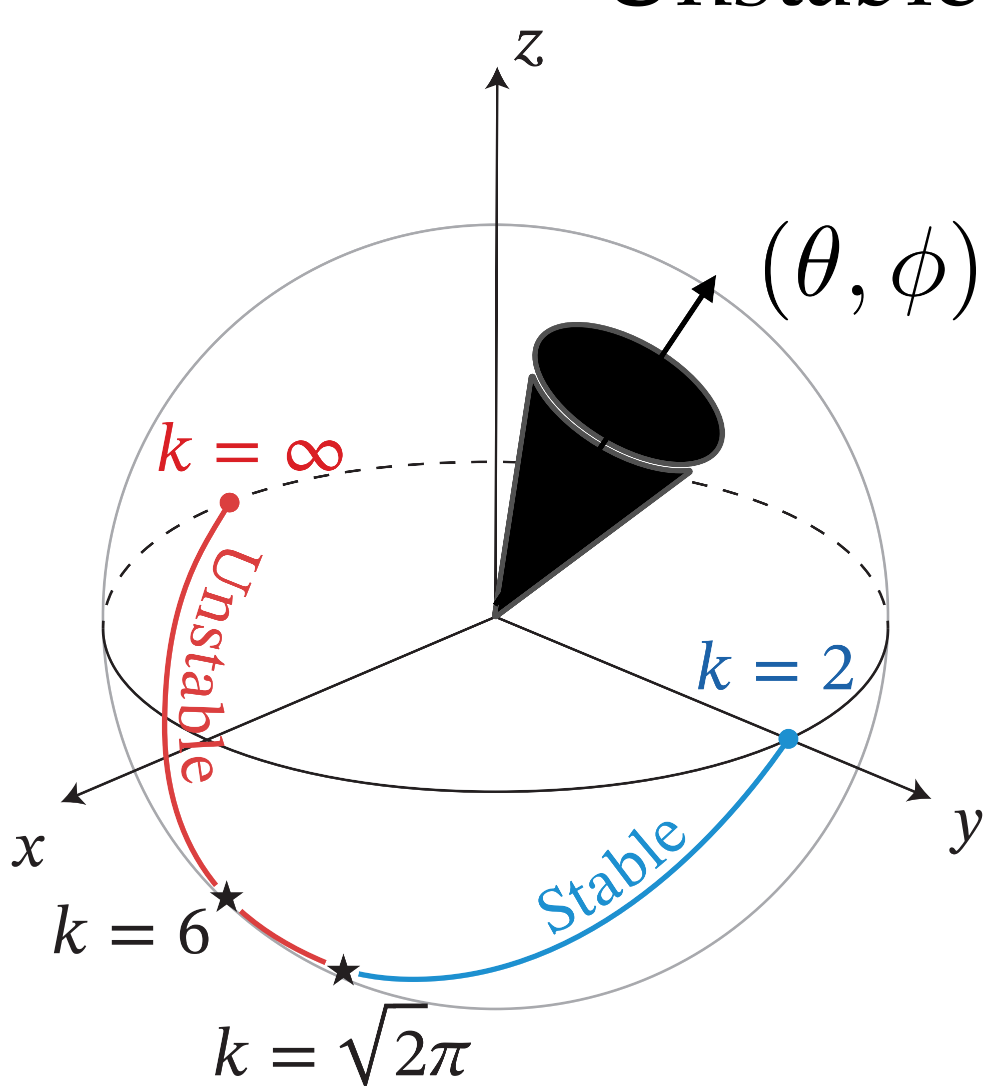


$$h(k) = \sin^2(kx_0/2) - \frac{kx_0}{2}\cot(kx_0/2)$$

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Becomes unstable when $h(k) = 1$

$$\cos(kx_0/2) = 0$$

$$\sin(kx_0/2) = 1$$

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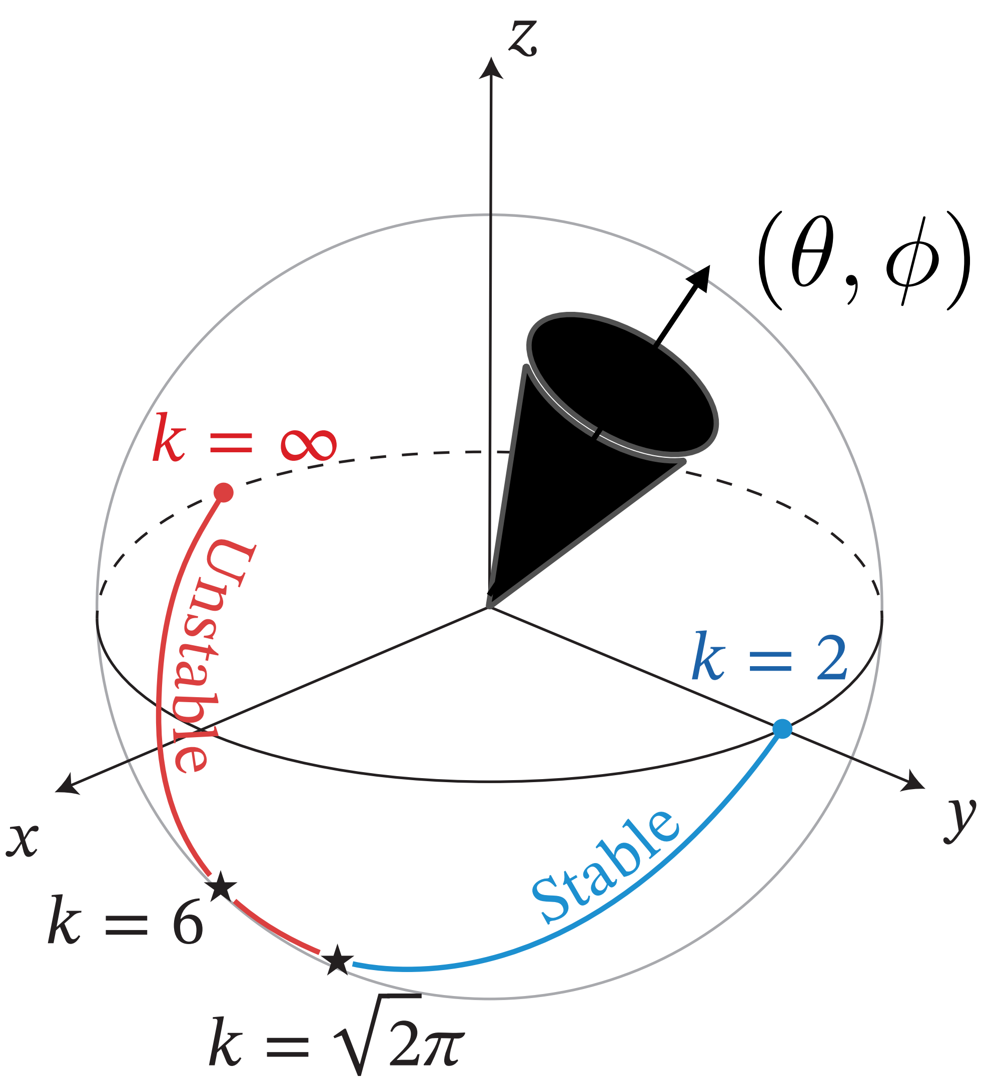


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$\cos(kx_0/2) = 0$ $\quad kx_0/2 = \pi/2$

$\sin(kx_0/2) = 1$ $\quad x_0^2 = 1/2$

$$k_c = \sqrt{2}\pi \approx 4.44$$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

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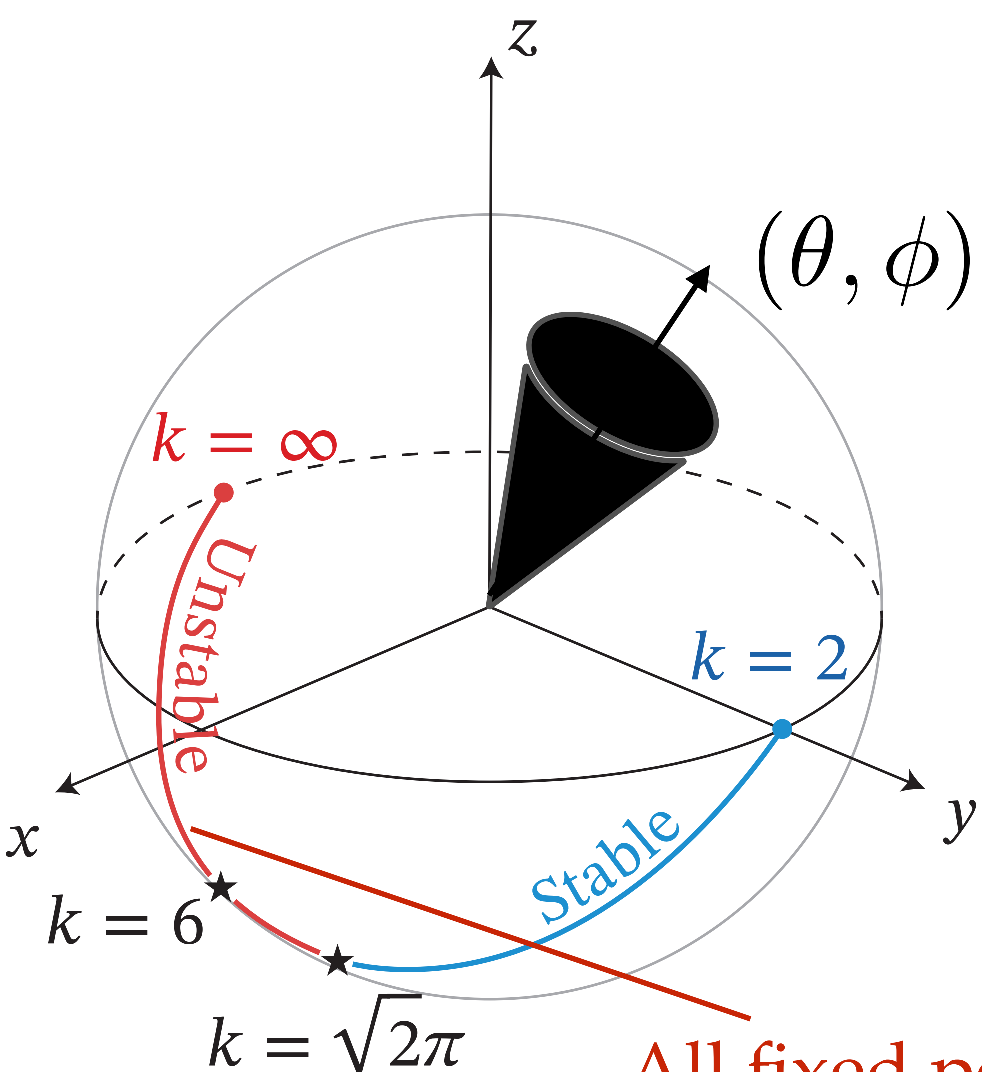


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Becomes unstable when $h(k) = 1$

$\cos(kx_0/2) = 0$ $\quad kx_0/2 = \pi/2$

$\sin(kx_0/2) = 1$ $\quad x_0^2 = 1/2$

All fixed points unstable $k > k_c$ $\quad k_c = \sqrt{2}\pi \approx 4.44$

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$$z_{n+1} = -x_n$$

$$\mathbf{r}_n = (x_n, y_n, z_n)$$

$$\mathbf{r}_{n+1} = f(\mathbf{r}_n)\mathbf{r}_n$$

Tangent map

$$\delta \mathbf{r}_n = \mathbf{r}_n - \mathbf{r}'_n$$

$$\delta \mathbf{r}_{n+1} = f'(\mathbf{r}_n)\delta \mathbf{r}_n$$

$$f'(\mathbf{J}_n) = \frac{\partial \mathbf{r}_{n+1}}{\partial \mathbf{r}_n} \qquad M_{ij}^n = \frac{\partial r_{n+1}^i}{\partial r_n^j}$$

$$\delta \mathbf{r}_{n+1} = \prod_{j=1}^{n} f'(\mathbf{r}_j)\delta \mathbf{r}_0$$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

Constantoudis and Theodorakopoulos PRE (1997)

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Lyapunov exponent

$$\delta \mathbf{r}_{n+1} = \prod_{j=1}^{n} f'(\mathbf{r}_j) \delta \mathbf{r}_0$$

$$f'(\mathbf{r}_n) = \begin{pmatrix} -kz_n \sin(kx_n) + ky_n \cos(kx_n) & \sin(kx_n) & \cos(kx_n) \\ -kz_n \cos(kx_n) - ky_n \sin(kx_n) & \cos(kx_n) & -\sin(kx_n) \\ -1 & 0 & 0 \end{pmatrix}$$

Define the largest exponent of the product $j_+(n)$

$$\lambda = \log( \lim_{n \to \infty} |j_+(n)|^{1/n})$$

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Define the largest exponent of the product $j_+(n)$

Largest eigenvalue of **f'**

$$\lambda = \log(\lim_{n\to\infty} |j_+(n)|^{1/n}) \approx \lim_{n\to\infty} \frac{1}{n} \sum_n \log(|j_+(n)^{\max}|)$$

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Constantoudis and Theodorakopoulos PRE (1997)

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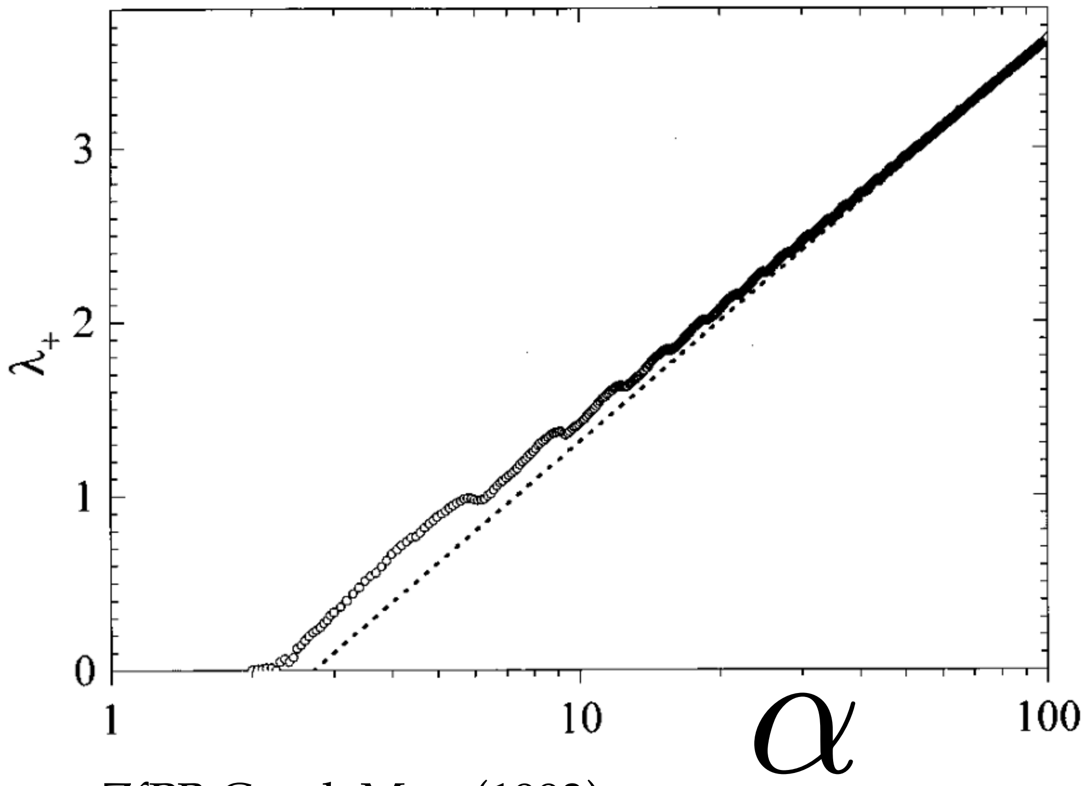
$$H_{\rm KT} = \alpha J_y + \frac{k J_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Lyapunov exponent

$$\lambda = \log(\lim_{n\to\infty} |j_+(n)|^{1/n}) \approx \lim_{n\to\infty} \frac{1}{n} \sum_n \log(|j_+(n)^{\max}|)$$




Large $\alpha$ limit

$$\lambda \approx \log(\alpha \sin(k/2))$$

Kus, Haake, and Eckhart, ZfPB Cond. Matt (1993)

Constantoudis and Theodorakopoulos PRE (1997)

# Chaotic Hamiltonian Systems: Heisenberg spin chain

So far only considered single particle problems

Now moving to many-body chaos

Das, Rao, Ramaswamy, EPL (2002)

Wijn, Hess, Fine, PRL (2012)

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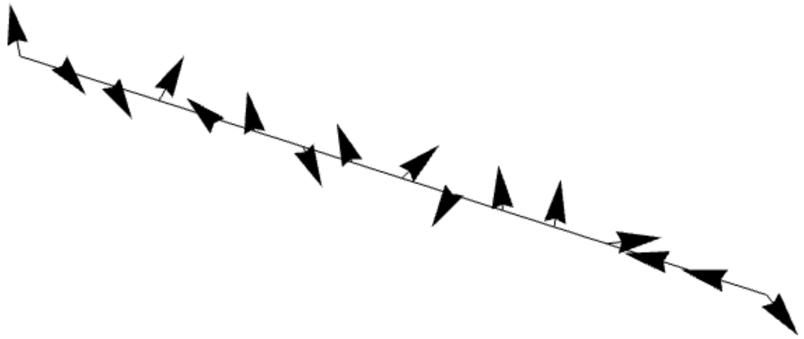

**Interacting spin chain**

$$H = -J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

$$\frac{d\mathbf{S}_j}{dt} = \{\mathbf{S}_j, H\} = -J(\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) \times \mathbf{S}_j$$

Das, Rao, Ramaswamy, EPL (2002)

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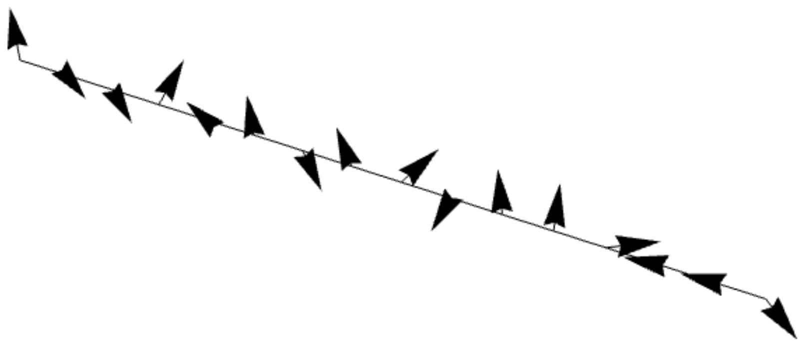

$$H = -J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

$$\frac{d\mathbf{S}_j}{dt} = \{\mathbf{S}_j, H\} = -J(\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) \times \mathbf{S}_j$$

Most initial states are chaotic $\lambda_{\max} \approx \frac{1}{4}\sqrt{2(J_x^2 + J_y^2 + J_z^2)}$

Das, Rao, Ramaswamy, EPL (2002)

Wijn, Hess, Fine, PRL (2012)

# Chaotic Hamiltonian Systems: Spin models

Now many-body chaos in higher dimensions

**Interacting spins models**

$$H = \sum_{\langle i,j \rangle} (J_x S_i^x S_j^x + J_y S_i^y S_j^y + J_z S_i^z S_j^z)$$

Sum over nearest neighbors

$$J_{\mathrm{max}} \equiv \max(|J_x|, |J_y|, |J_z|)$$

Das, Rao, Ramaswamy, EPL (2002)

Wijn, Hess, Fine, PRL (2012)

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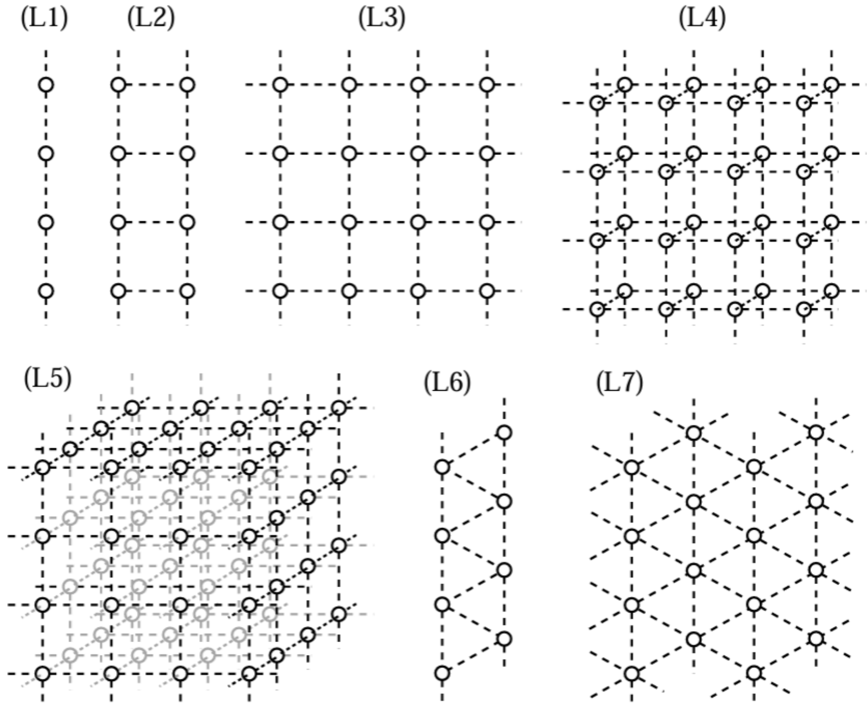
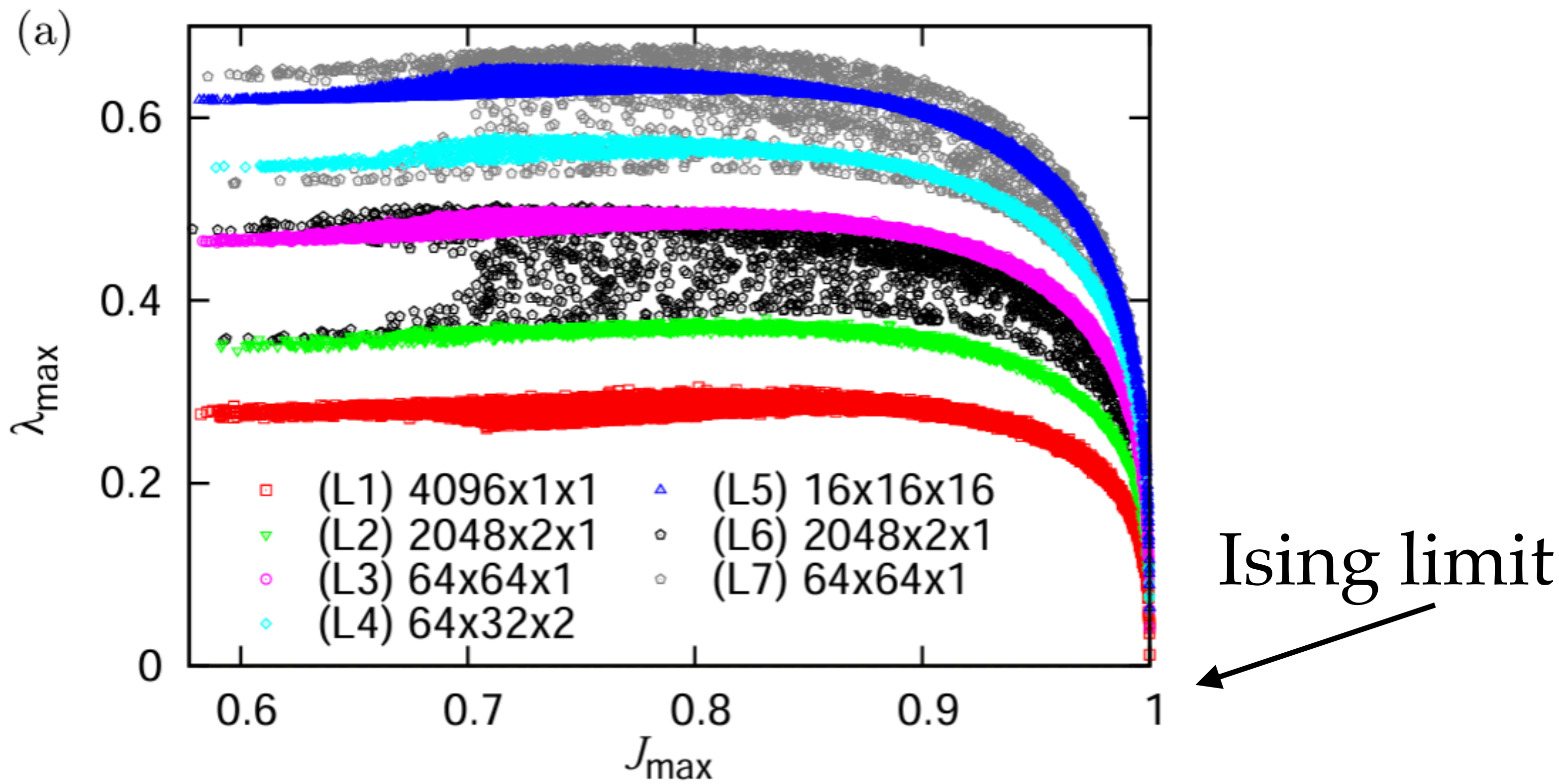



Das, Rao, Ramaswamy, EPL (2002)




$$J_{max} \equiv \max(|J_x|, |J_y|, |J_z|)$$

Wijn, Hess, Fine, PRL (2012)

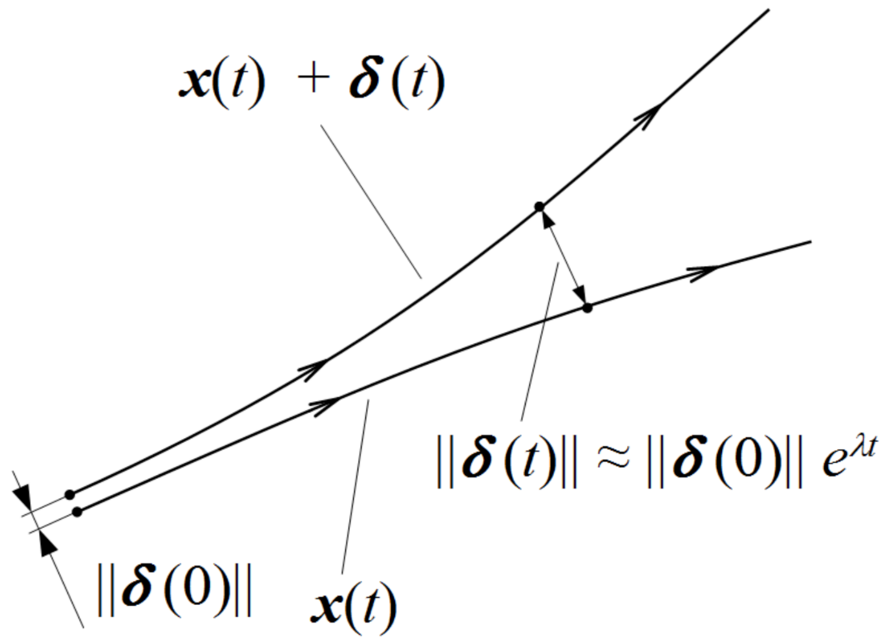
# Numerical Evaluation of the leading Lyapunov Exponent

Numerical estimates of Lyapunov exponents can be non-trivial, finite size effects and numerical round off error can be hard to track.

$$\mathbf{x}'(t=0) = \mathbf{x}(0) + \delta \qquad ||x(t) - x'(t)|| \sim e^{t\lambda_{\text{Lyapunov}}}$$

"Brute Force" $$\frac{1}{t}\log(||x(t) - x(t')|| = \lambda_{\text{Lyapunov}}$$ Bad way to go...

Trajectories eventually get too far apart and its not exponential deviation any longer



# Numerical Evaluation of the leading Lyapunov Exponent

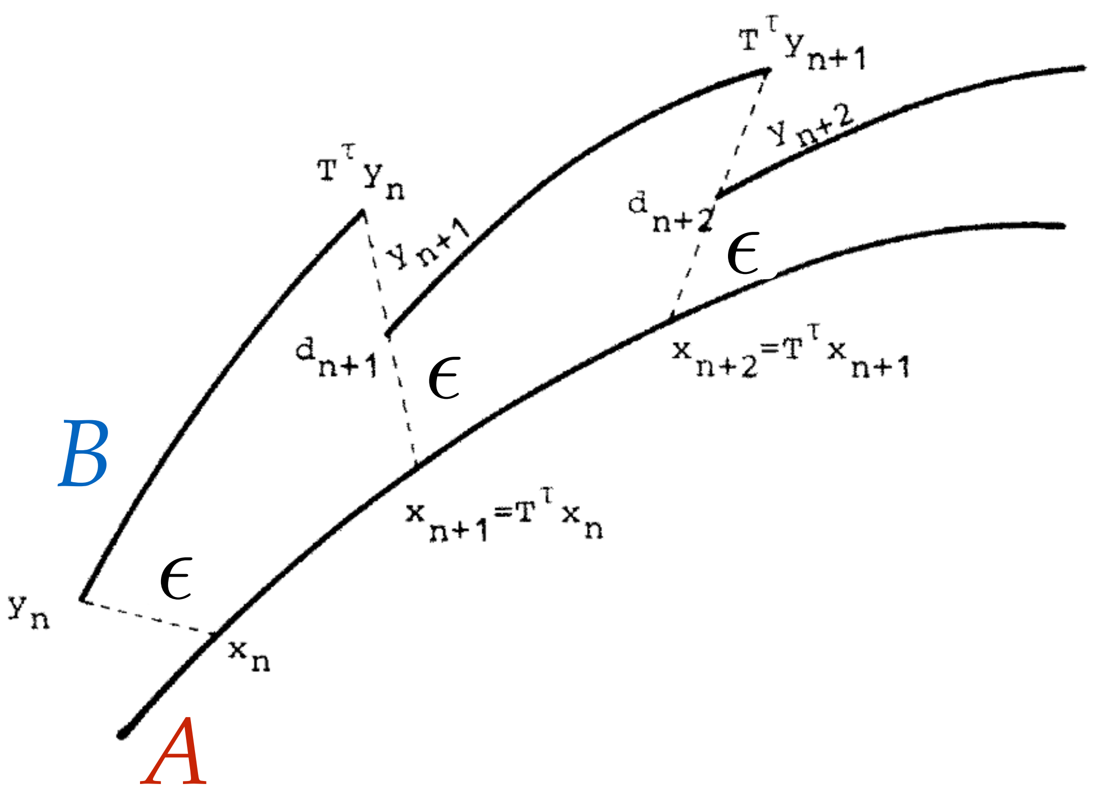
Optimal Numerical Approach: $$\lambda = \frac{1}{n\tau}\sum_{j=1}^{n} \ln\left(\frac{|d_j|}{\epsilon}\right)$$

Discretize the time evolution into $n$, $\tau$ steps

Make two copies, $A$ and $B$. Let A evolve under the time evolution of the problem,

After time $\tau$, reset $B$ to a distance $\epsilon$.

$$d_j = ||x_A(t_j) - x_B(t_j)||$$




Benettin, Galgani, Streclyn, PRA (1976)

# Summary of What We Have Learned

Chaotic dynamics in maps and Hamiltonian dynamics

Single particle models

Logistic map

Bernoulli map

Kicked rotor

Kicked top

Many-body models

Heisenberg spin chain

d-dimensional hyper cubic spin models

# Outline

I. Lecture series layout
II. Classical chaos
III. Quantum chaos
IV. Quantum thermalization
V. Evading thermalization

# Chaos in the Quantum limit?

Quantum systems lack the notion of a trajectory, **cannot** define a Lyapunov exponent from the separation between trajectories.

Unitarity enforces time dependent overlaps to be constant, no deviation between wave functions

$$\langle \phi(t)|\psi(t)\rangle = \langle \phi(0)|U_t^\dagger U_t|\psi(0)\rangle = \langle \phi(0)|\psi(0)\rangle$$

Can we define quantum notions of chaos that have a classical chaotic limit?

Are there quantum chaotic systems that don't have a simple classical limit?

*Random Matrices in Physics*, Wigner

*Quantum Signatures of Chaos*, Haake

# Chaos in the Quantum limit

Quantum Hamiltonians have an energy spectrum

$\hat{H}$ $E_i$

And similarly for Floquet Unitaries, their spectrum is phases

$\hat{U}_F$ $\lambda_i = e^{i\phi_i}$

The structure of these matrices and their eigenvalues distinguishes integrable and quantum chaotic systems

$\langle i|\hat{H}|j\rangle$ $\langle i|\hat{U}_F|j\rangle$

*Random Matrices in Physics*, Wigner

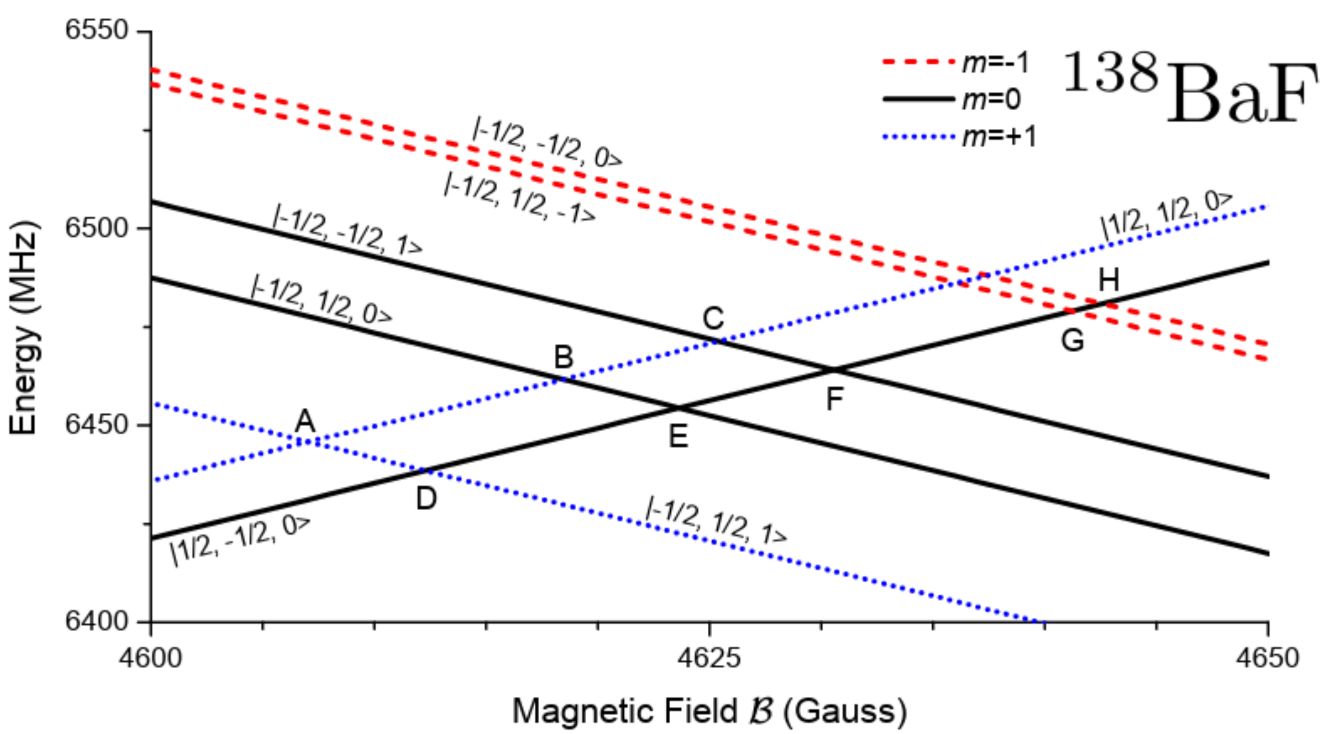
*Quantum Signatures of Chaos*, Haake

# Absence of Level Repulsion: Integrable Systems

Examples: $$\langle i|\hat{H}|j\rangle = \delta_{ij}E_i = \begin{pmatrix} E_1 & 0 & 0 & \dots \\ 0 & E_2 & 0 & \dots \\ 0 & 0 & E_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Integrable Hamiltonians are exactly solvable,
NO off-diagonal elements

## Level crossings between Zeeman multiplets




$$E_1, E_2, E_3, \dots$$

Can be become equal as we vary **one** parameter

Cahn, et al PRL (2014)

*Quantum Signatures of Chaos*, Haake

# Absence of Level Repulsion: Integrable Systems

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Integrable Hamiltonians are exactly solvable,
NO off-diagonal elements

$E_1, E_2, E_3, \dots$ Can be become equal as we vary **one** parameter

Treat as $N$ independent levels.

Level spacing $s_i = E_i - E_{i-1}$

What is the distribution of the level spacing? $P(s_i)$

*Random Matrices in Physics*, Wigner

*Quantum Signatures of Chaos*, Haake

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*P(s)*=Probability of finding the **nearest neighbor of level** *E* in *dE* at *E+s*
=*P(*we find a level between *E+s* and *E+s+ds* given one at E*)*
**P(*no other level is between *E* and *E+S)*
= ***P(s | E)******P(*no other level is between *E* and *E+S)*

*Random Matrices in Physics*, Wigner

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*P(s)*=Probability of finding the nearest neighbor of level *E* in *dE* at *E*+*s*
=***P*(we find a level between *E*+*s* and *E*+*s*+*ds* given one at E*)***
**P*(no other level is between *E* and *E*+*S*)
= ***P*(*s* | *E*)****P*(no other level is between *E* and *E*+*S*)

Assume a constant density of states

$P(s|E) = \mu(s)$ Independent of E, as the density of states is constant

$\mu(s) = \mu_0$ Assuming all levels are independent

*Random Matrices in Physics*, Wigner *Quantum Signatures of Chaos*, Haake

# Absence of Level Repulsion: Integrable Systems

Treat as $N$ independent levels.

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What is the distribution of the level spacing? $P(s_i)$

*P(s)*= *P(s | E)*****P*(no other level is between *E* and *E*+*S*)**

$$P(s) = \mu_0[1 - \int_0^s ds' P(s')] = \mu_0 \int_s^\infty ds' P(s')$$

Used the normalization $\int_0^\infty ds P(s) = 1$

Can now solve the differential equation for *P(s)*

*Random Matrices in Physics*, Wigner

*Quantum Signatures of Chaos*, Haake

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$$\frac{dP}{ds} = \mu_0(P(\infty) - P(s)) = -\mu_0 P(s)$$

$$P(s) = e^{-s}$$

Normalizing and scaling

$$P(s) = Ae^{-\mu_0 s}$$

$$\int_0^\infty ds P(s) = 1 \qquad \overline{s} = 1$$

$$P(0) > 0$$

*Random Matrices in Physics*, Wigner

*Quantum Signatures of Chaos*, Haake

# Level Repulsion: Hermitian Matrix

Consider interactions between level 1 and 2.

Start with a 2x2 matrix

$$H_{ij} = \begin{pmatrix} E_1 & V \\ V^* & E_2 \end{pmatrix}$$

Eigenvalues

$$E_\pm = \frac{1}{2}(E_1 + E_2) \pm \sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^*V}$$

Level spacing

$$|E_+ - E_-| = 2\sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^*V}$$

*Quantum Signatures of Chaos*, Haake

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Level spacing

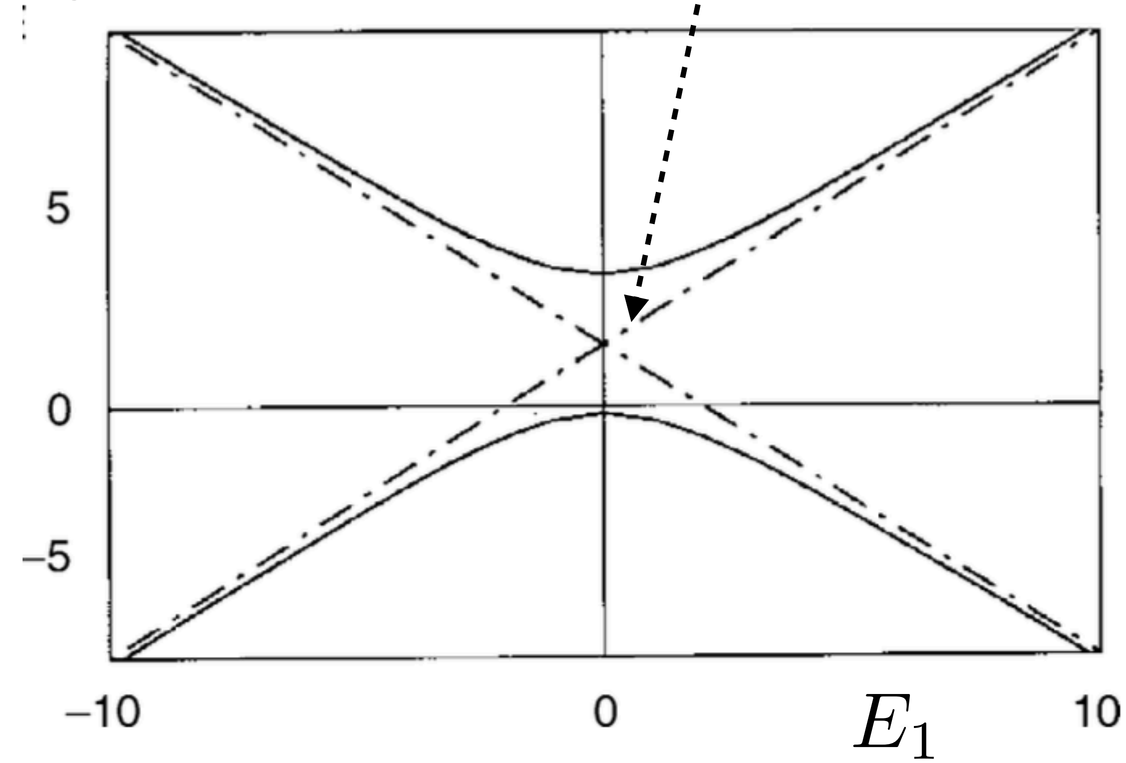
$$|E_+ - E_-| = 2\sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^* V}$$

Requires tuning 3 non-negative terms to make the levels cross $|E_+ - E_-| = 0$

Co-dimension of the avoided crossing $n = 3$

Tuning all three

$(E_1 - E_2), V^*, V$




*Quantum Signatures of Chaos*, Haake

# Level Repulsion: Real Matrix

Now make $V \in \mathbb{R}$ $H_{ij} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$

Eigenvalues

$$E_\pm = \frac{1}{2}(E_1 + E_2) \pm \sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^2}$$

Level spacing

$$|E_+ - E_-| = 2\sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^2}$$

*Quantum Signatures of Chaos*, Haake

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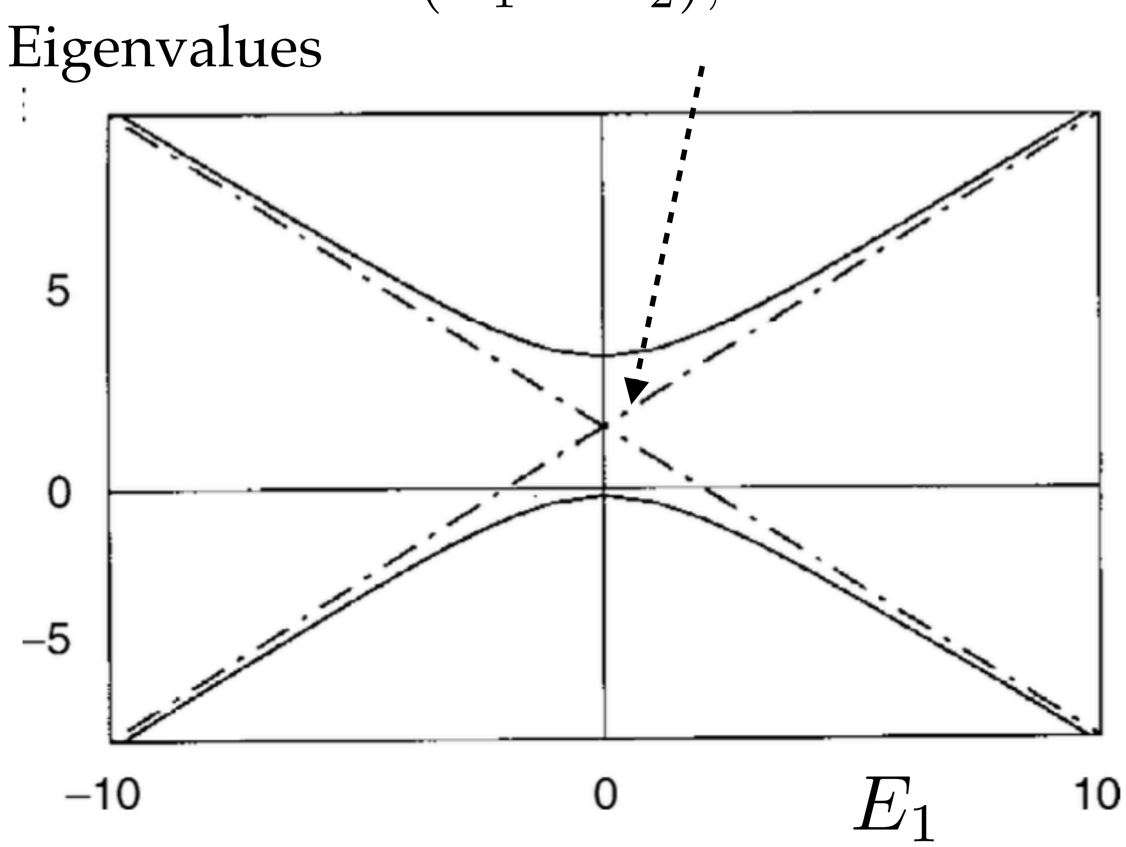
Level spacing

$$|E_+ - E_-| = 2\sqrt{\frac{1}{4}(E_1 - E_2)^2 + V^2}$$

Requires tuning 2 non-negative terms to make the levels cross $|E_+ - E_-| = 0$

Co-dimension of the avoided crossing $\quad n = 2$

Tuning two parameters $(E_1 - E_2), V$




*Quantum Signatures of Chaos*, Haake

# Level Repulsion, Symmetries, and Random Matrices

Thinking more broadly, this crossing will be embedded in a larger matrix. The levels will have contributions from some kind of randomness; e.g. disorder, kicking, measurements

This implies we need to think about:

(1)The symmetries of the Hamiltonian
(2)The statistics of random matrices

$$\begin{pmatrix} E_1 & V & & \\ V^* & E_2 & & \\ 0 & 0 & E_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

*Quantum Signatures of Chaos*, Haake

# Symmetry and Level Repulsion

(1) Consider the symmetry properties of the matrices formed from Hamiltonians matrix elements $H_{ij} = \langle i|\hat{H}|j\rangle$

The 10 fold Cartan classification, includes time reversal, spin-rotation, and particle-hole symmetries.

| | Symmetry | | | | Dimension $d$ | | | |
|---|---|---|---|---|---|---|---|---|
| AZ | $\mathcal{T}$ | $\Xi$ | $\Pi$ | 1 | 2 | 3 | 4 |
| Unitary (no TR) Hermitian Matrix: A | 0 | 0 | 0 | 0 | Z | 0 | Z |
| AIII | 0 | 0 | 1 | Z | 0 | Z | 0 |
| Orthogonal (has TR) Real matrix: AI | 1 | 0 | 0 | 0 | 0 | 0 | Z |
| BDI | 1 | 1 | 1 | Z | 0 | 0 | 0 |
| D | 0 | 1 | 0 | $Z_2$ | Z | 0 | 0 |
| DIII | −1 | 1 | 1 | $Z_2$ | $Z_2$ | Z | 0 |
| Symplectic (has TR): AII | −1 | 0 | 0 | 0 | $Z_2$ | $Z_2$ | Z |
| CII | −1 | −1 | 1 | Z | 0 | $Z_2$ | $Z_2$ |
| C | 0 | −1 | 0 | 0 | Z | 0 | $Z_2$ |
| CI | 1 | −1 | 1 | 0 | 0 | Z | 0 |

Time reversal $\mathcal{T}$

Particle-hole $\Xi$

Chiral $\mathcal{T}\Xi = \Pi$

Hasan and Kane, RMP (2010)

*Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

(2)The statistics of random matrices: Overwhelming evidence of universality between classically chaotic systems and their quantum limit having universal level statistics

Universality of level repulsion can be described using random matrix theory

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

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Universality of level repulsion can be described using random matrix theory

Interested in the probability distribution of the Hamiltonian's matrix elements

$P(H)=P(E_1)P(E_2)P(V)$ (for real symmetric matrices)

$P(H)=P(E_1)P(E_2)P(V,V^*)$ (for Hermitian matrices)

$$H_{ij} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

$$H_{ij} = \begin{pmatrix} E_1 & V \\ V^* & E_2 \end{pmatrix}$$

In general $$P(H) = Ce^{-A\mathrm{Tr}(H^2)}$$

Our aim is to use this to derive the distribution of level spacings

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

$$P(H) = Ce^{-A\mathrm{Tr}(H^2)}$$

Our aim is to use this to derive the distribution of level spacings

First relate non-diagonal and diagonal matrices via a unitary transformation

**For real symmetric matrices** this ensemble will be symmetric under all orthogonal transformations $P(H) = P(H')\,, \quad H' = OHO^{-1}$

Focus on 2x2 matrices $H_{ij} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$ $E_\pm = \frac{1}{2}(E_1+E_2) \pm \sqrt{\frac{1}{4}(E_1-E_2)^2 + V^2}$

H is diagonalized with the orthogonal transformation $O = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}$

$$\begin{aligned} E_1 &= E_+\cos^2\Theta + E_-\sin^2\Theta \\ E_2 &= E_+\sin^2\Theta + E_-\cos^2\Theta \\ V &= (E_+ - E_-)\cos\Theta\,\sin\Theta \end{aligned}$$

$$P(E_+, E_-) = \det\frac{\partial(E_1, E_2, V)}{\partial(E_+, E_-, \Theta)} P(E_1, E_2, V)$$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

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$$= C|E_+ - E_-|e^{-A(E_+^2 + E_-^2)}$$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

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**For Hermitian matrices** this ensemble will be symmetric under all unitary transformations $P(H) = P(H') \quad H' = UHU^{-1}$

H is diagonalized with the unitary transformation $U = \begin{pmatrix} \cos\Theta & -e^{-i\phi}\sin\Theta \\ e^{i\phi}\sin\Theta & \cos\Theta \end{pmatrix}$

$$\begin{aligned} E_1 &= E_+ \cos^2\Theta + E_- \sin^2\Theta \\ E_2 &= E_+ \sin^2\Theta + E_- \cos^2\Theta \\ V &= (E_+ - E_-)\, e^{i\phi} \cos\Theta \sin\Theta \end{aligned}$$

$$P(E_+, E_-) = \det \frac{\partial(E_1, E_2, V, V^*)}{\partial(E_+, E_-, \Theta, \phi)} P(E_1, E_2, V, V^*)$$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

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$$= C|E_+ - E_-|^2 e^{-A(E_+^2 + E_-^2)}$$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

Summary

**For real symmetric matrices, Gaussian Orthogonal ensemble (GOE)**

$$P(E_+, E_-) = \det\frac{\partial(E_1, E_2, V)}{\partial(E_+, E_-, \Theta)} P(E_1, E_2, V) = C|E_+ - E_-|e^{-A\mathrm{Tr}H^2}$$

**For Hermitian matrices, Gaussian Unitary ensemble (GUE)**

$$P(E_+, E_-) = \det\frac{\partial(E_1, E_2, V, V^*)}{\partial(E_+, E_-, \Theta, \phi)} P(E_1, E_2, V, V^*) = C|E_+ - E_-|^2 e^{-A\mathrm{Tr}H^2}$$

Power law is set by the **co-dimension**
of the avoided level crossing (***n-1***), general result

$$P(E_+, E_-) = C|E_+ - E_-|^{n-1} e^{-A\mathrm{Tr}H^2}$$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

Focused on 2x2 matrices so far

$$P(E_+, E_-) = C|E_+ - E_-|^{n-1} e^{-A\,\mathrm{Tr}\,H^2}$$

Power law is set by the co-dimension of the avoided level crossing, general result

Generalizing to *NxN* matrices

$$P(E_1, E_2, \ldots, E_N) = \text{const} \prod_{\mu<\nu}^{1\ldots N} |E_\mu - E_\nu|^{n-1} \exp\left(-A \sum_{\mu=1}^{N} E_\mu^2\right)$$

$n = 2$ Orthogonal, GOE

$n = 3$ Unitary, GUE

$n = 5$ Symplectic, GSE

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Random Matrix Theory and Ensembles

Focused on 2x2 matrices so far

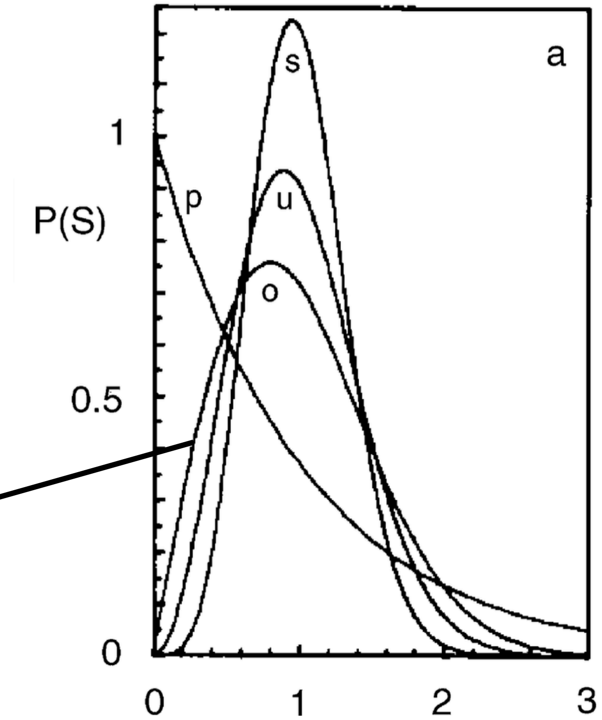
$$P(E_+, E_-) = C|E_+ - E_-|^{n-1} e^{-A\mathrm{Tr}H^2}$$

From this, we can compute

$$P(s) = \int_{-\infty}^{\infty} dE_+ \int_{-\infty}^{\infty} dE_- \delta(s - |E_+ - E_-|) P(E_+, E_-)$$

$$= \frac{\pi}{2} s e^{-\frac{\pi s^2}{4}} \quad \text{Orthogonal}$$

$$= \frac{32}{\pi^2} s^2 e^{-\frac{4 s^2}{\pi}} \quad \text{Unitary}$$




Level repulsion appears through $P(s=0) = 0$

*Random Matrices in Physics*, Wigner; *Level Clustering in the Regular Spectrum*, Barry and Tabor; *Quantum Signatures of Chaos*, Haake

# Statistics and Level Repulsion

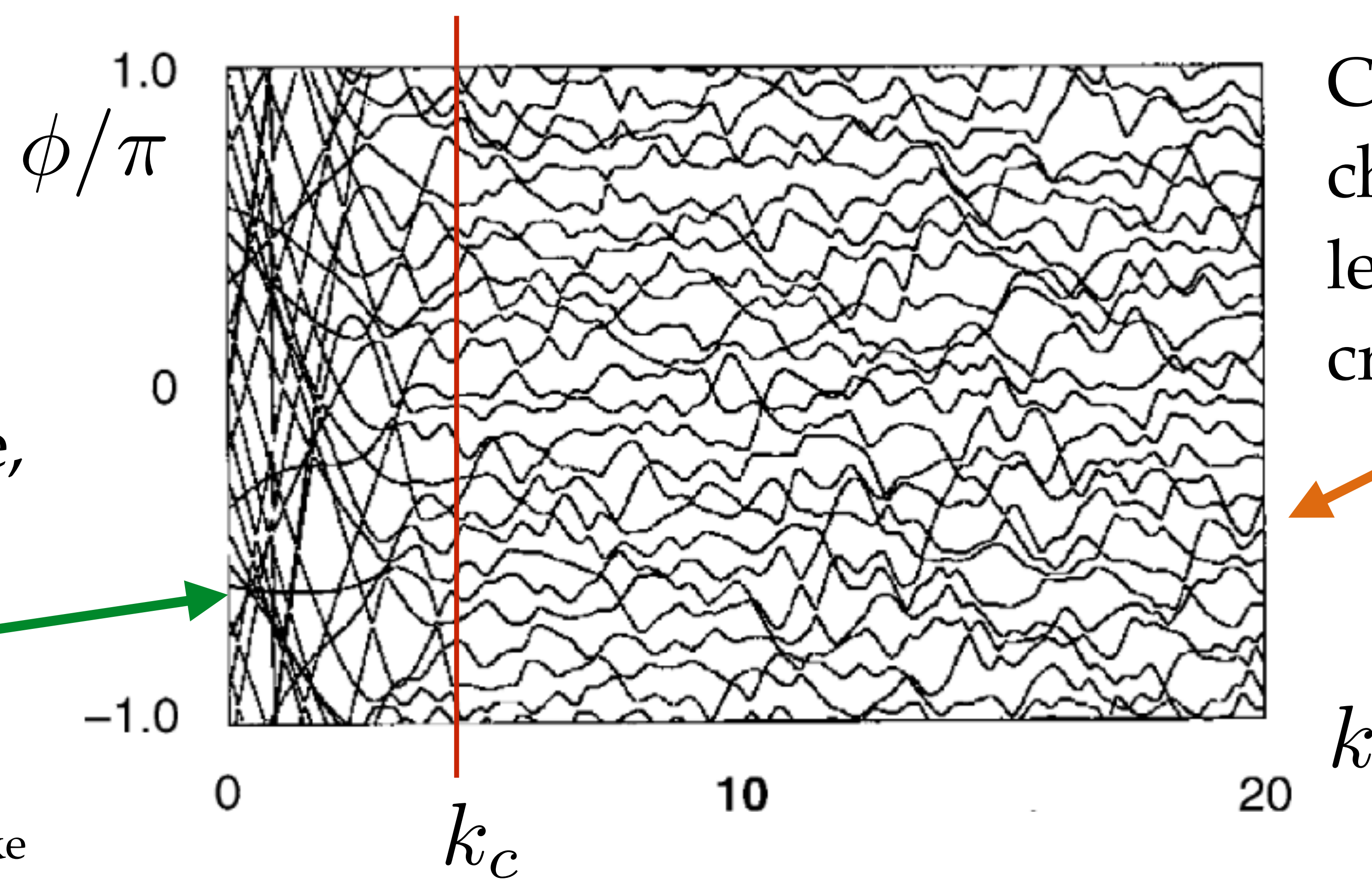
(2)The statistics of random matrices: Overwhelming evidence of universality between classically chaotic systems and their quantum limit having universal level statistics

*Quantum Signatures of Chaos*, Haake

# Statistics and Level Repulsion

(2)The statistics of random matrices: Overwhelming evidence of universality between classically chaotic systems and their quantum limit having universal level statistics

Quantum kicked top $$\hat{H}_{\mathrm{KT}} = \alpha \hat{J}_y + \frac{k\hat{J}_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$ $$\hat{U}_{\mathrm{KT}} = e^{-i\alpha \hat{J}_y T} e^{-ik\hat{J}_z^2/(2S)}$$




Classically chaotic regime, levels DO NOT cross

Classically integrable, Quantum levels are crossing

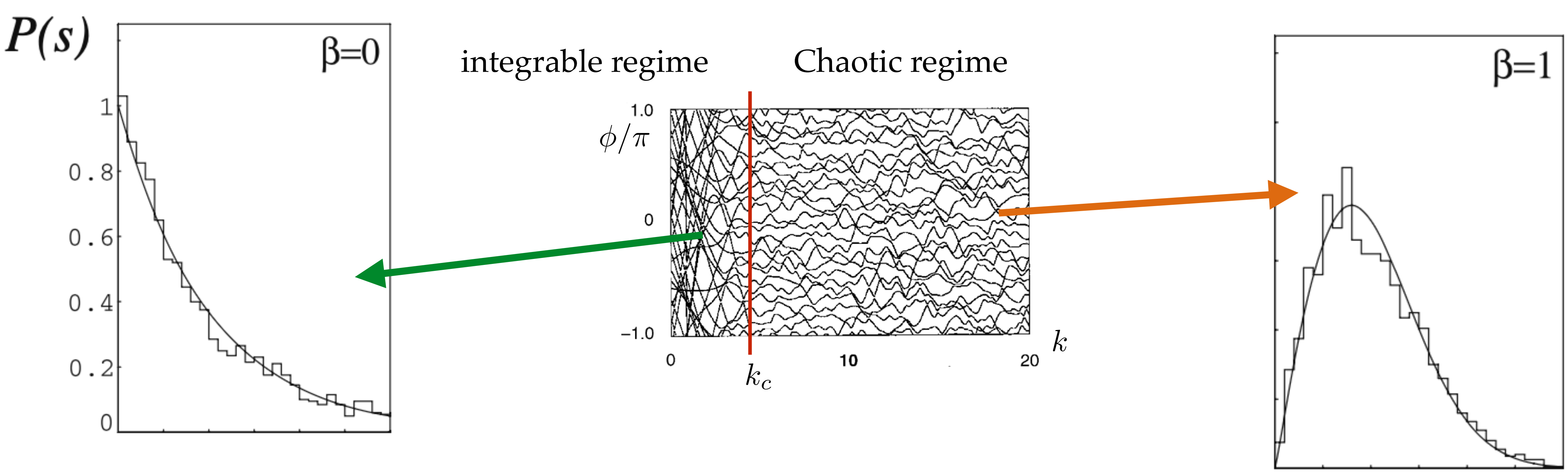
*Quantum Signatures of Chaos*, Haake

# Statistics and Level Repulsion

(2)The statistics of random matrices: Overwhelming evidence of universality between classically chaotic systems and their quantum limit having universal level statistics

Quantum kicked top $\hat{H}_{\mathrm{KT}} = \alpha \hat{J}_y + \frac{k\hat{J}_z^2}{2S} \sum_{n=-\infty}^{\infty} \delta(t - nT)$ $\hat{U}_{\mathrm{KT}} = e^{-i\alpha \hat{J}_y T} e^{-ik\hat{J}_z^2/(2S)}$

Consider the distribution of level spacings $P(s_i)$ $s_i = E_i - E_{i-1}$



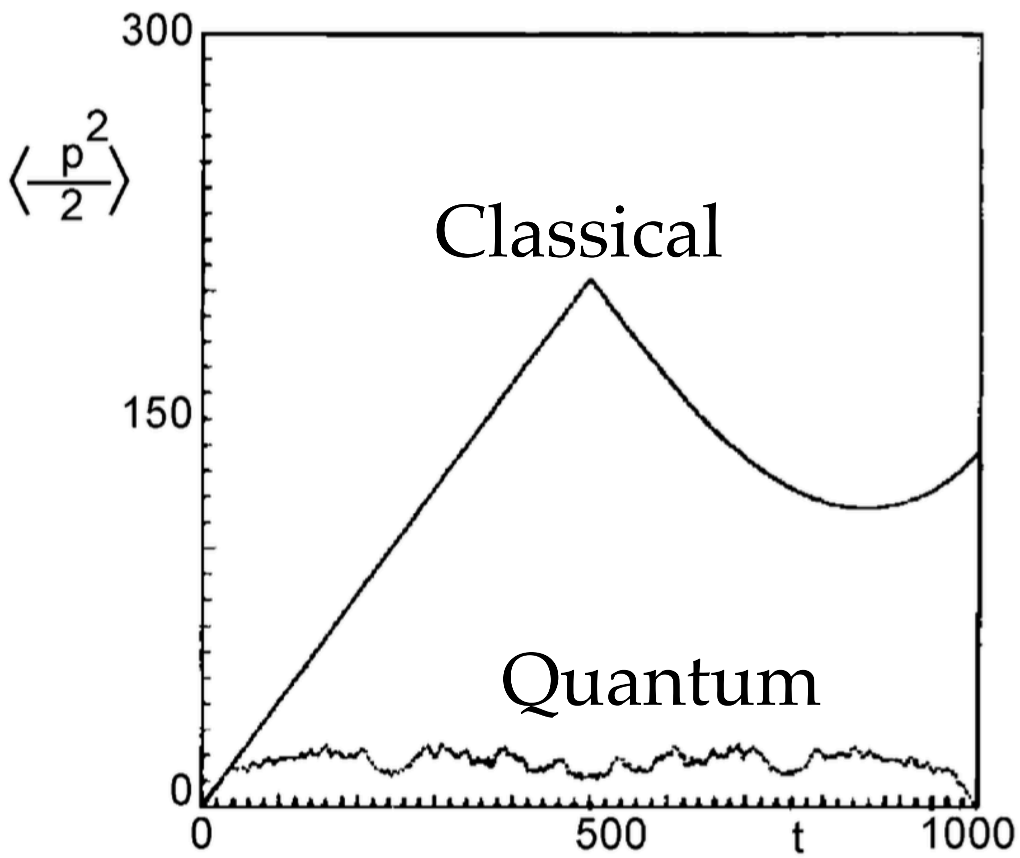

*Quantum Signatures of Chaos*, Haake

# Not so fast: Absence of Quantum Chaos in the Rotor

Lets begin with single particle models with canonical quantization

Quantum kicked rotor $$\hat{H}_{\mathrm{KR}} = \frac{\hat{p}^2}{2I} + k\cos(\hat{\theta}) \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$U_{KR} = e^{-i\hat{p}^2 T/2I} e^{-ik\cos(\hat{\theta})}$$




Classical model has diffusion

Quantum model undergoes dynamical localization

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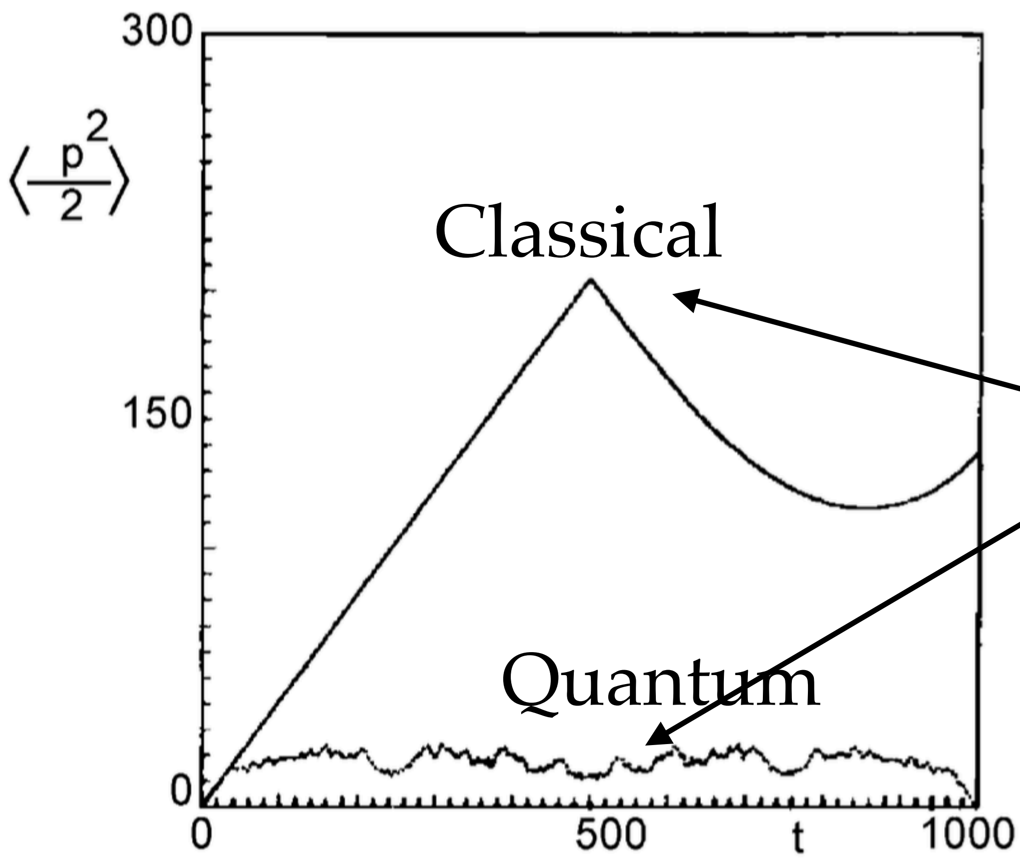


Classical model has diffusion

Motion is reversed, exponentially sensitive
Only in the classical limit

Quantum model undergoes dynamical localization

*Quantum Signatures of Chaos*, Haake

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$$U_{KR} = e^{-i\hat{p}^2 T/2I} e^{-ik\cos(\hat{\theta})}$$

We wish to transform this eigenvalue equation

$$U|\phi_j\rangle = e^{-iE_j T}|\phi_j\rangle$$

*Quantum Signatures of Chaos*, Haake

# Not so fast: Absence of Quantum Chaos in the Rotor

Floquet operator $$U_{KR} = e^{-i\hat{p}^2 T/2I} e^{-ik\cos(\hat{\theta})}$$

Eigenvalue equation $$U|\phi_j\rangle = e^{-iE_j T}|\phi_j\rangle$$

$$e^{-i(\hat{p}^2/2-E_j)T} e^{-ik\cos(\hat{\theta})}|\phi_j\rangle = |\phi_j\rangle$$

Define a new vector $$|\tilde{\phi}_j\rangle = e^{ik\cos(\hat{\theta})}|\phi_j\rangle = e^{-iT(\hat{p}^2/2-E_j)}|\phi_j\rangle$$

Decompose the Floquet operator $$e^{ix} = \frac{1+i\tan(x/2)}{1-i\tan(x/2)}$$

$$e^{-ik\cos(\hat{\theta})} = \frac{1+iW(\hat{\theta})}{1-iW(\hat{\theta})}$$

*Quantum Signatures of Chaos*, Haake

# Not so fast: Absence of Quantum Chaos in the Rotor

$$|\tilde{\phi}_j\rangle = e^{ik\cos(\hat{\theta})}|\phi_j\rangle = e^{iT(\hat{p}^2/2-E_j)}|\phi_j\rangle$$

$$|\phi_j\rangle = e^{-ik\cos(\hat{\theta})}|\tilde{\phi}_j\rangle = \frac{1+iW(\hat{\theta})}{1-iW(\hat{\theta})}|\tilde{\phi}_j\rangle$$

$$\frac{1}{1+iW(\hat{\theta})}|\phi_j\rangle = \frac{1}{1-iW(\hat{\theta})}|\tilde{\phi}_j\rangle \equiv |u_j\rangle$$

$$|\phi_j\rangle = (1+iW(\hat{\theta}))|u_j\rangle \qquad |\tilde{\phi}_j\rangle = (1-iW(\hat{\theta}))|u_j\rangle = e^{iT(\hat{p}^2/2-E_j)}|\phi_j\rangle$$

$$(1-iW(\hat{\theta}))|u_j\rangle = e^{iT(\hat{p}^2/2-E_j)}(1+iW(\hat{\theta}))|u_j\rangle$$

*Quantum Signatures of Chaos*, Haake

# Not so fast: Absence of Quantum Chaos in the Rotor

$$(1 - iW(\hat{\theta}))|u_j\rangle = e^{iT(\hat{p}^2/2 - E_j)}(1 + iW(\hat{\theta}))|u_j\rangle$$

In the theta basis $u_j(\theta) = \langle\theta|u\rangle$

$$(1 - iW(\theta))u_j(\theta) = e^{iT((-i\partial_\theta)^2/2 - E_j)}(1 + iW(\theta))u_j(\theta)$$

Fourier transforming to the momentum basis $\hat{p}|m\rangle = m|m\rangle$

$$W(\theta) = \sum_{m=-\infty}^{\infty} W_m e^{im\theta} \qquad u_j(\theta) = \sum_{m=-\infty}^{\infty} u_j^m e^{im\theta} \qquad u_j^m = \langle m|u\rangle$$

$$T_m u_j^m + \sum_{r \neq 0} W_r u_j^{m+r} = -W_0 u_j^m$$

*Quantum Signatures of Chaos*, Haake

# Not so fast: Absence of Quantum Chaos in the Rotor

We have mapped the quantum dynamics to a discrete Schrodinger equation in momentum space

$$T_m u_j^m + \sum_{r\neq 0} W_r u_j^{m+r} = -W_0 u_j^m$$

On-site energies

$$T_m = \tan[(E_j - m^2/2)/2]$$

Long-range
Hopping between site
$m$ and $r$

$$W_r = -\int_0^{2\pi} \frac{d\theta}{2\pi} e^{ir\theta} \tan(k\cos\theta/2)$$

Energy eigenvalue

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Pseudorandom numbers

The sequence $(E_j - m^2/2) \bmod \pi$

Is ergodic in the interval $[0, \pi]$

Density of levels $\rho(T_m)dT_m = dE_j/\pi$

$$\frac{dT_m}{dE_j} = \frac{1}{2}\cos^{-2}((E_j - m^2/2)/2) = \frac{1}{2}(1 + T_m^2)$$

$$\rho(T_m) = \frac{1}{\pi(1 + T_m^2)}$$

(Normalizing)

*Quantum Signatures of Chaos*, Haake

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For $k < \pi$ $W_r \sim e^{-r}$ Can be proven

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Energy eigenvalue

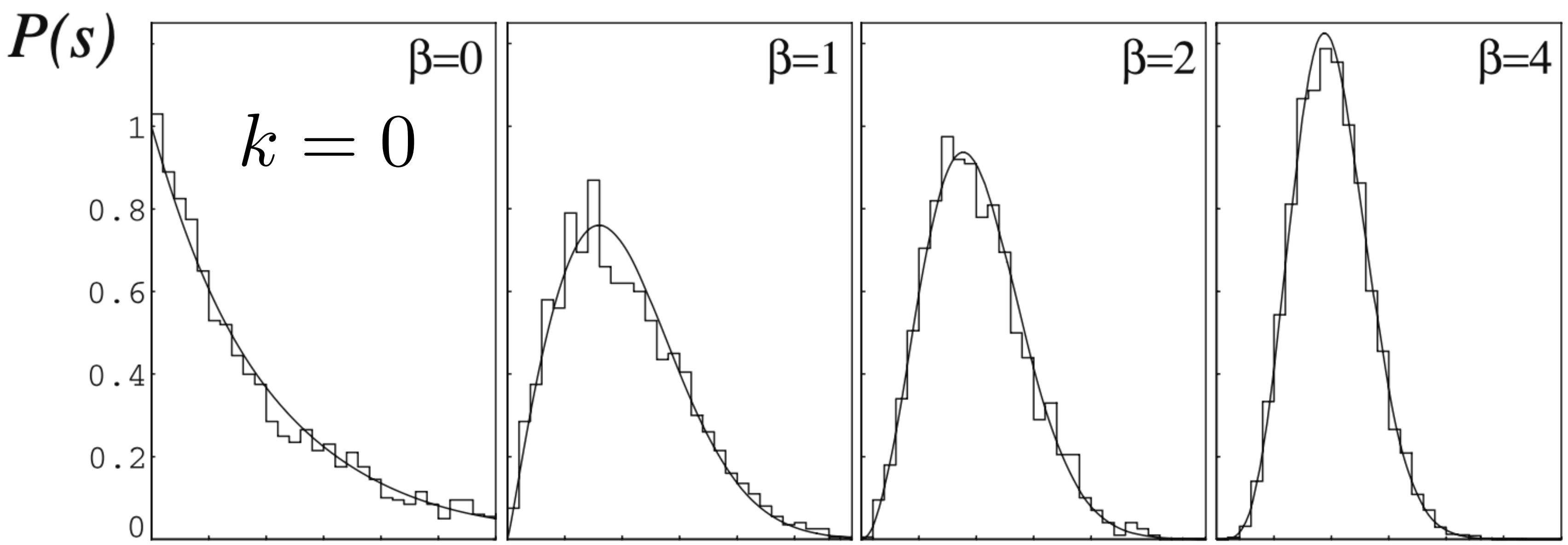
For $k < \pi$ $W_r \sim e^{-r}$ Can be proven

This is a disordered 1D hopping model, Anderson localizes!

*Quantum Signatures of Chaos*, Haake

# Summary So Far

What we have learned so far:
The kicked top serves as a quintessential model for quantum chaos, the 3 symmetry classes, GOE, GUE, GSE can be probed




*Quantum Signatures of Chaos*, Haake

The kicked rotor becomes dynamically localized and does not represent chaotic behavior.

What about the quantum spin chain?

# Quantum Spin Chains

Focusing on the antiferromagnetic Heisenberg spin chain

$$H = J \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1}$$

It turns out to be **integrable**, solvable using the Bethe ansatz

Simply taking the quantum limit does not yield a chaotic system.

Add next nearest neighbor coupling

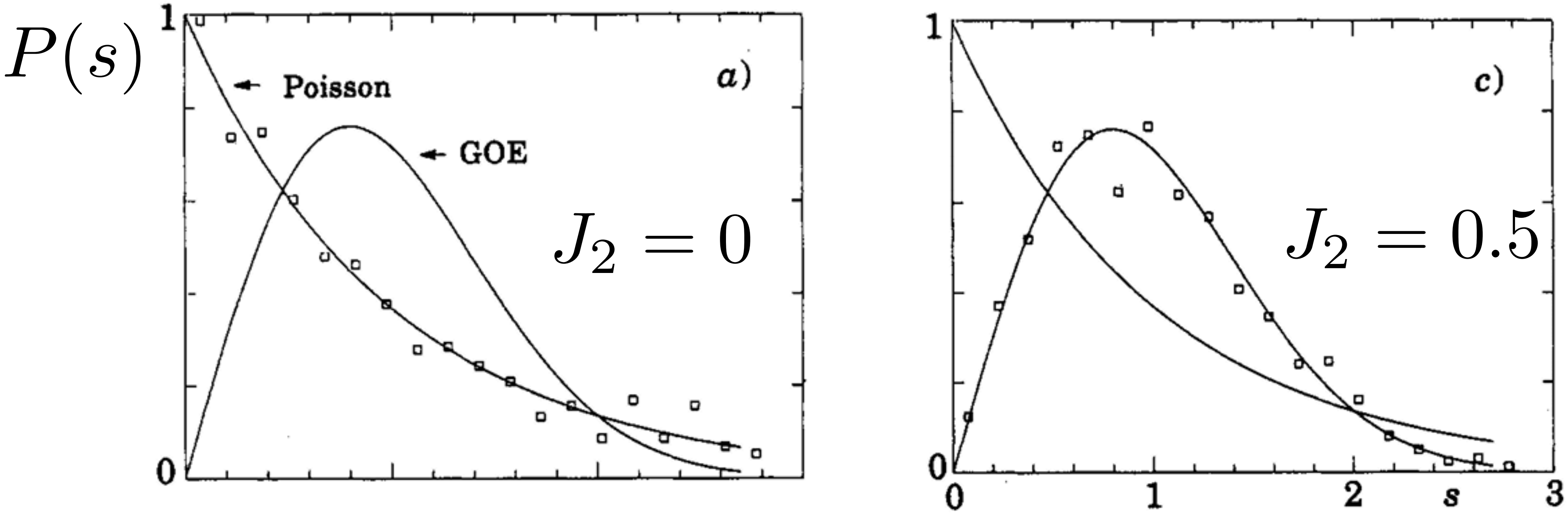
$$H = J_1 \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1} + J_2 \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+2}$$

Do the many-body levels now look chaotic?

# Quantum Spin Chains

$$H = J_1 \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1} + J_2 \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+2}$$

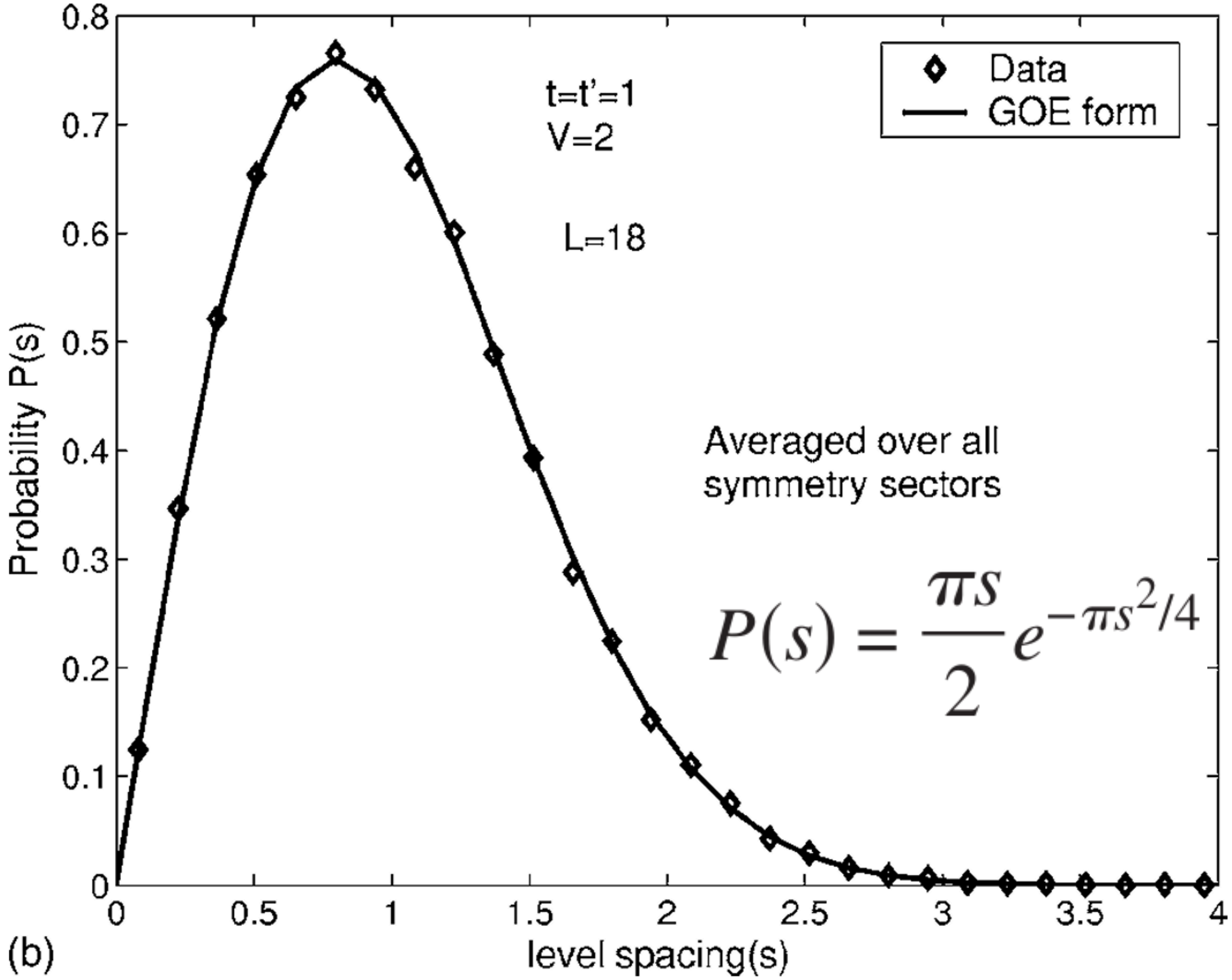
Away from integrability the many-body levels satisfy random matrix theory




Poilblanc et al EPL, (1993)

# Strongly interacting 1D Fermions

$$H = -t\sum_j c_j^+ c_{j+1} - t'\sum_j c_j^+ c_{j+2} + \text{h.c.} + V\sum_j n_j n_{j+1}$$




The many-body energy levels

$$s_i = E_i - E_{i-1}$$

Well described using the GOE ensemble

Mukerjee, Oganesyan, Huse, PRB (2006)

# Need methods to Describe Many-Body Quantum Chaos

Random matrix theory is useful but its not enough.

Many-body quantum chaotic systems should thermalize:
How is this described?
How does this take place?

In addition to correlations, we will use the entanglement entropy (and several other notions from quantum information)

# Characterizing Quantum Many Body Systems

A useful characterization for quantum many-body systems beyond correlations is the nature of their **entanglement structure**

$$\langle\Psi|\mathcal{O}(x,t)\mathcal{O}(0,0)|\Psi\rangle$$

# CHARACTERIZING QUANTUM MANY BODY SYSTEMS

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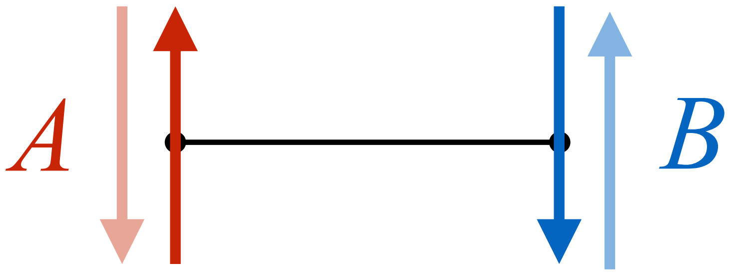
Example: 2 spin-1/2s in a singlet




$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow_A\rangle |\downarrow_B\rangle - |\downarrow_A\rangle |\uparrow_B\rangle \right)$$

"1 bit of entanglement"

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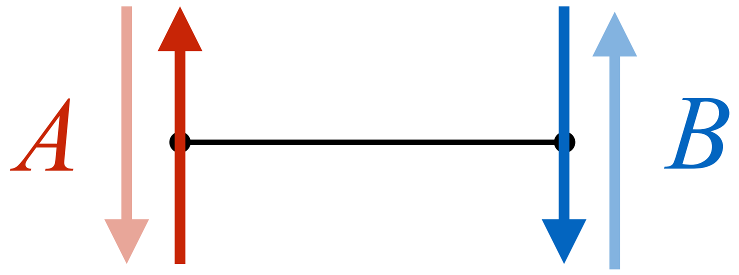


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Von Neumann entanglement entropy

$$\rho = |\psi\rangle\langle\psi|$$

$$S_E = -\mathrm{Tr}_{\mathrm{A}}(\rho_A \log \rho_A)$$

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$

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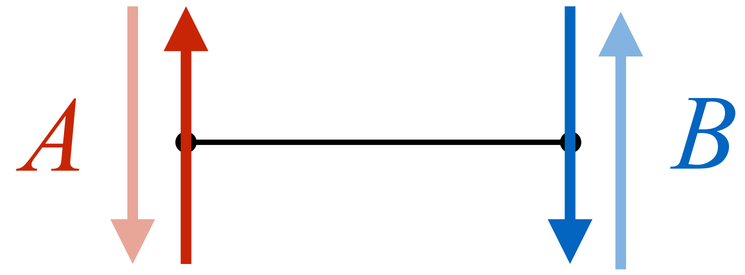


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$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$S_E = -\mathrm{Tr}_{\mathrm{A}}(\rho_A \log \rho_A)$$

$$= \log 2$$

"1 bit of entanglement"

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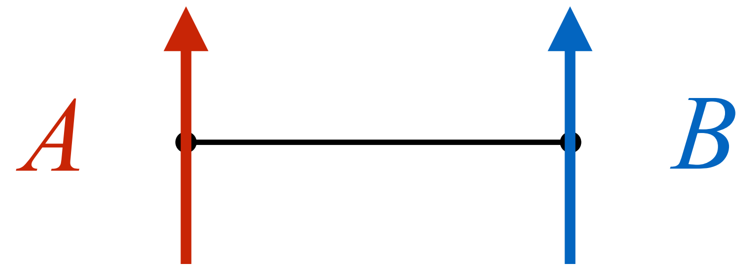
$$\langle \Psi | \mathcal{O}(x,t)\mathcal{O}(0,0) | \Psi \rangle$$

Example: 2 spin-1/2s aligned




$$|\psi\rangle = |\uparrow_A\rangle |\uparrow_B\rangle$$

Von Neumann entanglement entropy

$$S_E = -\mathrm{Tr}_\mathrm{A}(\rho_A \log \rho_A)$$
$$= 0$$

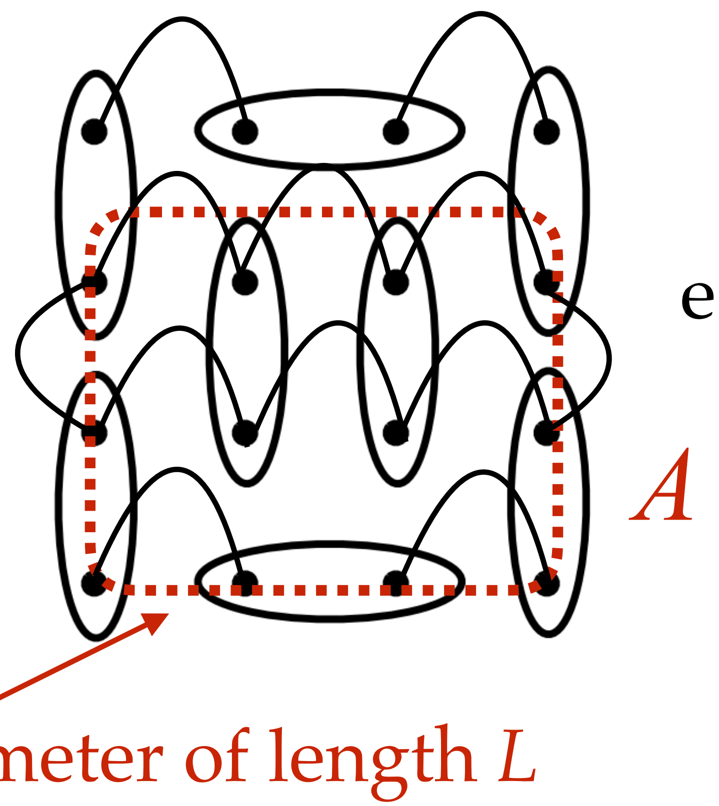
No entanglement!

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

# Characterizing Quantum Many Body Systems

Now consider a many body problem
e.g. a large number of spin-1/2s interacting




**a weakly entangled state**
short range collection of singlets:
e.g. ground state of a valence bond solid

$S_E \sim L^{d-1}$ "area law"

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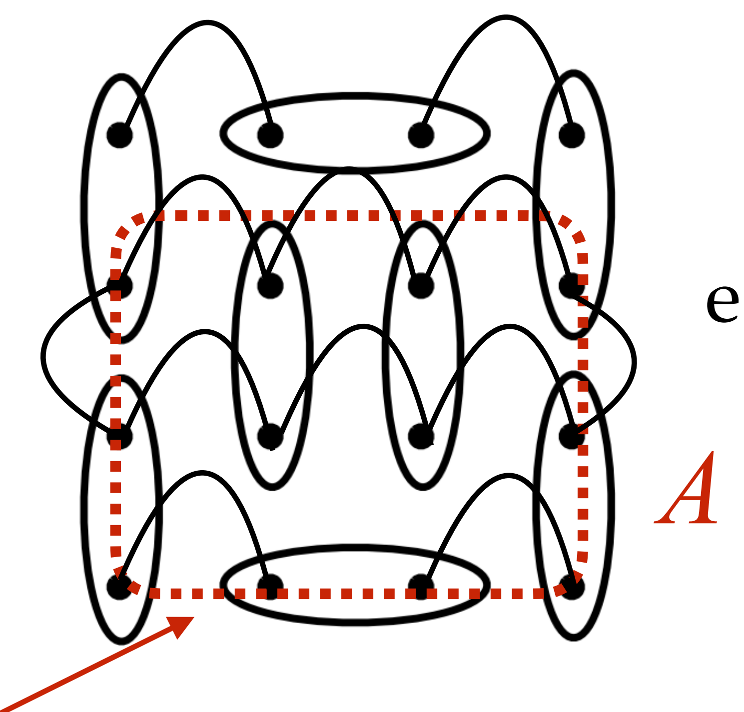


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short range collection of singlets:
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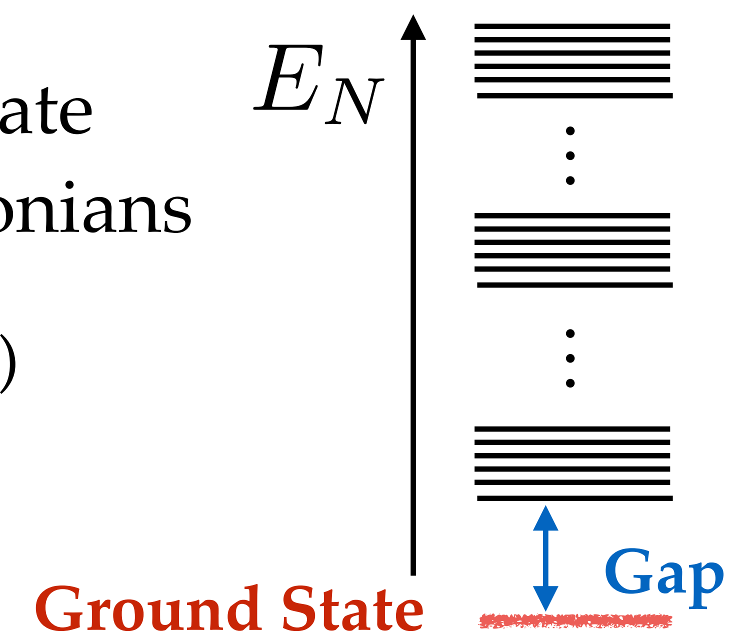
$$S_E \sim L^{d-1}$$ "area law"

characteristic of the ground state entanglement of gapped Hamiltonians

proven in 1D, Hasings JSTAT (2007)

many-body
Energy spectra



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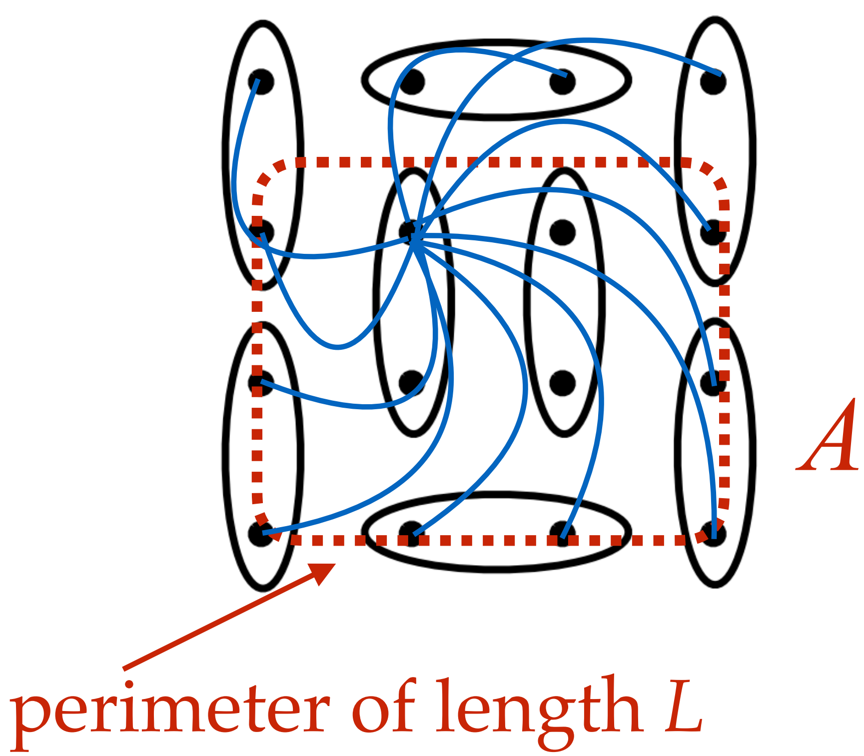
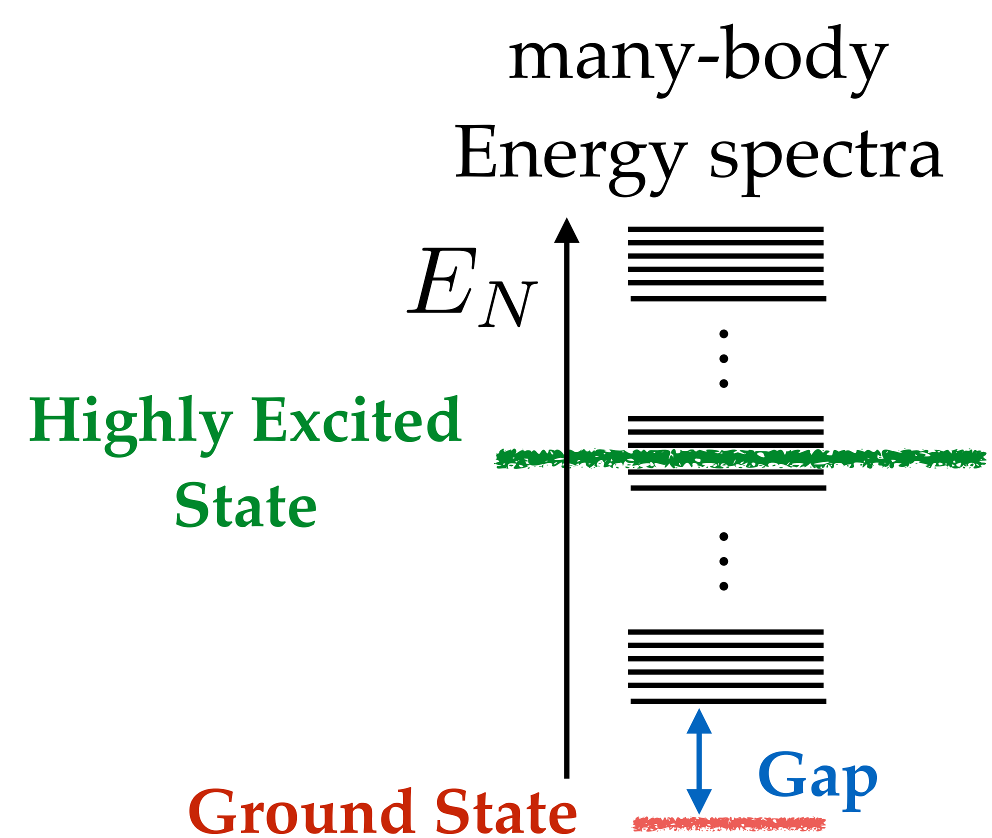



A **highly entangled state**
at **finite energy density**

$$S_E \sim L^d \quad \text{"volume law"}$$



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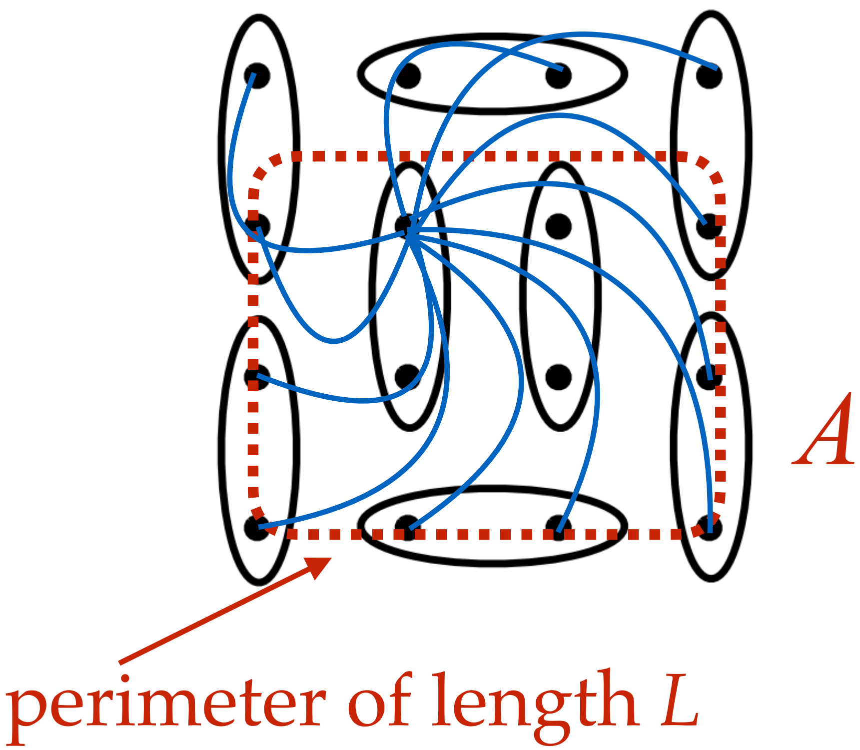


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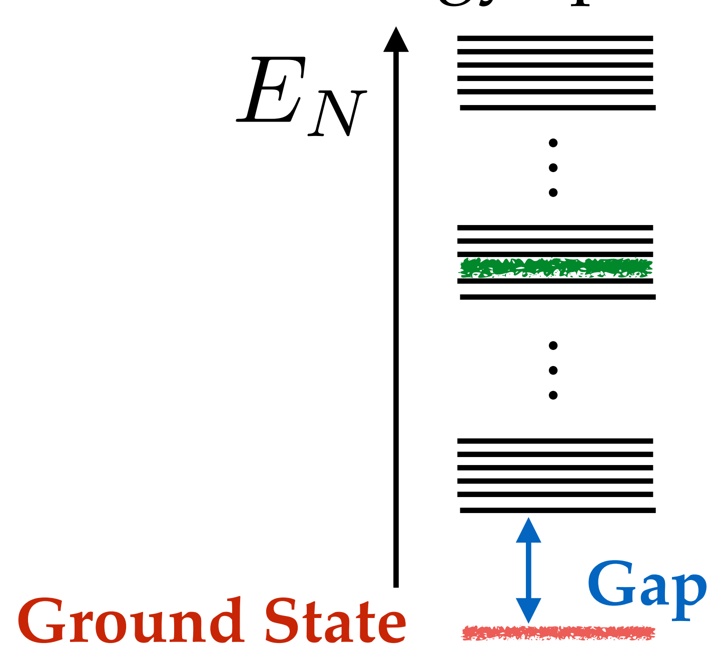


The long time-dynamics of
a many body Hamiltonian

$$|\Psi_0\rangle = |\uparrow\downarrow\ldots\uparrow\downarrow\rangle$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

# Characterizing Quantum Many Body Systems

In summary, quantum states can be broadly classified by their entanglement structure.

**weakly** entangled $S_E \sim L^{d-1}$ "area law"

**highly** entangled $S_E \sim L^d$ "volume law"

Many-body quantum chaotic states are highly entangled

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In summary, quantum states can be broadly classified by their entanglement structure.

**weakly** entangled $S_E \sim L^{d-1}$ "area law"

**highly** entangled $S_E \sim L^d$ "volume law"

Many-body quantum chaotic states are highly entangled

In *d=1*, critical states acquire a log-correction

$$S_E \sim \log L$$

Calabrese and Cardy, J. Stat. Mech (2004)

# Outline

I. Lecture series layout
II. Classical chaos
III. Quantum chaos
IV. Quantum thermalization
V. Evading thermalization

# Thermalization in Classical Systems

Classical thermalization takes place through the "ergodic hypothesis"

Ergodic systems can explore all of their phase space.

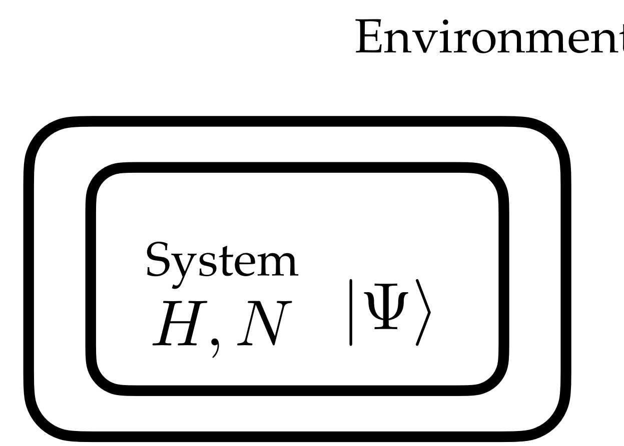
Chaotic systems need not be ergodic, and ergodic systems need not be chaotic.

*Ergodic Theory and Differentiable Dynamics*, Mane

# From Many-Body Quantum Chaos To Thermalization

How does an isolated quantum system reach thermal equilibrium?

When a system has thermalized it can be described by a few parameters, e.g. chemical potential and temperature




$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

Deutsch PRA (1991) Srednicki PRE (1994) Rigol, Dunjko, and Olshanii, Nature (2008) Linden, et al PRE (2009)

Cardy PRL (2014) Garison and Grover PRX (2018) Kaufman et al, Science (2016)

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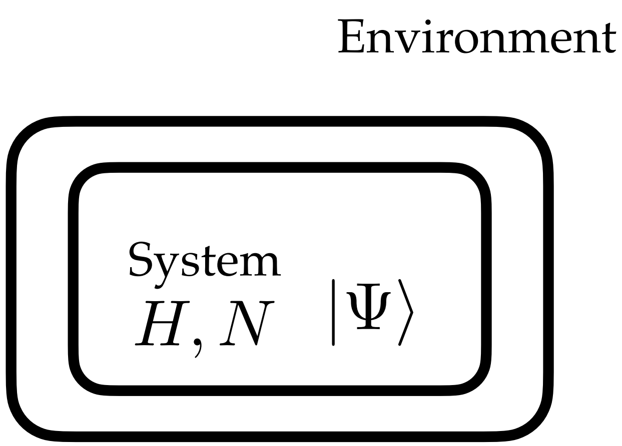


$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

But Hamiltonian time evolution is unitary, and it cannot erase information, (only picks up phases)

All of the information of the initial state must be preserved in the evolution. It appears to be a paradox….

Deutsch PRA (1991) Srednicki PRE (1994) Rigol, Dunjko, and Olshanii, Nature (2008) Linden, et al PRE (2009)
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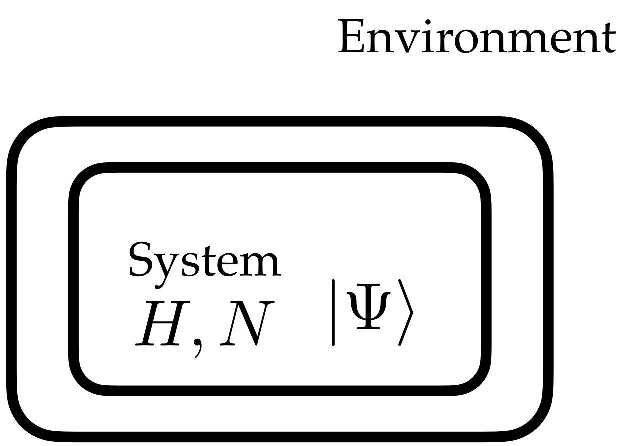
How does an isolated quantum system reach thermal equilibrium?

The resolution to this putative paradox
Is that the unitary evolution has spread the
local information about the initial state
across the entire system.

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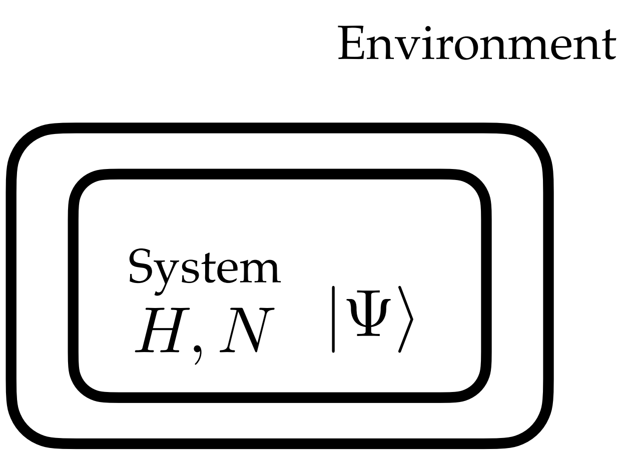


$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

For this to be possible it has to be
sufficiently entangled and locally chaotic.

At late times, this local information has
been hidden and cannot be retrieved by
local measurements.

Deutsch PRA (1991)    Srednicki PRE (1994)    Rigol, Dunjko, and Olshanii, Nature (2008)    Linden, et al PRE (2009)

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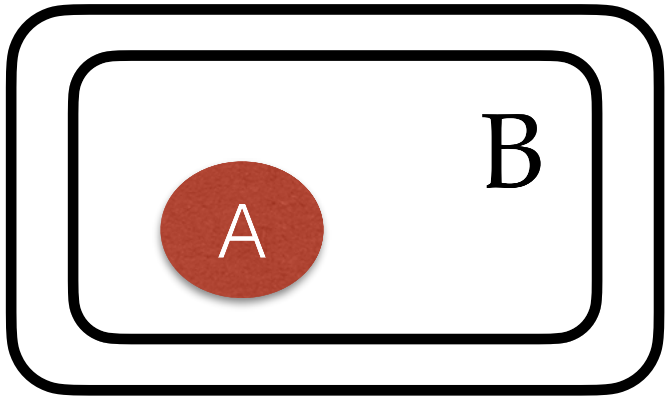
# Eigenstate Thermalization Hypothesis (ETH)

Lets now focus on local observables in a small region A and ask if region A can thermalize while region B acts like a reservoir




Consider an initial state with weight over a large number of energy eigenstates

$$|\Psi(0)\rangle = \sum_n c_n |n\rangle$$

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle$$

Rigol, Dunjko, and Olshanii, Nature (2008)

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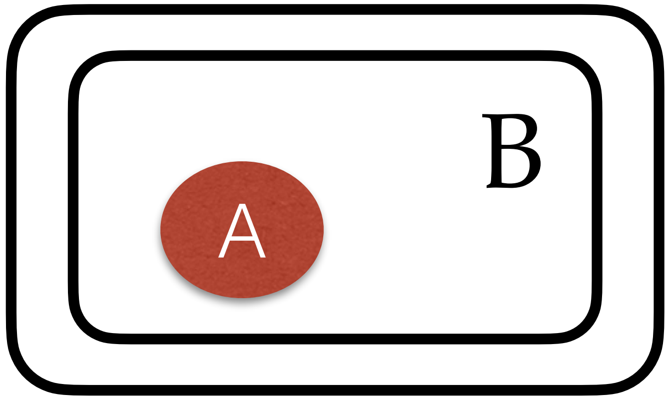


Consider an initial state with weight over a large number of energy eigenstates

$$|\Psi(0)\rangle = \sum_n c_n |n\rangle$$

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle$$

For a local operator (in A)

$$\lim_{t\to\infty} \frac{1}{t} \langle \Psi(t)|\hat{O}|\Psi(t)\rangle = \sum_n A_{nn} O_{nn} \equiv \langle \hat{O} \rangle_{\mathrm{eq}}$$

such a steady state has "equilibrated"

Linden, Popescu, Short, Winter PRE (2009)

Rigol, Dunjko, and Olshanii, Nature (2008)

# Eigenstate Thermalization Hypothesis (ETH)

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For the system to have thermalized, ETH requires

$$\langle\hat{O}\rangle_{\mathrm{MC}} = \frac{1}{\mathcal{N}}\sum_{|E_n-E|\leq\Delta} O_{nn} \approx \langle\Psi(E)|\hat{O}|\Psi(E)\rangle$$

Micro-canonical ensemble

$$\langle\hat{O}\rangle_{\mathrm{eq}} = \langle\hat{O}\rangle_{\mathrm{MC}}$$

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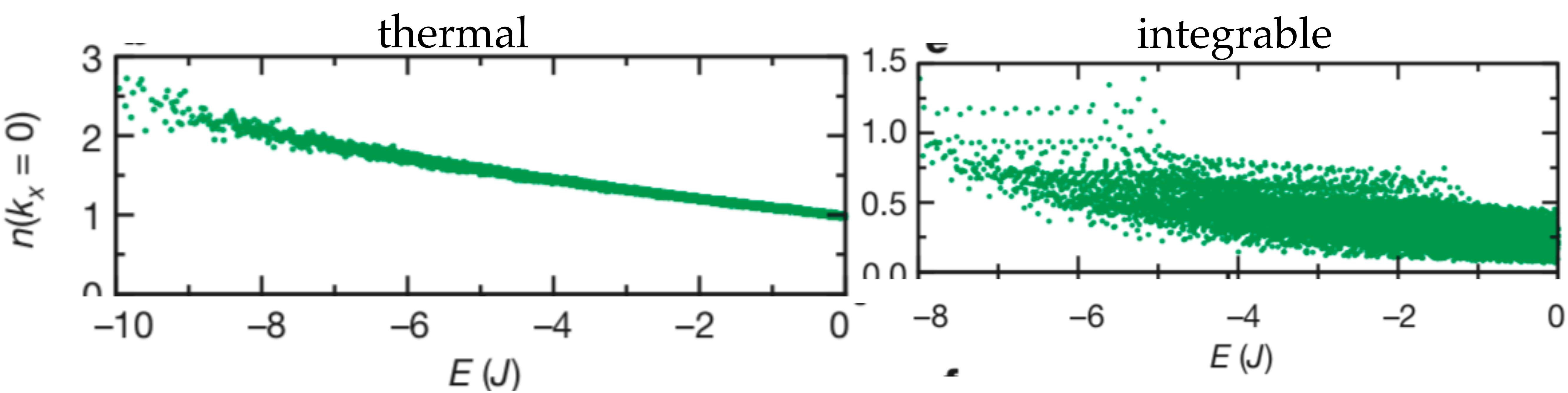

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Linden, Popescu, Short, Winter PRE (2009)

$$O(t) \equiv \langle\psi(t)|\hat{O}|\psi(t)\rangle = \sum_{m,n} C_m^* C_n e^{i(E_m-E_n)t} O_{mn}$$

$$= \sum_m |C_m|^2 O_{mm} + \sum_{m,n\neq m} C_m^* C_n e^{i(E_m-E_n)t} O_{mn}$$

ETH can also be stated as an asatz for matrix elements

$$\langle\hat{O}\rangle_{\text{eq}} = \langle\hat{O}\rangle_{\text{MC}}$$

$$O_{mn} = O\left(\bar{E}\right)\delta_{mn} + e^{-S(\bar{E})/2} f_O\left(\bar{E},\omega\right) R_{mn}$$

Rigol, Dunjko, and Olshanii, Nature (2008)

# A Stronger Version of ETH

For a local operator

$$\lim_{t\to\infty} \frac{1}{t}\langle\Psi(t)|\hat{O}|\Psi(t)\rangle = \sum_n A_{nn} O_{nn} \equiv \langle\hat{O}\rangle_{\text{eq}}$$

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$$\langle\hat{O}\rangle_{\text{MC}} = \frac{1}{\mathcal{N}} \sum_{|E_n - E|\leq\Delta} O_{nn}$$

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$$\langle\hat{O}\rangle_{\text{C}} = \frac{1}{Z}\text{Tr}(e^{-\beta H}\hat{O})$$

Canonical ensemble

temperature is set by the avg energy

$$\langle\Psi|H|\Psi\rangle = \text{Tr}(He^{-\beta H})/\text{Tr}(e^{-\beta H})$$

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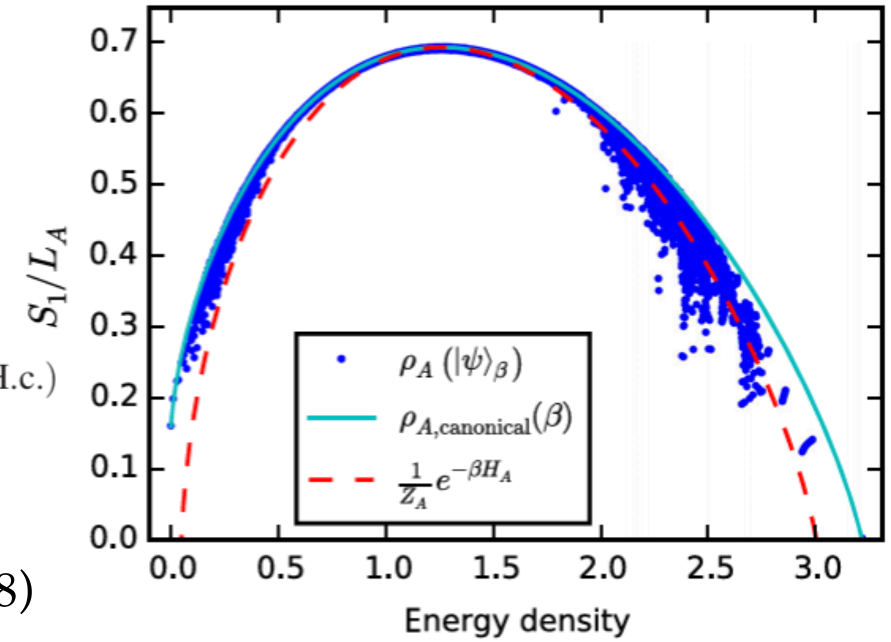
## ETH requires

$$\langle\hat{O}\rangle_{\text{eq}}=\langle\hat{O}\rangle_{\text{C}}=\langle\hat{O}\rangle_{\text{MC}}$$

Then we can conclude

$$\rho_A\sim\text{Tr}_B(e^{-\beta H})$$

$$H=-\sum_i(tb_i^\dagger b_{i+1}+t'b_i^\dagger b_{i+2}+\text{H.c.})+\sum_i(Vn_in_{i+1}+V'n_in_{i+2}),$$



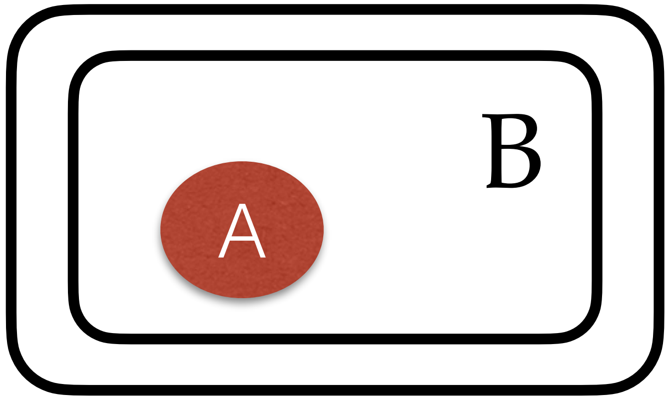

Rigol, Dunjko, and Olshanii, Nature (2008)   Garrison and Grover, PRX (2018)

# An even Stronger Version of ETH

Conventional thermalization

$$\rho_A \sim \mathrm{Tr}_B(e^{-\beta H})$$

Canonical ensemble




Deep thermalization: Consider making an extensive number of local measurements in B. The wave function in A is projected and weighted by the Born probability of the measurement, and the reduced density matrix in A is now:

**Projected ensemble**: The ensemble of all possible states in A after the measurement and Born probabilities in B

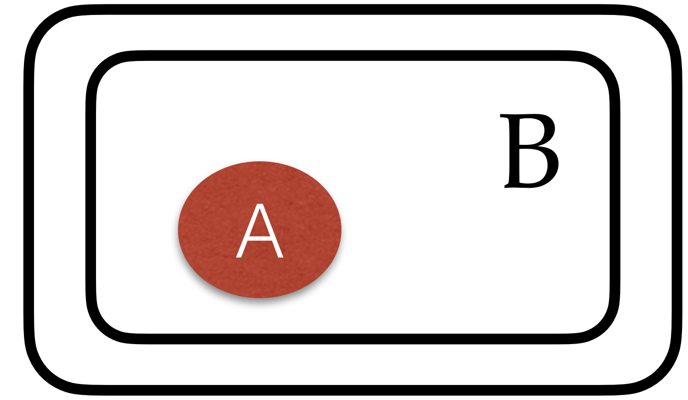
$$\rho_A^{DT} = \sum_{\mathbf{m}} p_{\mathbf{m}} |\psi_A^{(\mathbf{m})}\rangle\langle\psi_A^{(\mathbf{m})}|$$

Born probability of measurement outcome in B

Wavefunction in A after the measurement in B

# An even Stronger Version of ETH

**Projected ensemble**: The ensemble of all possible states in A after the measurement and Born probabilities

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The projected ensemble has the full distribution of the reduced density matrix

Cotler et al, PRX Quantum (2023)

Ippoliti and Ho, PRX Quantum (2023)

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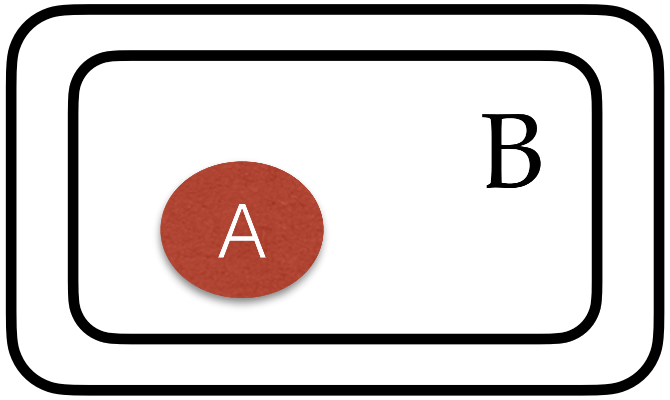


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Higher moments can be estimated from it.

$$(\rho_A^{DT})^k = \sum_{\mathbf{m}} p_{\mathbf{m}} |\psi_A^{(\mathbf{m})}\rangle\langle\psi_A^{(\mathbf{m})}|^{\otimes k}$$

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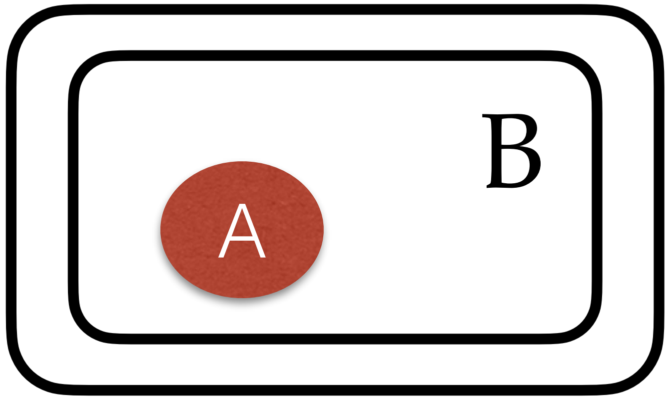


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$$(\rho_A^{DT})^k = \sum_{\mathbf{m}} p_{\mathbf{m}} |\psi_A^{(\mathbf{m})}\rangle\langle\psi_A^{(\mathbf{m})}|^{\otimes k}$$

The mean reduces to ETH

$$(\rho_A^{DT})^{k=1} = \rho_A$$

In the absence of any symmetries or conservation laws, at infinite temperature

$$\rho^{(k)}(t) \to \rho_H^{(k)}$$

$$\rho_H^{(k)} = \int_{\psi\sim\mathrm{Haar}(\mathcal{H}_A)} d\psi\, (|\psi\rangle\langle\psi|)^{\otimes k}$$

# What about the process of thermilization

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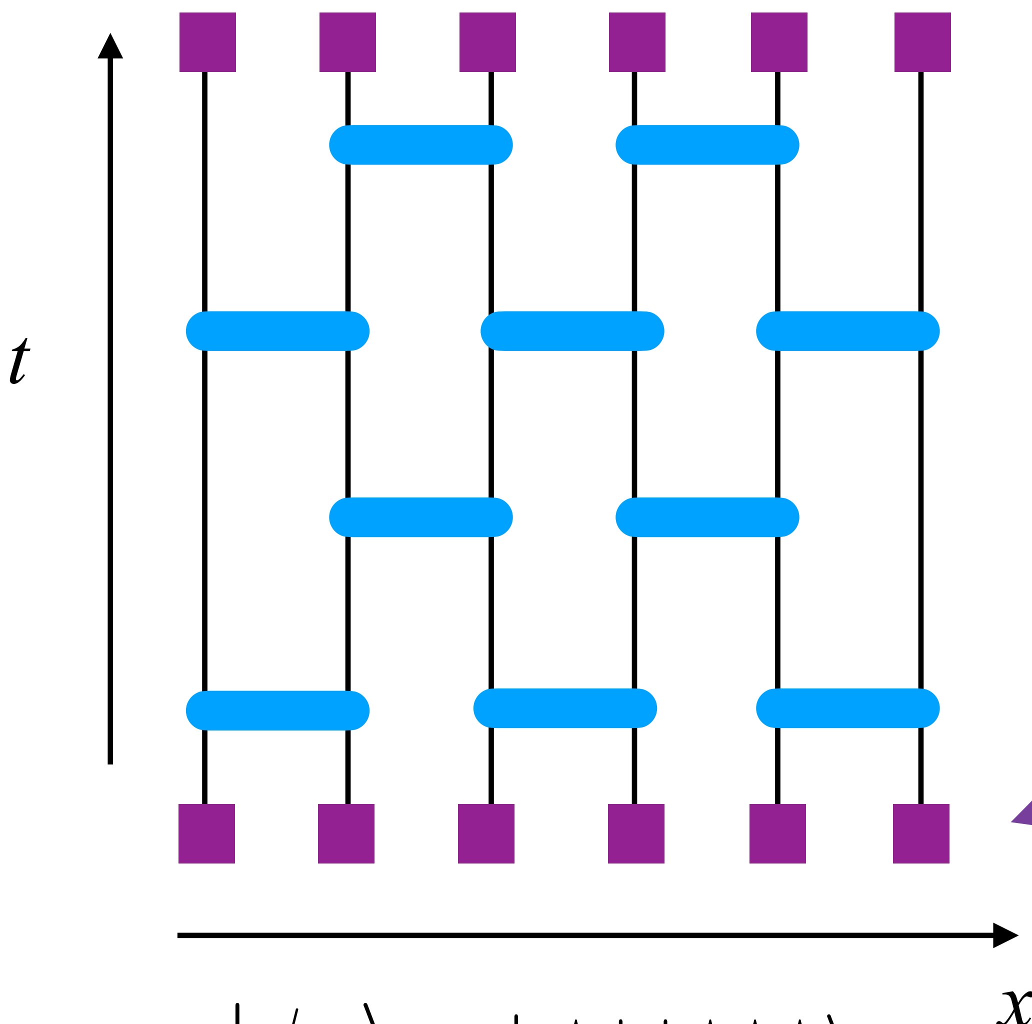
Random unitary circuits have emerged as minimal models to address this question.

von Keyserlingk, Rakovszky, Pollmann, and Sondhi PRX (2018)

A. Nahum, S. Vijay, and J. Haah PRX (2018)

# Random Unitary Circuits




$$|\psi_0\rangle = |\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\rangle$$

e.g. some random product state

*q*-states "spins"

e.g. $q$=2 is a spin-1/2 chain

**Analytically tractable** for $q \to \infty$

A. Nahum, S. Vijay, and J. Haah PRX (2018)
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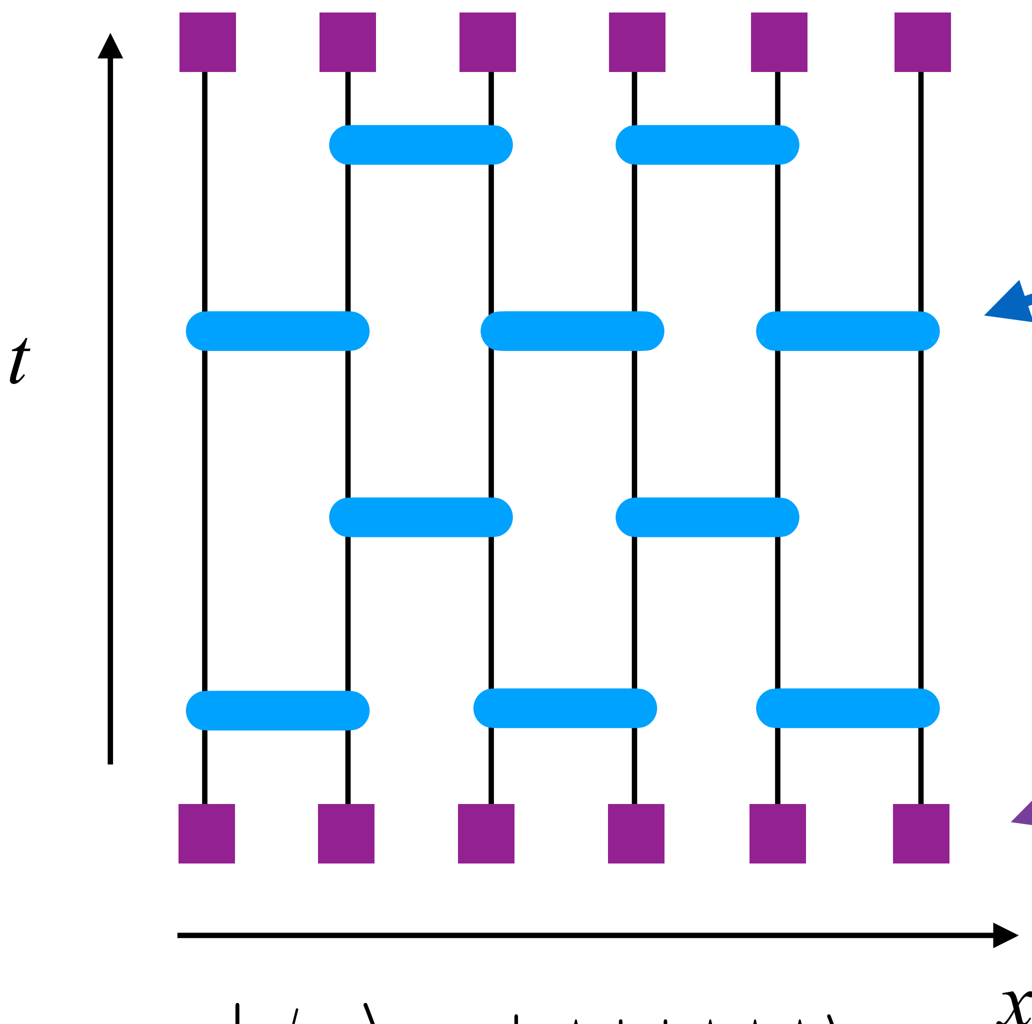


$U_{i,i+1}$

unitary chosen from
a Haar random distribution

$q$-states "spins"

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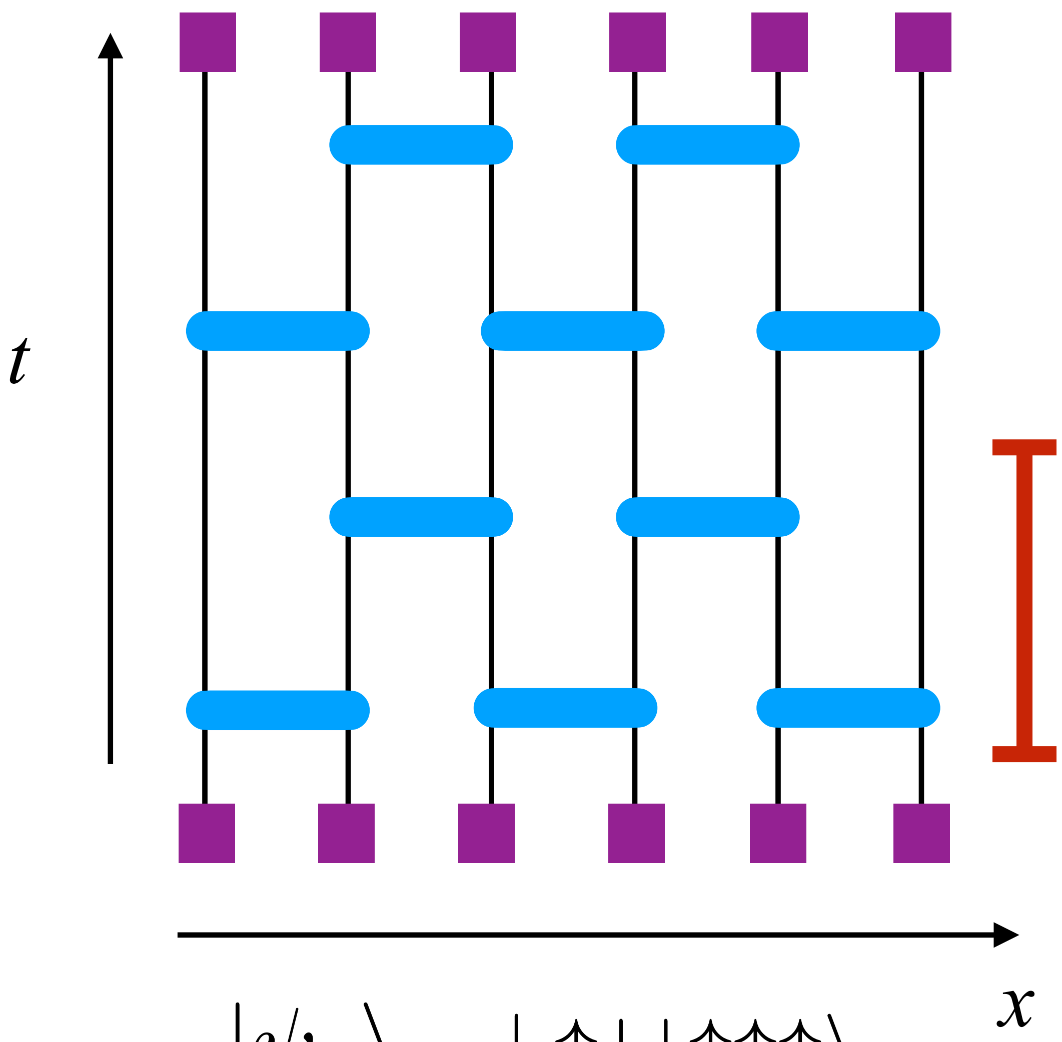


= one time step

$$U(t',t'-1)=\bigotimes_{i\in 2\mathbb{Z}} U_{i,i+1}(t',t'-1)\bigotimes_{i\in 2\mathbb{Z}+1} U_{i,i+1}(t',t'-1)$$

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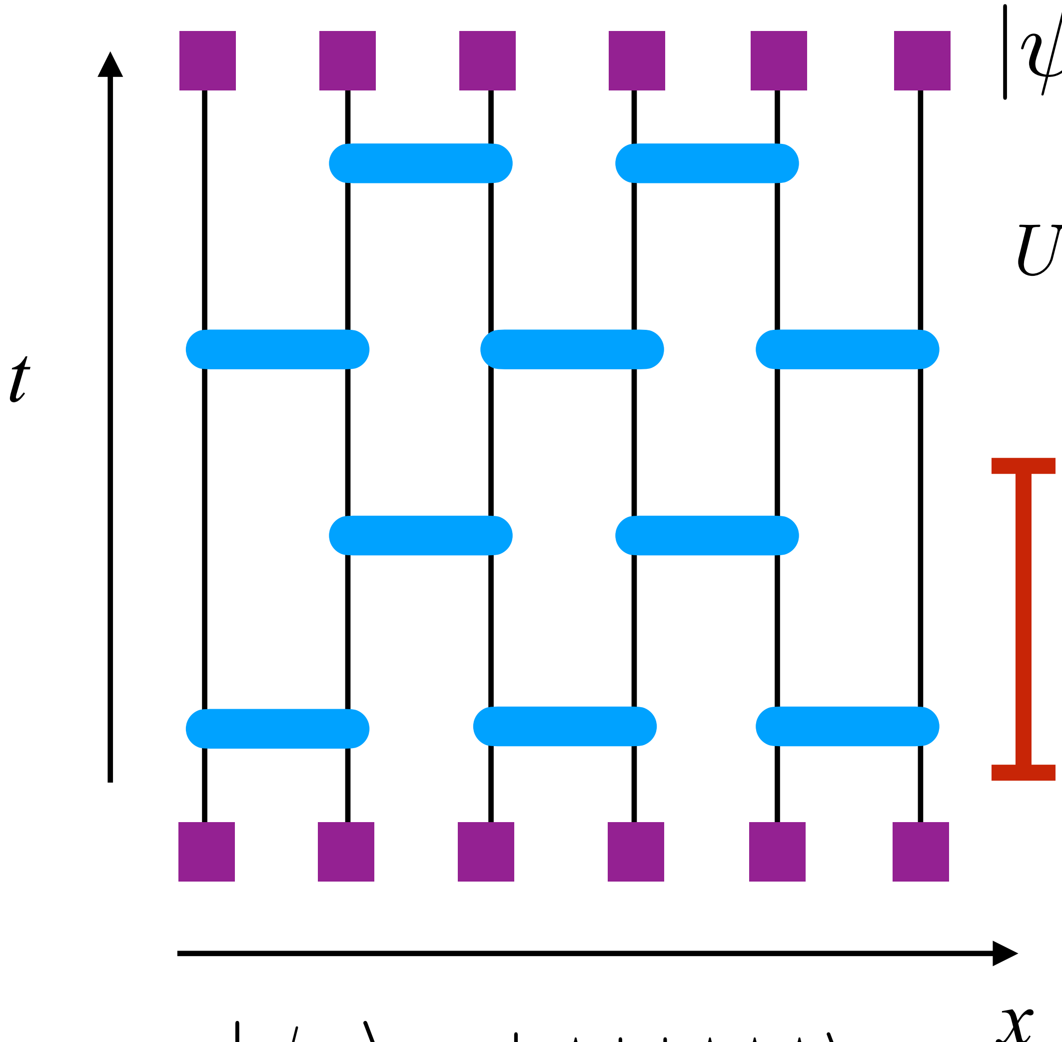


$$|\psi(t)\rangle = U(t)|\psi_0\rangle$$

$$U(t) = U(t,t-1)U(t-1,t-2)\dots U(1,0)$$

= one time step

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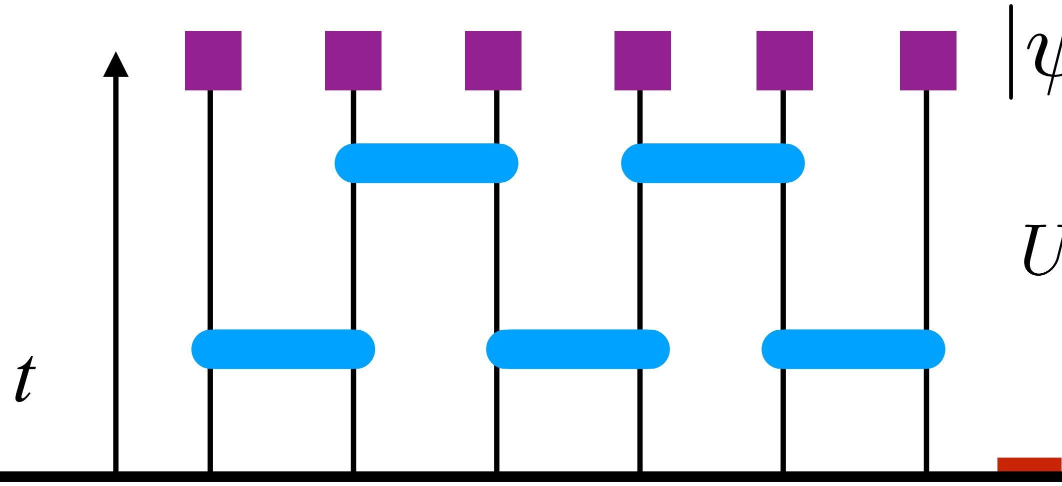


$$|\psi(t)\rangle = U(t)|\psi_0\rangle$$

$$U(t) = U(t, t-1)U(t-1, t-2)\ldots U(1,0)$$

It is expected / argued that the behavior found here is generic to many-body quantum dynamics

e time step

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# Statistics of the Reduced Density Matrix

The reduced density matrix for a random pure state (i.e. Page state) $|\Psi\rangle = \sum_i \alpha_i |C_i\rangle$

$\alpha_i$ → random complex numbers

$\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ is a **Wishart** random matrix

Znidaric, J. Phys. A: Math. Theor. (2007)

Yang et al PRL (2016)

Chen and Ludwig PRB (2018)

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$$P(\{Y_{iJ}\}) = \mathcal{N}^{-1} \exp\{-\frac{\beta}{2} N_B \mathrm{Tr}(YY^\dagger)\}$$

$Y$ is a complex random matrix

$$\hat{\rho}_A \equiv \frac{YY^\dagger}{\mathrm{Tr}(YY^\dagger)} \qquad W = YY^\dagger$$

$W$ is from the Wishart random matrix ensemble

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**Yang et al PRL (2016)**

Chen and Ludwig PRB (2018)

Captures **states** that satisfy **ETH** $\hat{\rho}_A \equiv \frac{YY^\dagger}{\mathrm{Tr}(YY^\dagger)}$

$$W = YY^\dagger$$

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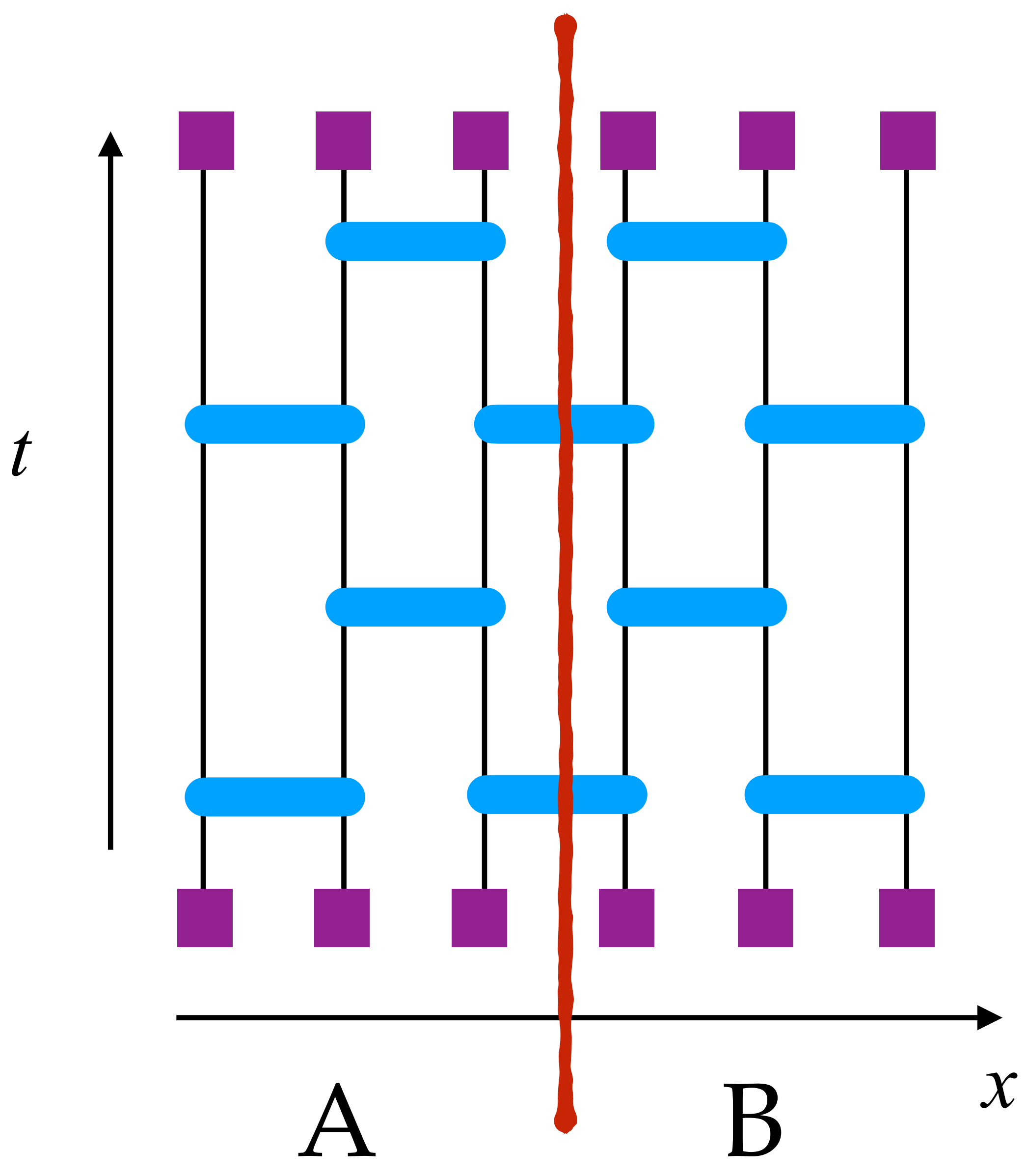
And the density of levels follows the **Marchenko-Pastur** distribution $P(\lambda) \sim 1/\sqrt{\lambda}$

# Entanglement Spectra

Entanglement spectrum of the reduced density matrix is analogous to a set of "Energy levels"

Li and Haldane PRL (2008)



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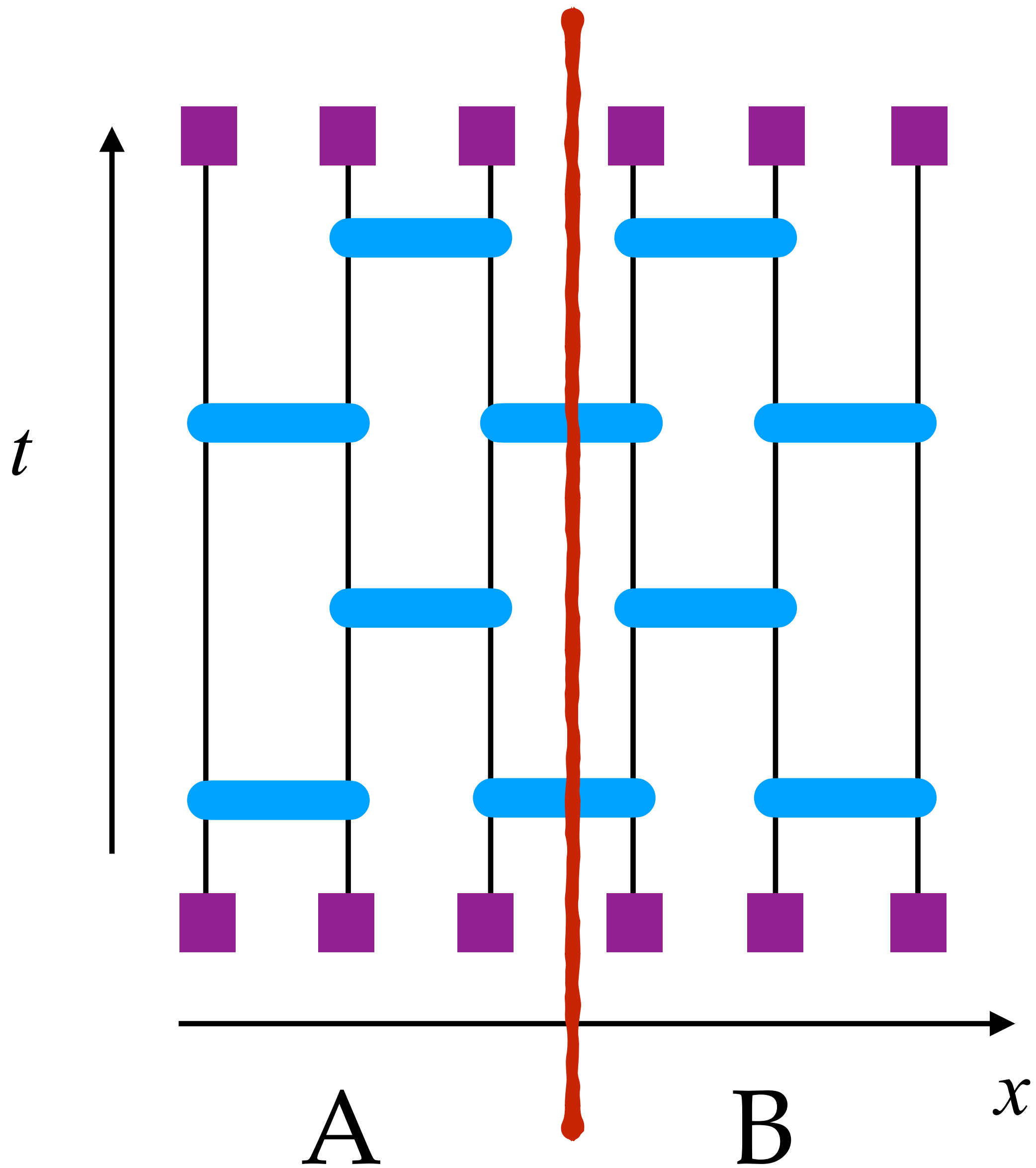


Li and Haldane PRL (2008)

$$\hat{\rho}_A = \mathrm{Tr}_B\left[\hat{U}(t)|\Psi_0\rangle\langle\Psi_0|\hat{U}^\dagger(t)\right]$$

Eigenvalues of the reduced density matrix $\{\lambda_n\}$

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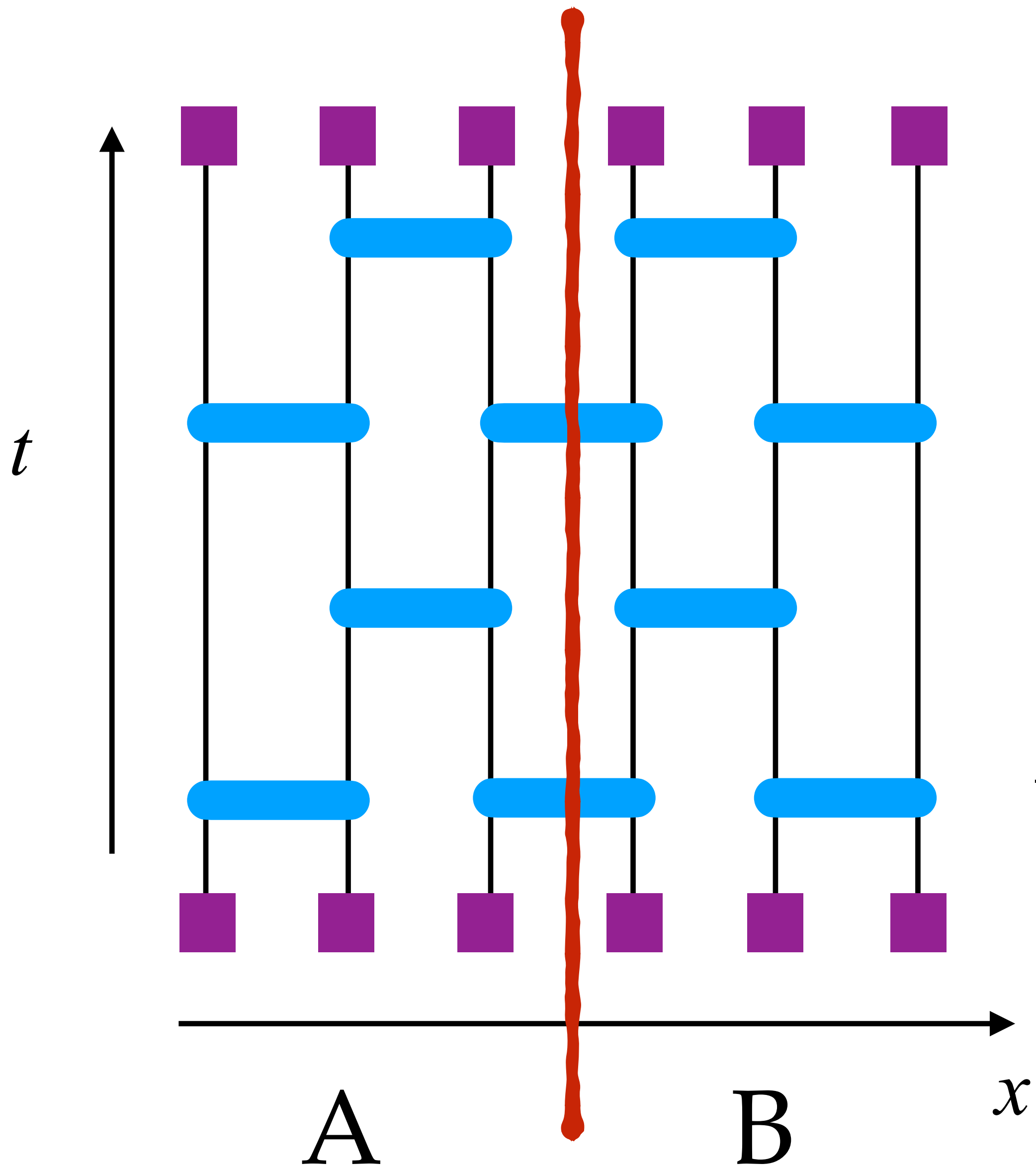


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Eigenvalues of the reduced density matrix $\{\lambda_n\}$

**Entanglement spectra** $\{E_n\} = \{-\log \lambda_n\}$

**Renyi entropies** $S_\alpha = \frac{1}{\alpha - 1} \log \left( \sum_n \lambda_n^\alpha \right)$

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For Hamiltonian dynamics in a system that satisfies ETH

$$\rho_A \sim \mathrm{Tr}_B(e^{-\beta H})$$

Thus the entanglement spectrum inherits the spectral statistics of the projected Hamiltonian (up to boundary terms)

# Scrambling at short times

Starting from an initial product state, the **rank** of the reduced density matrix grows as a function of time

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

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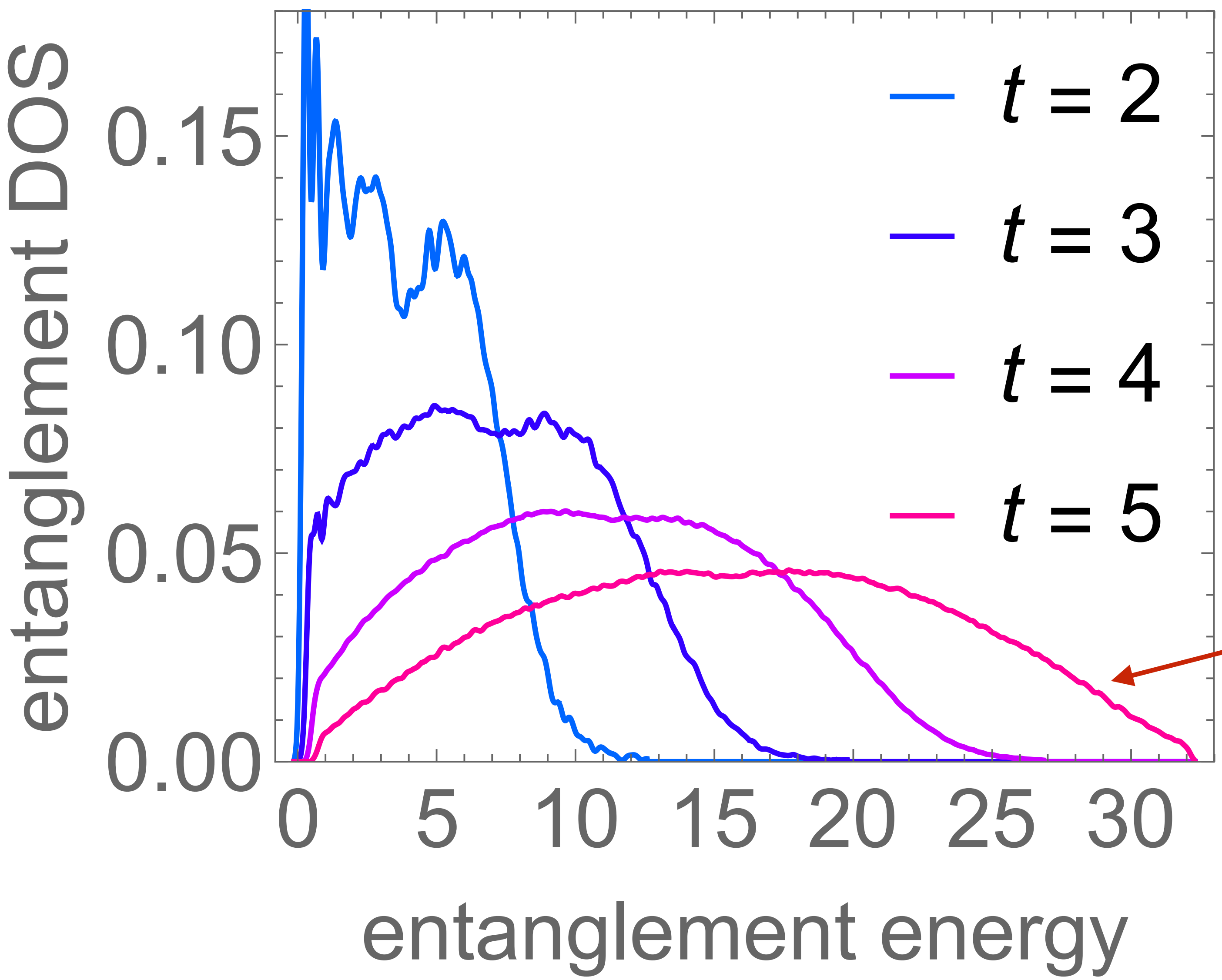
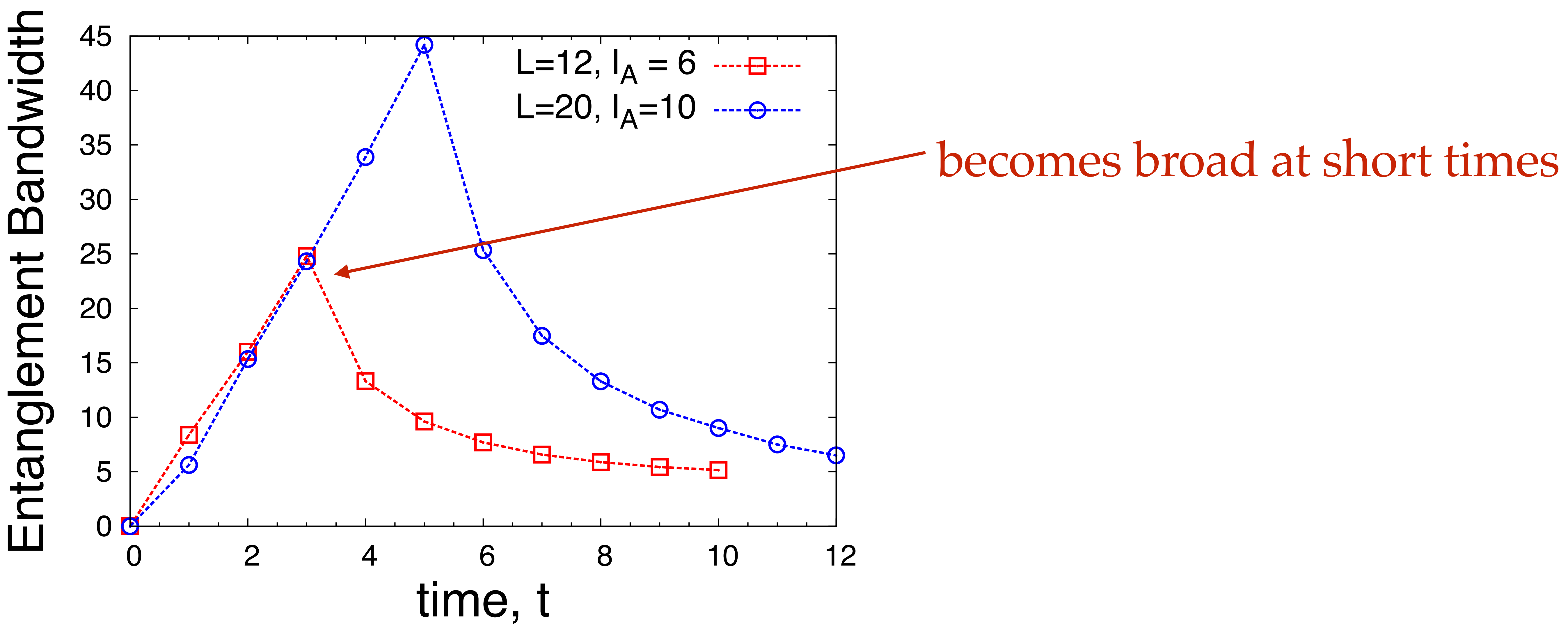



becomes broad at short times

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

# Entanglement Entropy vs Bandwidth

Growth of the rank of the reduced density matrix captured by the entanglement bandwidth!

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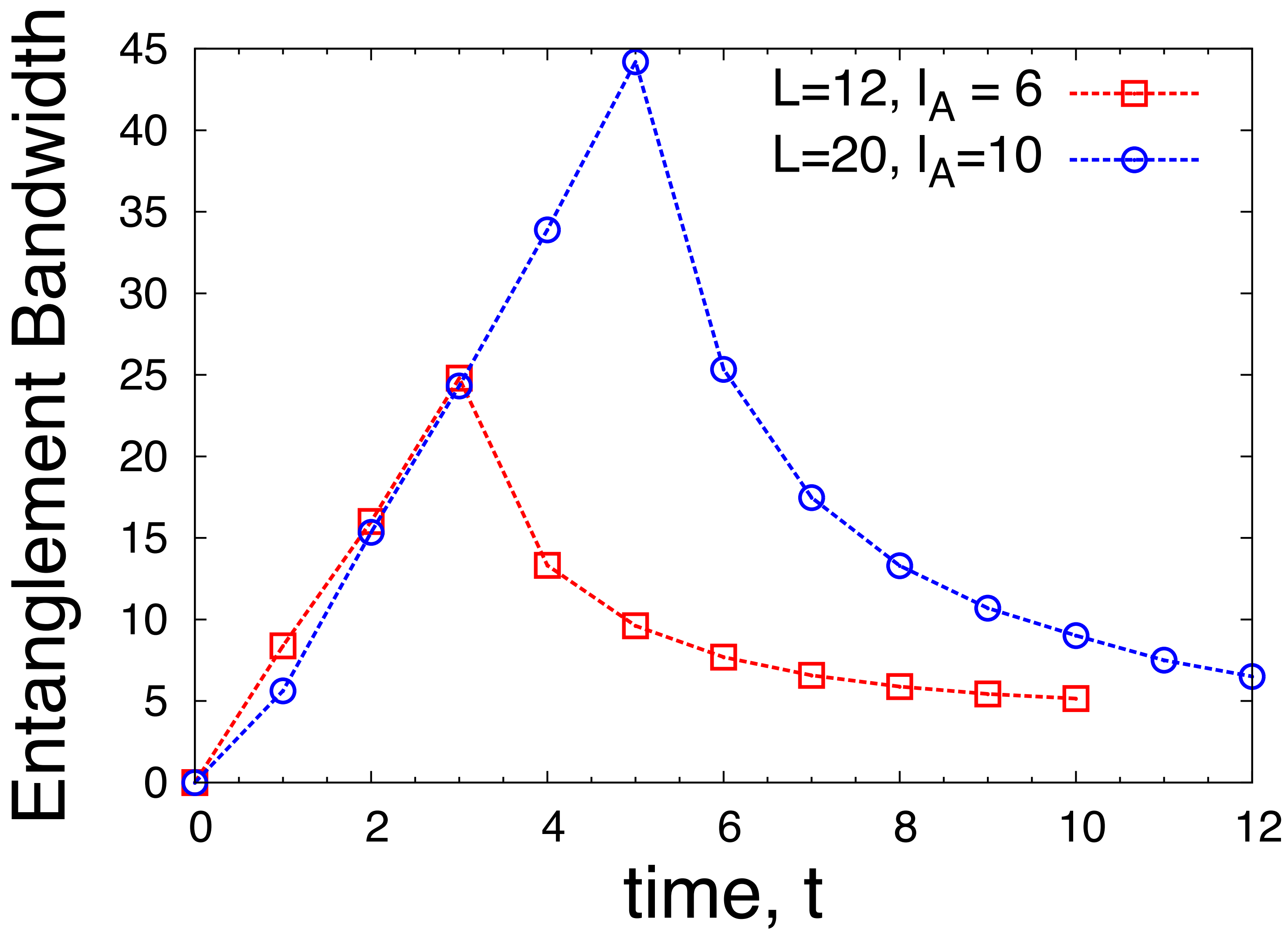


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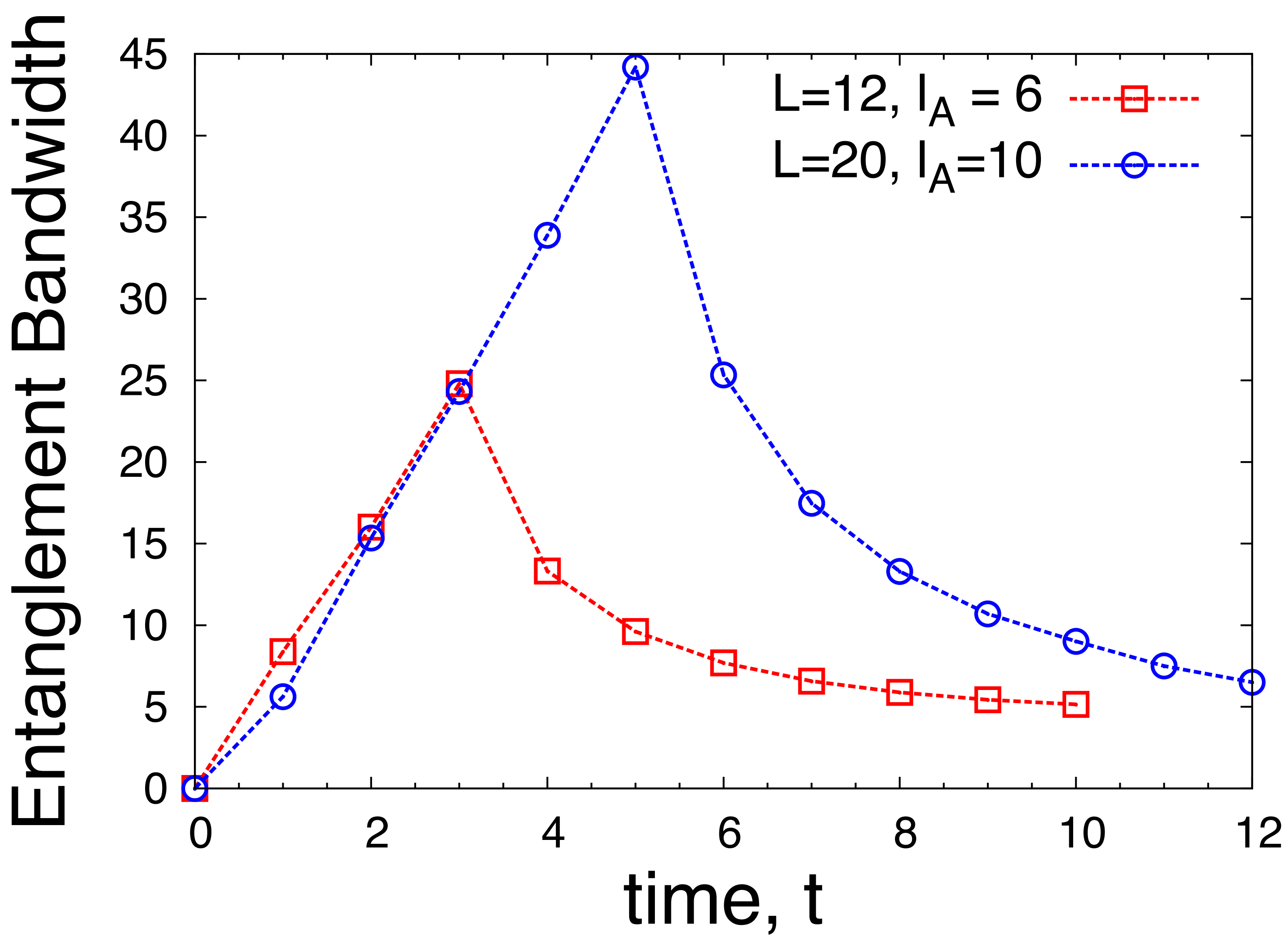


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Operator carries weight

$$\sim \exp\{-[t(v_{LC} - v_B)]^2/(Dt)\}$$

~RDM eigenvalues

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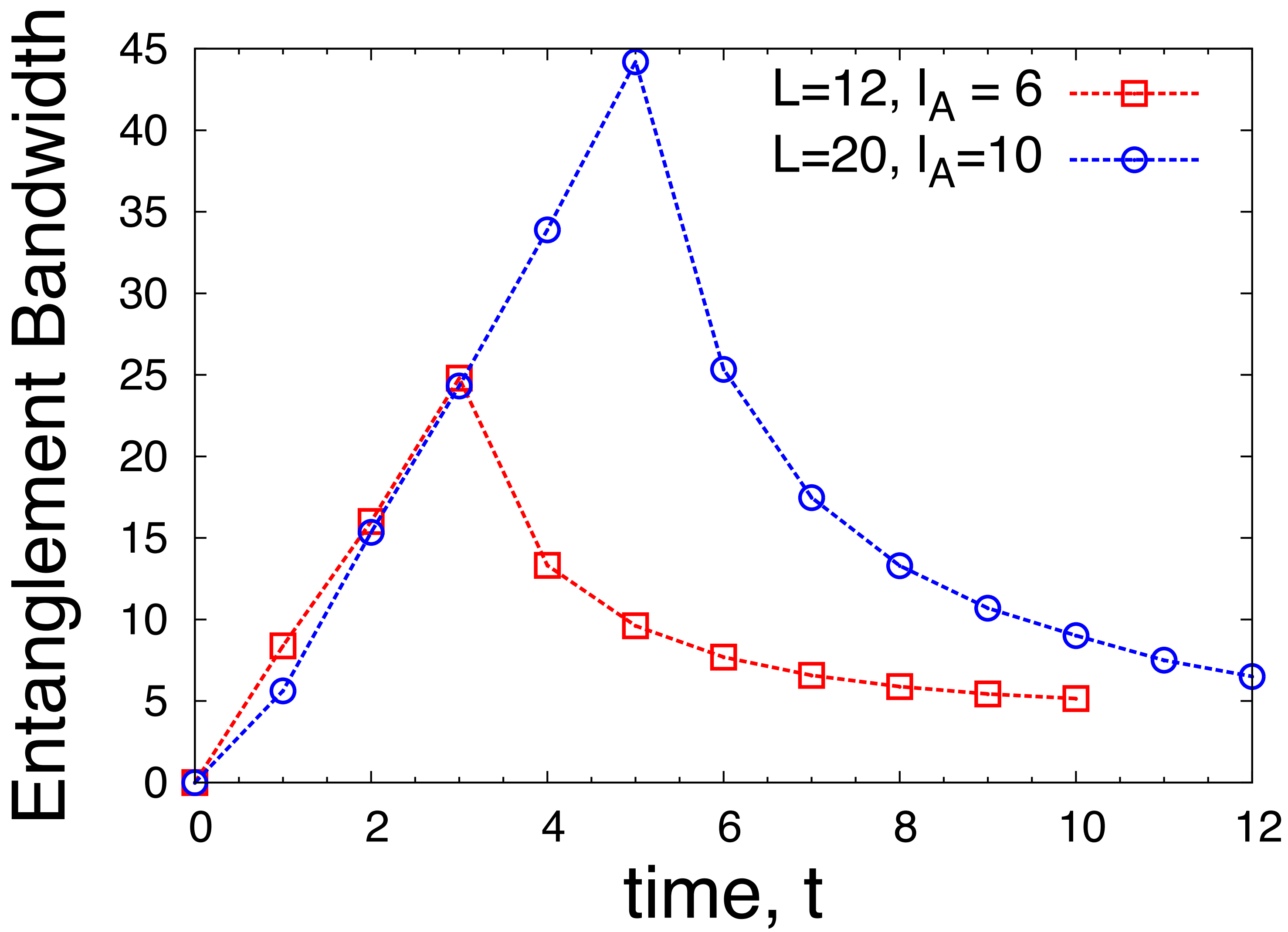


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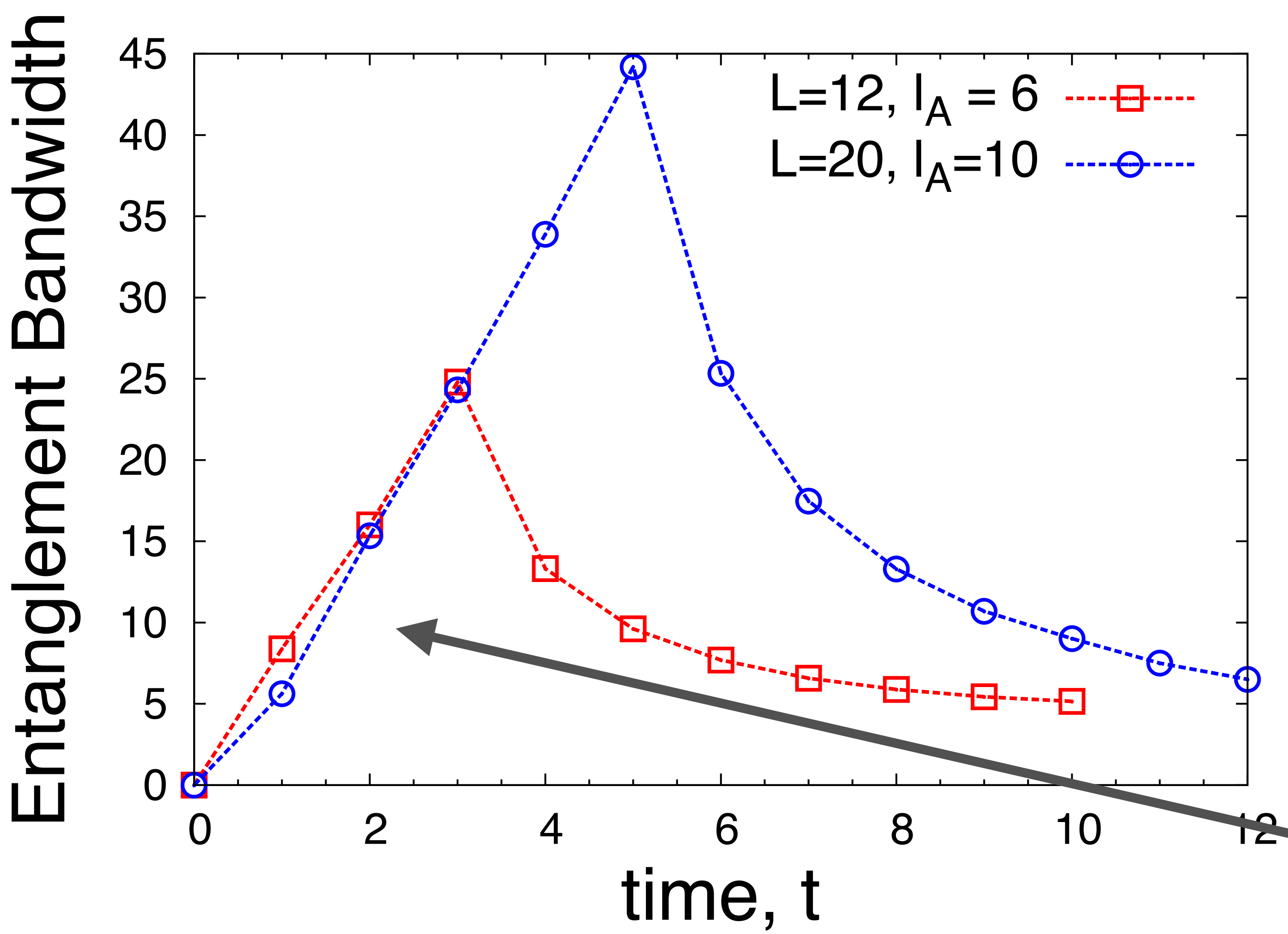


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bandwidth grows linearly!

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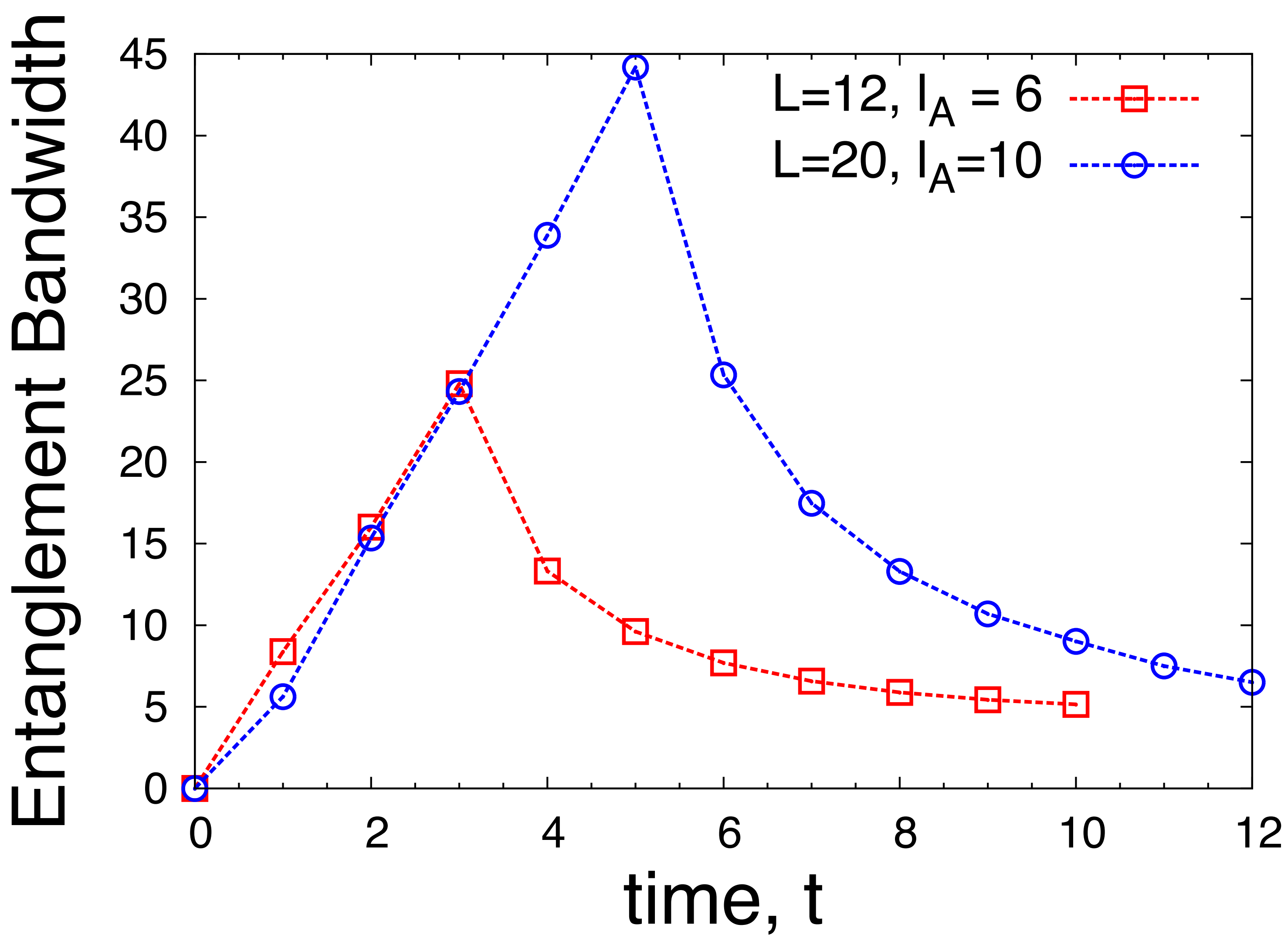


Can be understood via operator spreading

$$v_{LC} = \Delta x / \Delta t = 2$$

Light cone hits the edge of the subsystem at time

$$t = l_A / v_{LC} = l_A / 2$$

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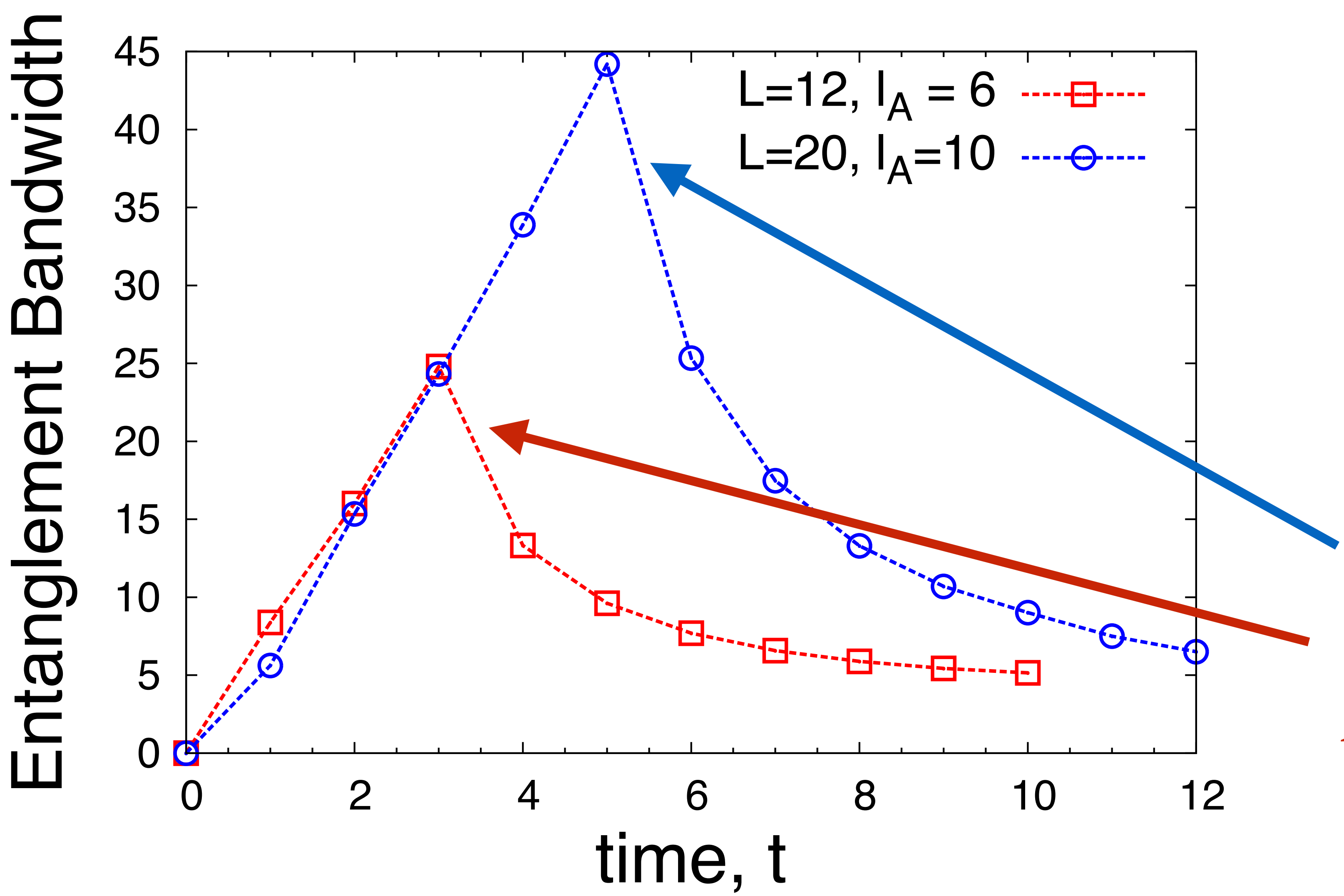


Can be understood via operator spreading

$$v_{LC} = \Delta x/\Delta t = 2$$

Light cone hits the edge of the subsystem at time

$$t = l_A/v_{LC} = l_A/2$$

Entanglement spectra's bandwidth grows until the light cone hits the edge of the subsystem

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

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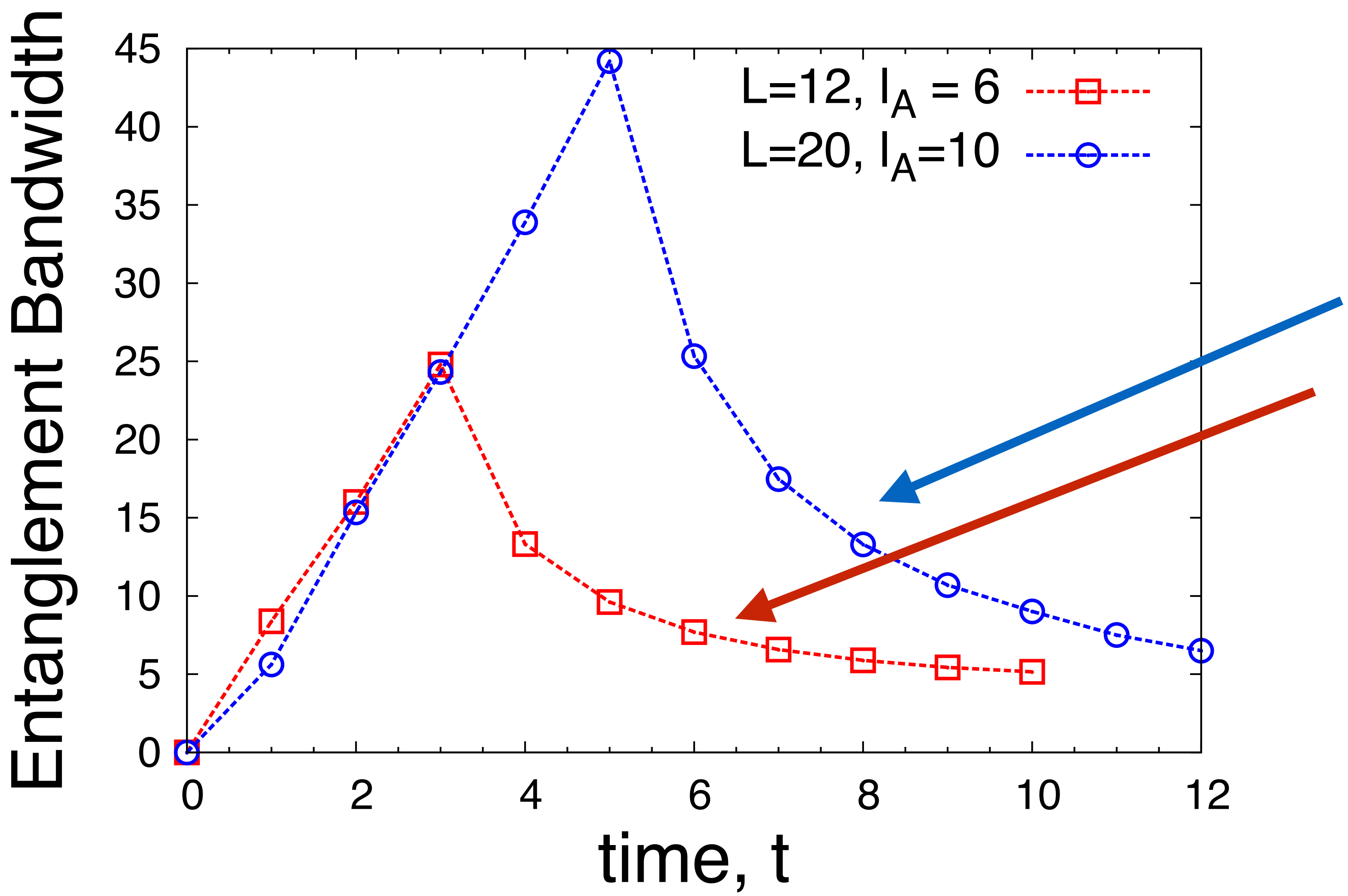
Growth of the rank of the reduced density matrix captured by the entanglement bandwidth!




For later times the RDM is full rank

$$= \min(q^{2t}, q^{l_A})$$

and the BW shrinks like

$$\sim 1/(t - t_0)$$

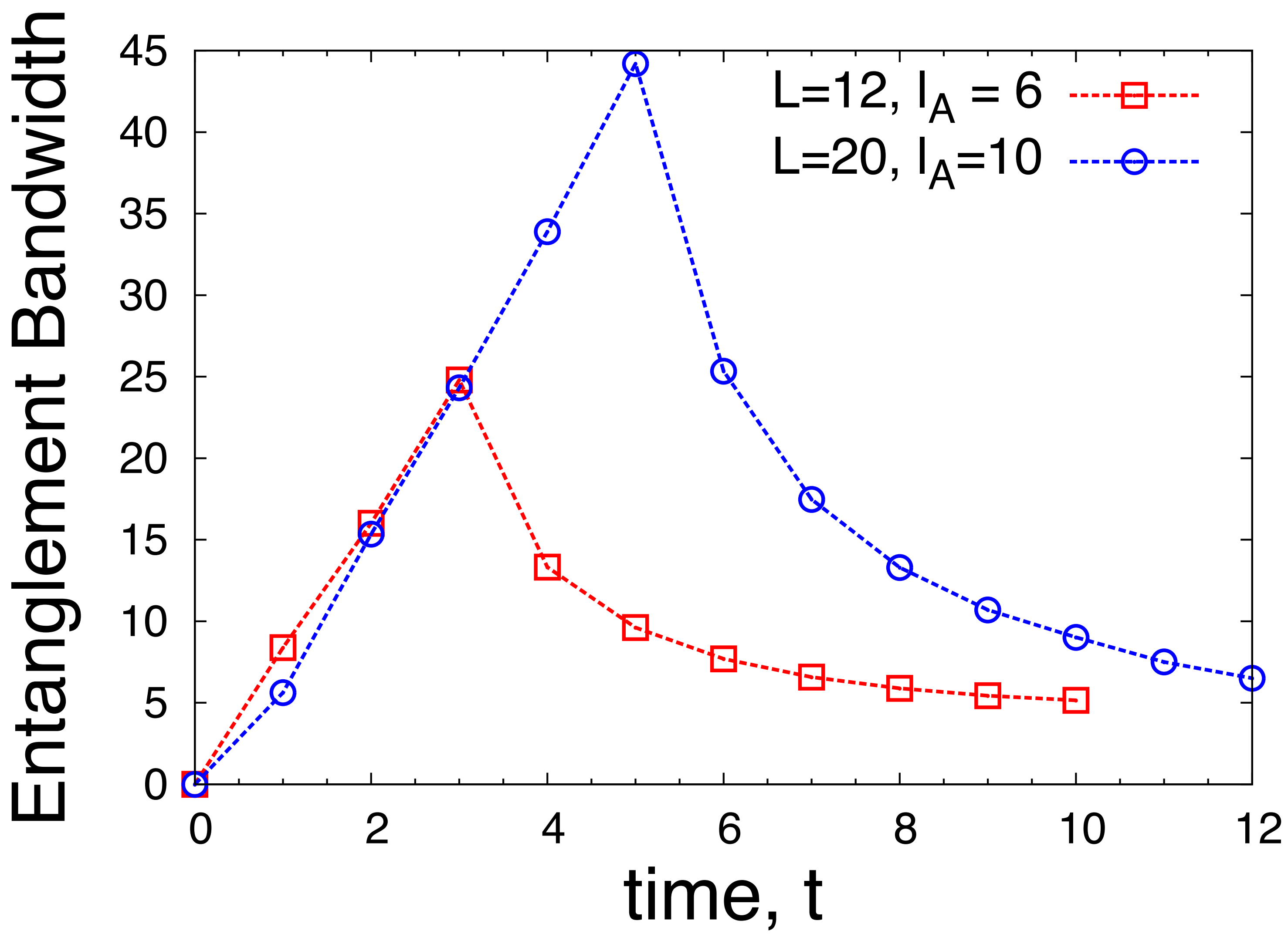
saturating to a value

$$\sim 1/\sqrt{q^{l_A}}$$

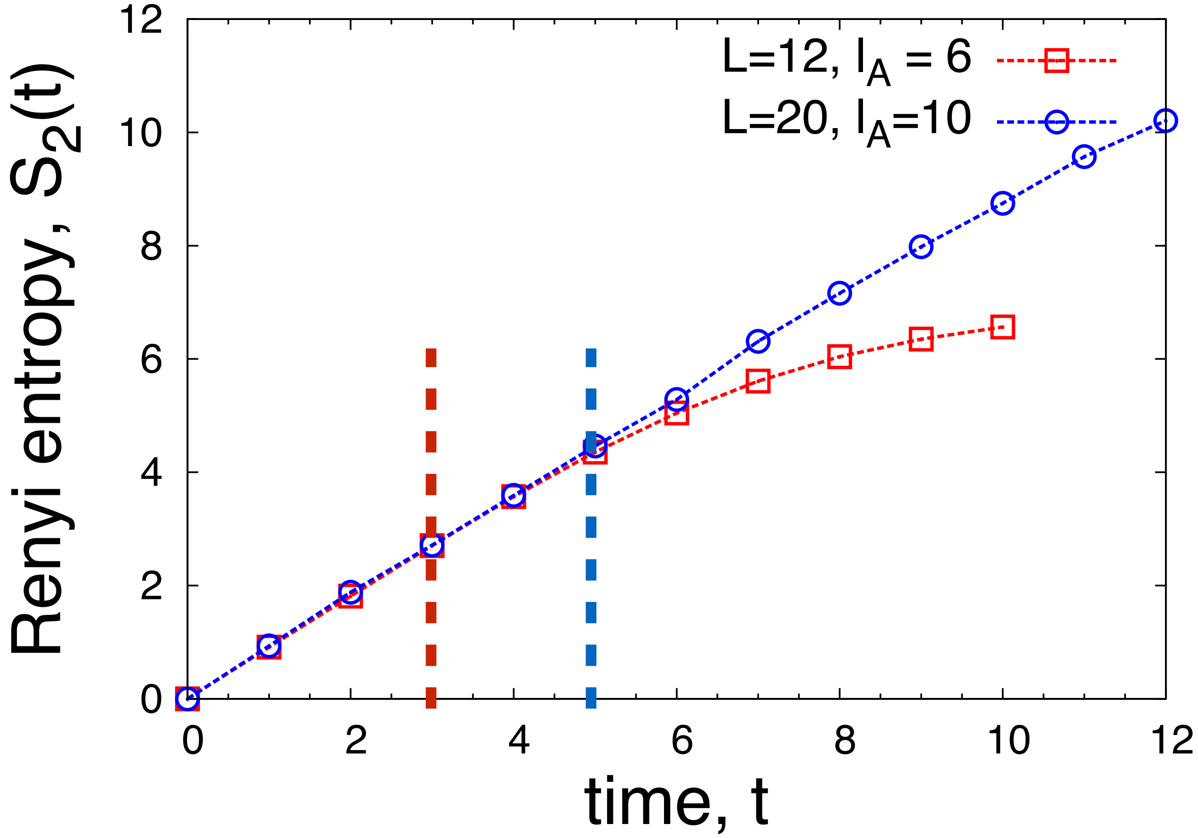
Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

# Entanglement Entropy vs Bandwidth

Growth of the rank of the reduced density matrix captured by the entanglement bandwidth!




This time scale is **invisible** in the **entanglement entropy**




Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

# Scrambling at short times, universality?

At these short times

$$t < l_A / v_{LC}$$

is the distribution of reduced density matrix eigenvalues universal?

Consider the **distribution of reduced density matrix eigenvalues**

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

# Scrambling at short times, universality?

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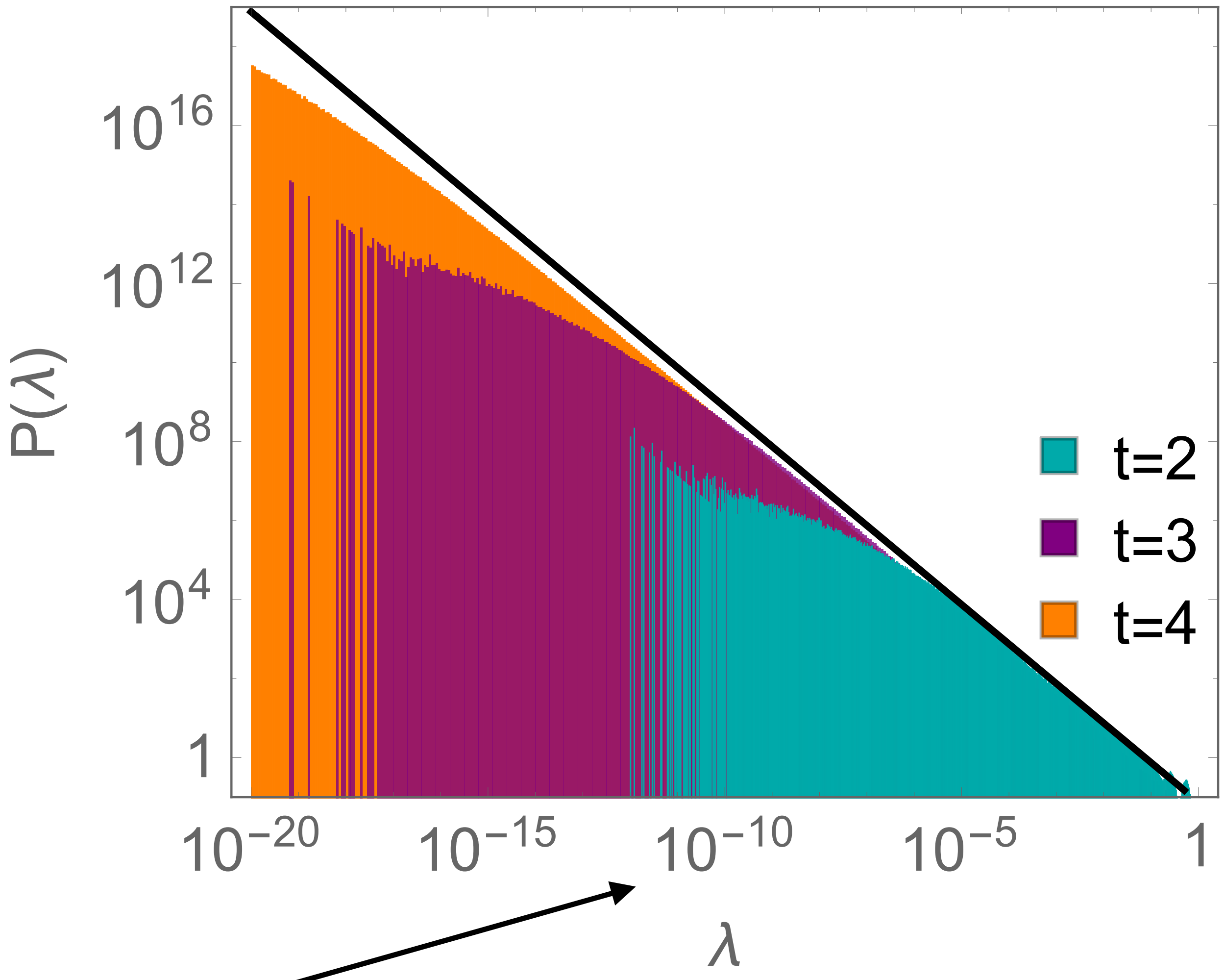
$t < l_A/v_{LC}$
is the distribution of reduced density matrix eigenvalues universal?

We find YES!

Consider the **distribution of reduced density matrix eigenvalues**

**Reduced density matrix DOS**




**several decades** of a scale free RDM eigenvalue distribution $P(\lambda) \sim 1/\lambda$

Demonstrated this for Hamiltonian and Floquet dynamics as well

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

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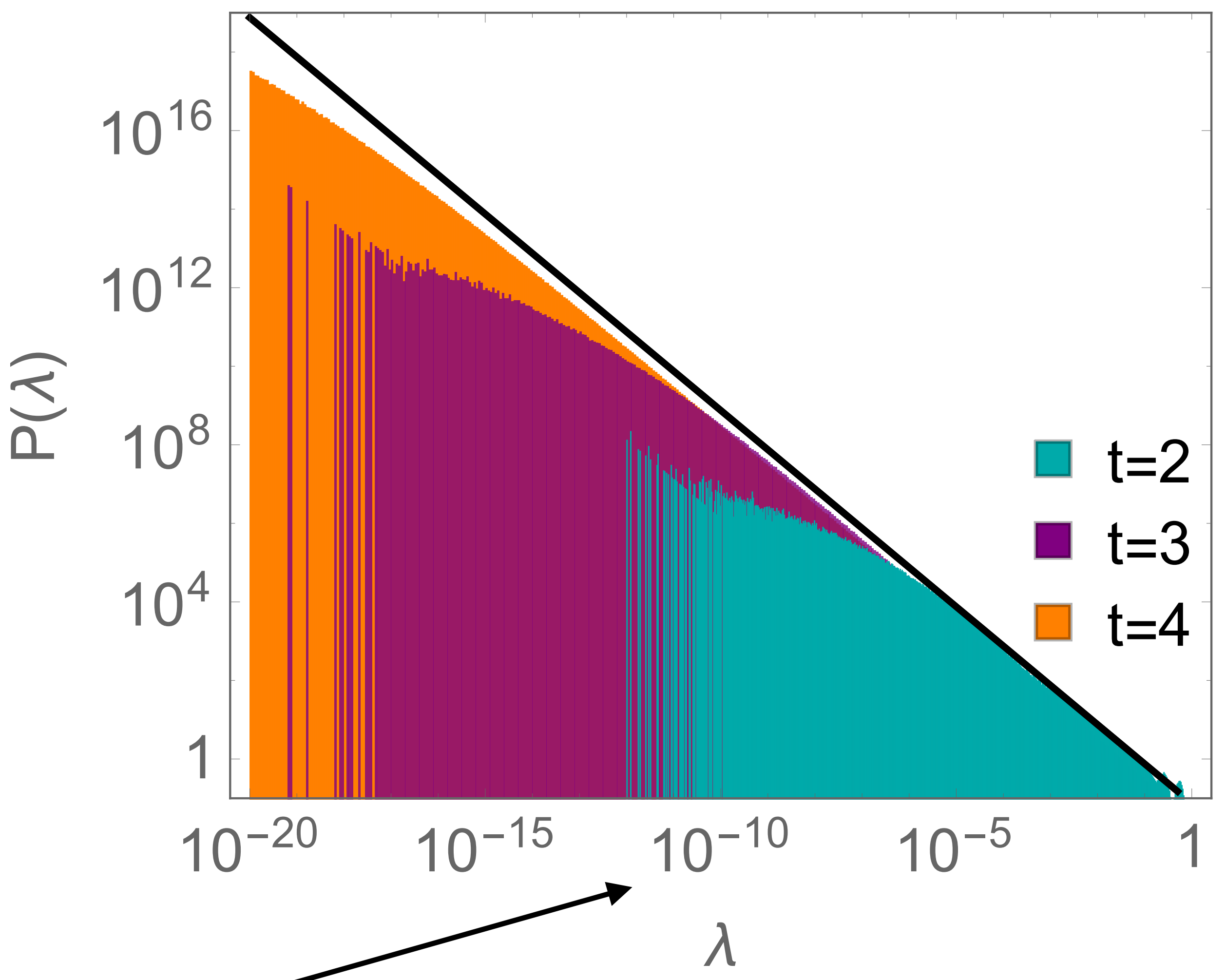


**several decades** of a scale free RDM eigenvalue distribution $P(\lambda) \sim 1/\lambda$

**Distinct** from a random thermal state follows Marchenko-Pastur $P_{MP}(\lambda) \sim 1/\sqrt{\lambda}$

# Summary: Time scales on approach to equilibrium

Relevant time scales:

(1) An O(1) time scale, capturing local thermalization

(2) When the light cone has hit the edge of the sub-system, $t \sim l_A$

$$0 < t < l_A / v_{LC}$$

$$P(\lambda) \sim 1/\lambda$$

(3) Global thermalization time, when the entanglement has saturated $t \sim L$

$$P_{MP}(\lambda) \sim 1/\sqrt{\lambda}$$

Po-Yao Chang, Xiao Chen, Sarang Gopalakrishnan, JHP PRL (2019)

# Outline

I. Lecture series layout
II. Classical chaos
III. Quantum chaos
IV. Quantum thermalization
V. Evading thermalization

# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

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Many body localization (MBL): Strongly disordered or quasiperiodic many body Hamiltonians or Floquet unitaries

$$H = \sum_{i=1}^{L} [h_i \hat{S}_i^z + J \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1}]$$

Pal and Huse (2017)

$$\begin{aligned} H^{\mathrm{QP}/R} = {} & J \sum_{i=1}^{L-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z \sum_{i=1}^{L-1} S_i^z S_{i+1}^z \\ & + \sum_{i=1}^{L} W \cos(2\pi k i + \phi_i^{\mathrm{QP}/R}) S_i^z \\ & + J' \sum_{i=1}^{L-2} (S_i^x S_{i+2}^x + S_i^y S_{i+2}^y), \end{aligned}$$

Khemani, Sheng, Huse (2017)

Setiawan, Deng, Pixley (2017)

Nandkishore and Huse, Annual Review of CMP Physics (2015)

Abanin, Altman,Bloch, Serbyn, Rev Mod Phys (2019)

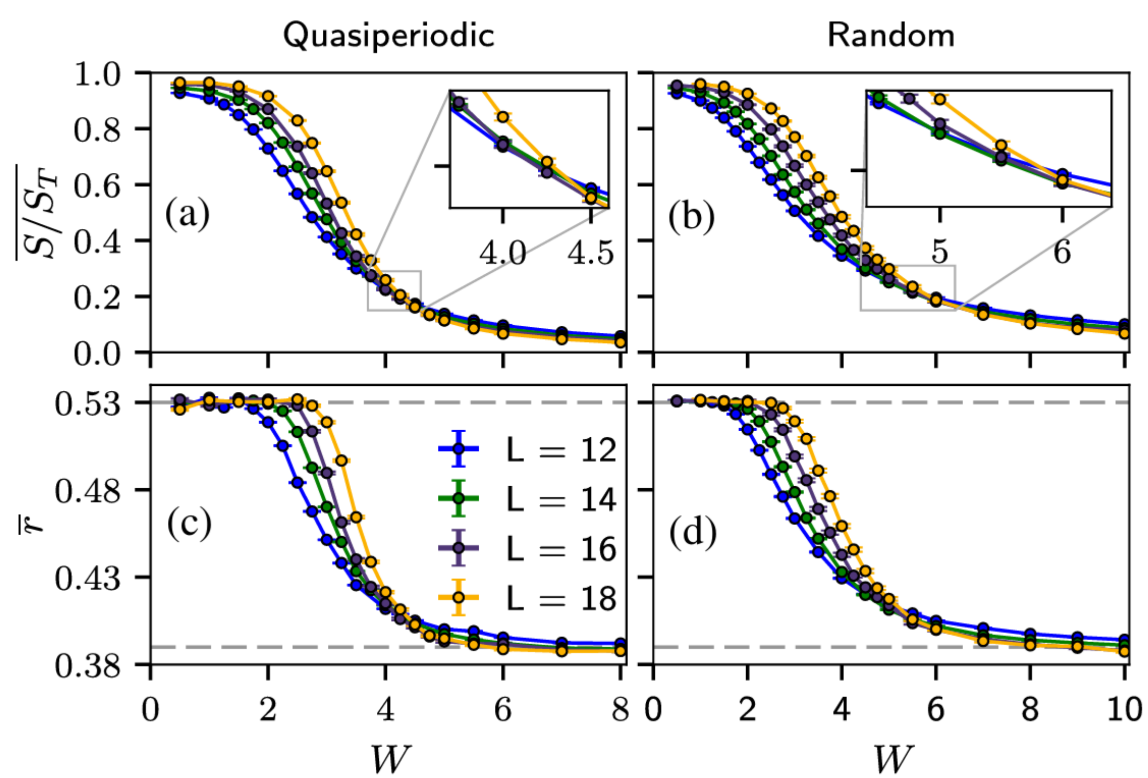
# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

Many body localization (MBL): Strongly disordered or quasiperiodic many body Hamiltonians or Floquet unitaries

Spectral statistics on small sizes have found clear indication of either a transition or cross over in the entanglement and between random matrix theory and Poisson level statistics

Oganesyan and Huse, PRB (2007)




Khemani, Sheng, Huse (2017)

$$\delta_n = E_{n+1} - E_n$$

$$r_n = \min\{\delta_n, \delta_{n-1}\}/\max\{\delta_n, \delta_{n-1}\}$$

Nandkishore and Huse, Annual Review of CMP Physics (2015)

Abanin, Altman,Bloch, Serbyn, Rev Mod Phys (2019)

# How to stop systems from thermalizing?

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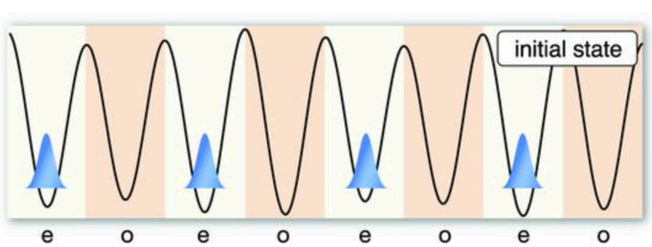
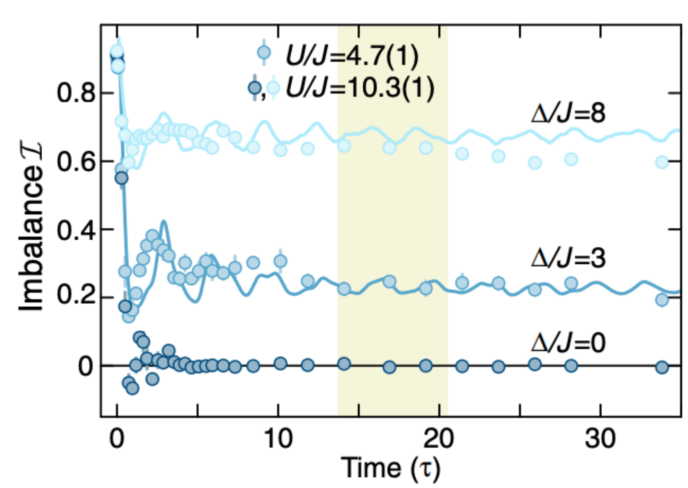
Many body localization (MBL): Strongly disordered or quasiperiodic many body Hamiltonians or Floquet unitaries

Experimental data on ultra cold atoms and trapped ions have found signatures of a transition, but are limited to finite time.
(Will focus on this in some detail in the next lecture, Friday June 5th)

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

$|\Psi_0\rangle =$ 


Schrieber et al, Science (2015)

Choi, et al Science (2016)

Smith, et al Nature Phys (2016)

Bordia, et al PRX (2017)

Guo, et al Nature Phys (2021)

# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

Many body localization (MBL): Strongly disordered or quasiperiodic many body Hamiltonians or Floquet unitaries

Problem still strongly debated. Has been argued that the relaxation time is exponentially large but finite, where as MBL has an infinite relaxation time.

Disordered MBL not stable in dimensions greater then one.

De Roeck and Huveneers, PRB (2017)

Proof of MBL in 1D relies on one tacit assumption, not clear if it is an ironclad proof.

Imbrie, Jour. Stat. Phys. (2016)

Huse, Physics Viewpoint (2016)

# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

Quantum scars: Non-integrable Hamiltonians with a spectrum generating algebra $([H, \hat{Q}^{\dagger}] - \omega\hat{Q}^{\dagger})W = 0,$

Invariant under Q and is a subspace of the full Hilbert space

Starting from some eigenstate $|\psi_0\rangle$

$(\hat{Q}^{\dagger})^n|\psi_0\rangle$ Generates scar states that are **sub-thermal**

With energy $E_0 + n\hbar\omega$

Chandran, Iadecola, Khemani, Moessner, Annual Review of CMP (2022)

Serbyn, Abanin, Papic (2021)

# How to stop systems from thermalizing?

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Quantum scars: Non-integrable Hamiltonians with a spectrum generating algebra $([H, \hat{Q}^\dagger] - \omega \hat{Q}^\dagger) W = 0,$

Known examples

Spectrum-generating algebra

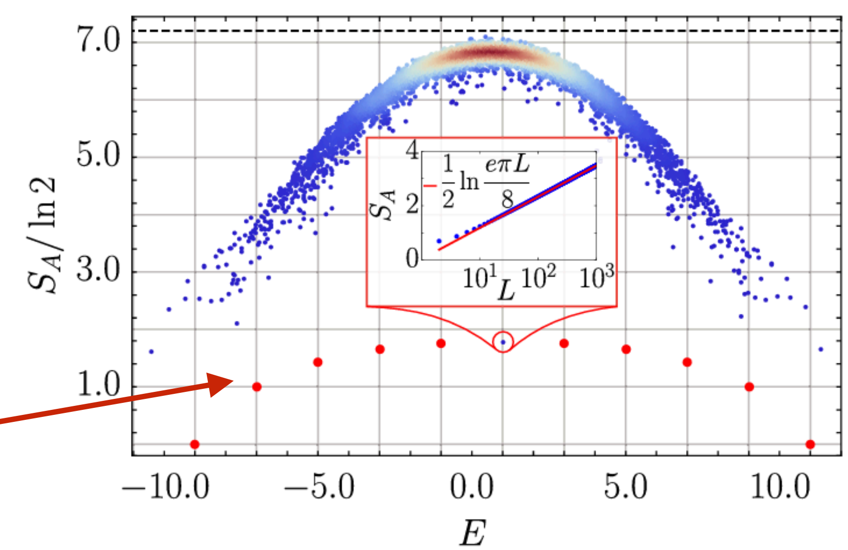
AKLT model, PXP model, spin-1 XY magnets, extended Hubbard model

Has exact scar states

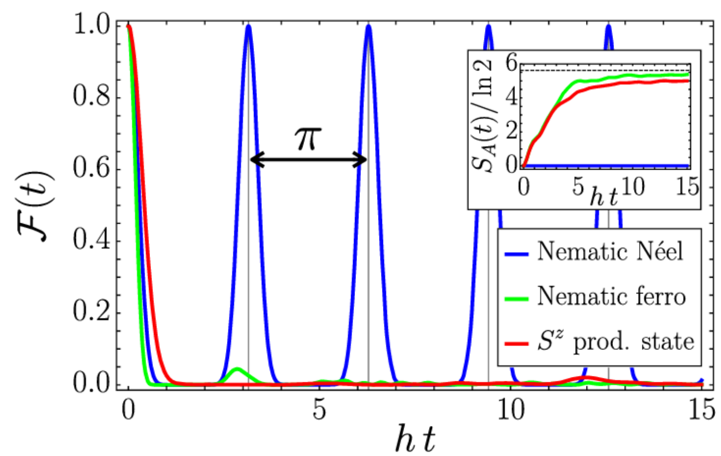
$$H = J\, H_{\mathrm{XY}} + h\, S^z$$
$$= J \sum_r (S^x_r S^x_{r+1} + S^y_r S^y_{r+1}) + h \sum_r S^z_r$$

Schecter and Iadecola, PRL (2019)

Scar states are log entangled subthermal




Many-body revival of the Neel state




Chandran, Iadecola, Khemani, Moessner, Annual Review of CMP (2022)

Serbyn, Abanin, Papic (2021)

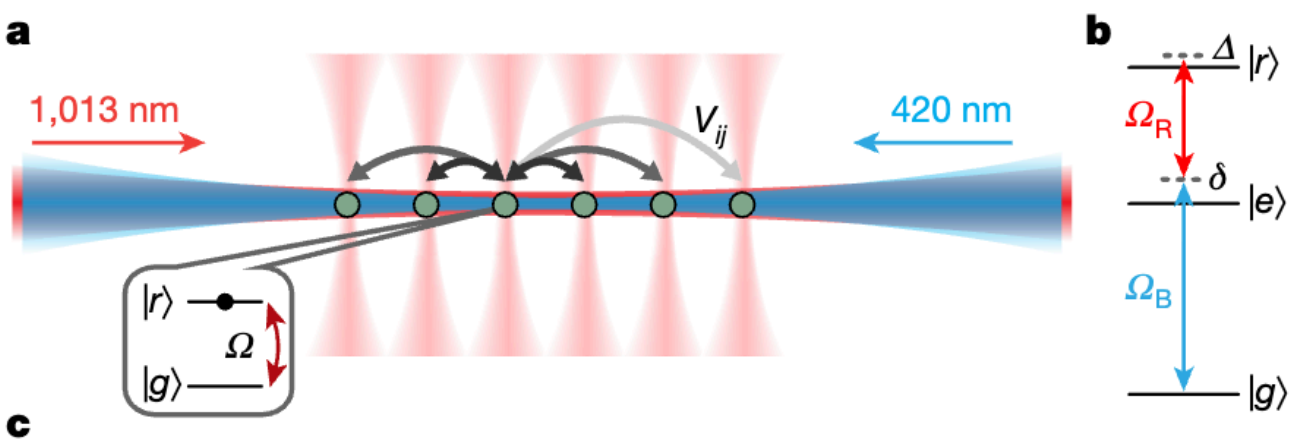
# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

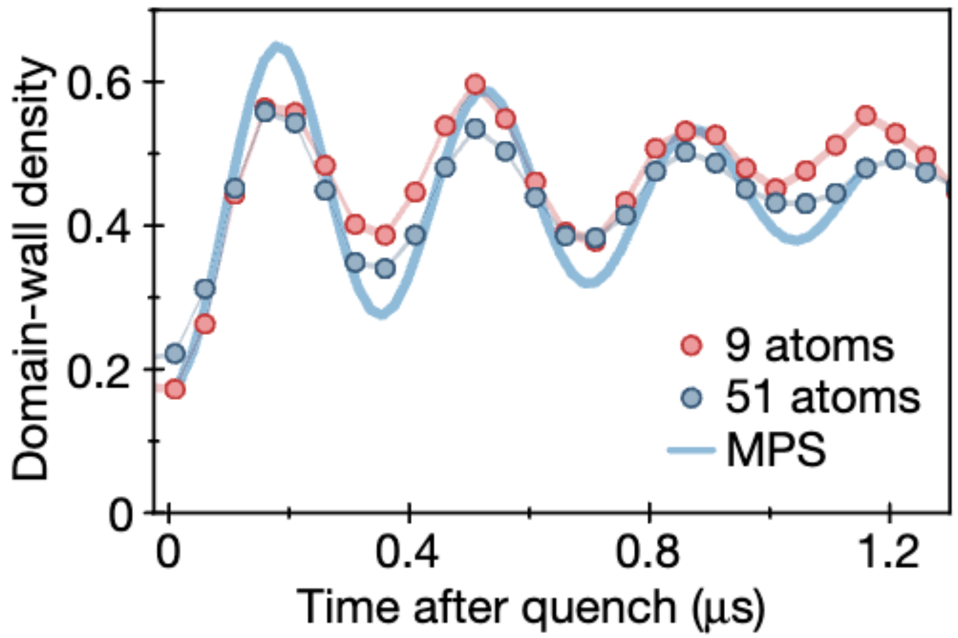
Effects of the scars seen in experiments observed anomalous relaxation in a Rydberg atom array

$$\frac{\mathcal{H}}{\hbar} = \sum_i \frac{\Omega_i}{2}\sigma_x^i - \sum_i \Delta_i n_i + \sum_{i<j} V_{ij} n_i n_j \qquad \sigma_x^i = |g_i\rangle\langle r_i| + |r_i\rangle\langle g_i|$$

Rydberg atom tweezer arrays




Bernien et al Nature (2017)




Chandran, Iadecola, Khemani, Moessner, Annual Review of CMP (2022)

Serbyn, Abanin, Papic (2021)

# How to stop systems from thermalizing?

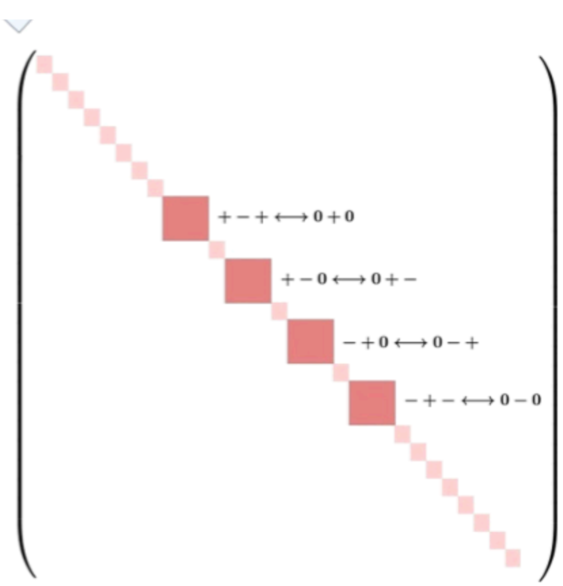
To stop a quantum system from thermalizing we need to remove/reduce the entanglement

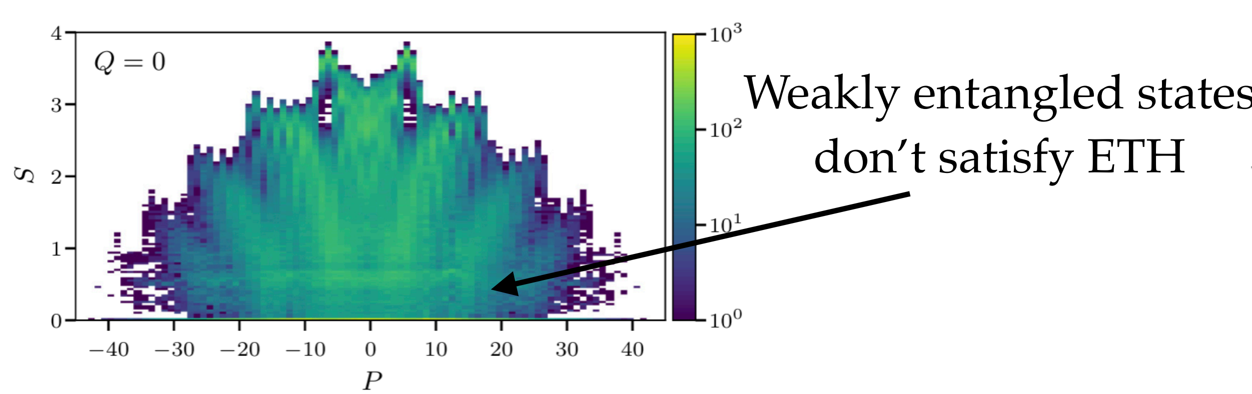
Hilbert space fragmentation: Constraints split the Hilbert space into many disconnected Krylov sectors. Prevents the dynamics from being ergodic.

Can be induced by a finite number of conservation laws

e.g. conserving charge and dipole

$$Q \equiv \sum_n S_n^z \quad \text{and} \quad P_{n_0} \equiv \sum_n (n - n_0) S_n^z$$

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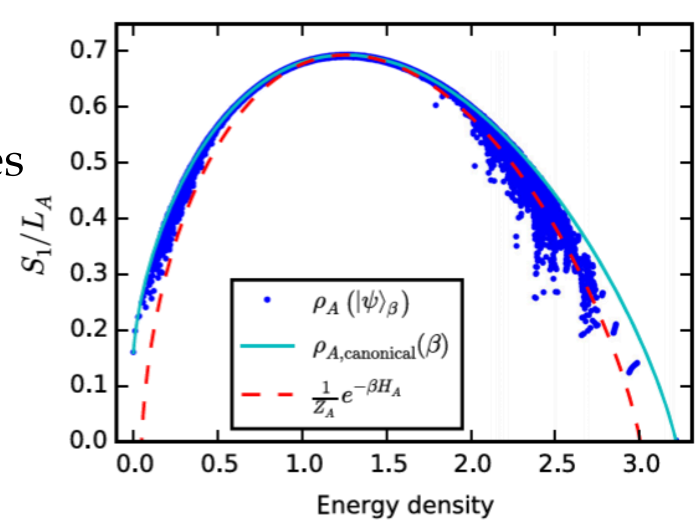


Sala et al PRX (2020) Khemani, Hermele, Nandkishore, PRB Garrison and Grover, PRX (2018)

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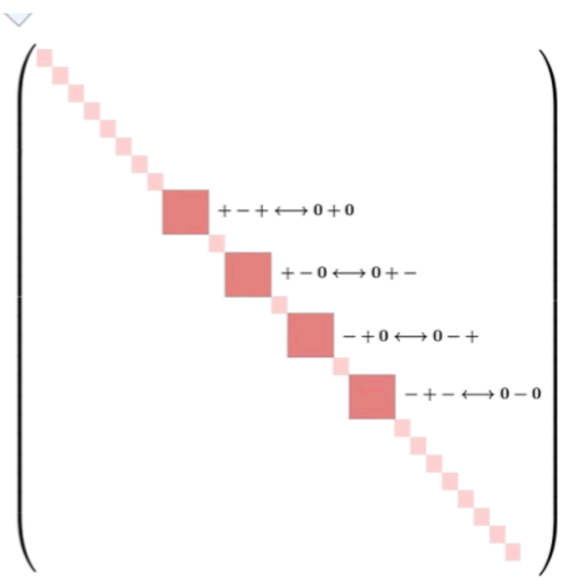


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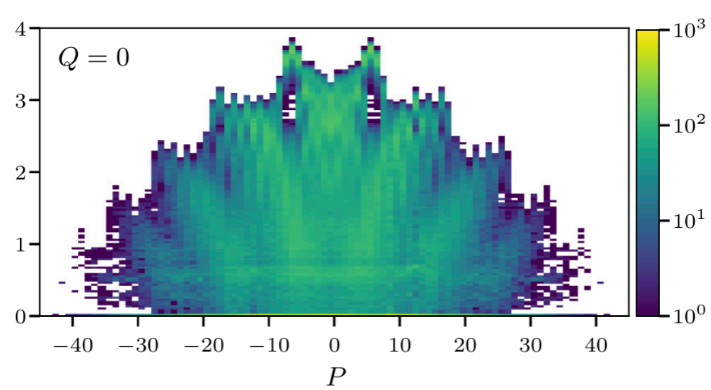


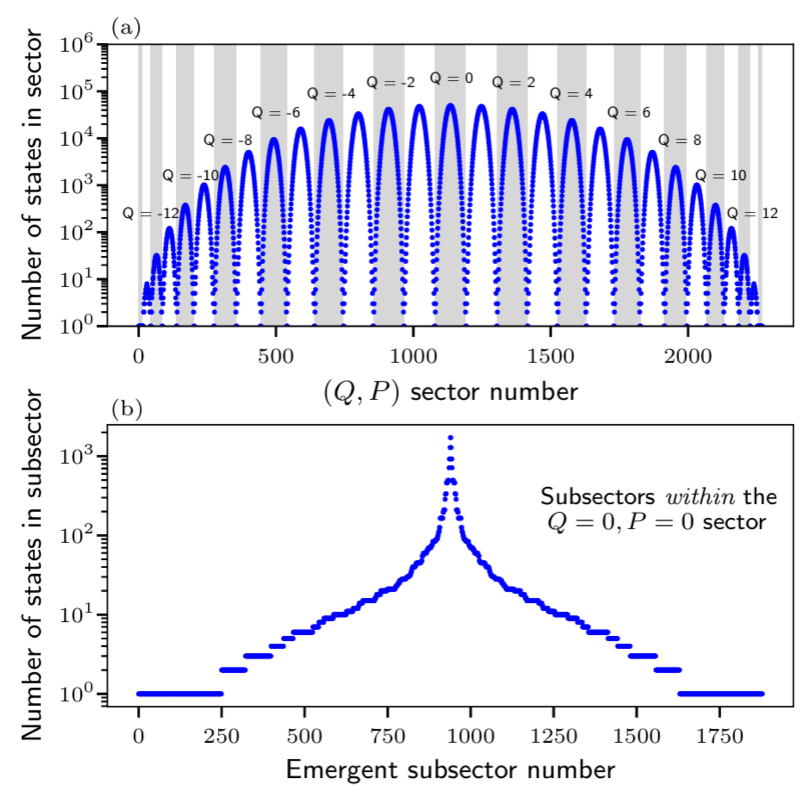


Sala et al PRX (2020) Khemani, Hermele, Nandkishore, PRB

# How to stop systems from thermalizing?

To stop a quantum system from thermalizing we need to remove/reduce the entanglement

Measurement: Strong projective measurements collapse the wave function and remove entanglement. By measuring a particles location we have localized it in a dynamical sense.

Focus of lecture 3 (June 8)

Feedback/feedforward: Based on the outcome of the measurement perform an additional operation on the state

Focus of lecture 4 (June 10)

# Conclusions

- Classical chaos is ubiquitous in nature.
- Quantum chaos of single particle systems remains well defined through random matrix theory.
- Many-body quantum chaos remains a challenging problem to define sharply as it lacks a simple classical limit.
- Thermalization in isolated many-body quantum systems takes place through locally scrambling information.
- Three relevant time scales on the approach to thermal equilibrium.
- Evading thermalization remains a topic of great interest.