Symmetric states and PPT entanglement

Based on the work
https://arxiv.org/abs/2501.04786

Aabhas Gulati (with Ion Nechita and Satvik Singh)

February 6, 2025

Table of contents

Mixed state entanglement

Entanglement in pure quantum states

- Quantum states of bipartite systems $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ satisfy

$$||\psi||^2 = 1$$

- The **separable** (product!) matrices,

$$|\psi\rangle = |v\rangle \otimes |w\rangle$$

- Any vector not of this form is **entangled**!

And in mixed states?

- Quantum states of bipartite systems $\rho \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ satisfy

$$\rho \geq 0 \text{ and } \mathrm{Tr}(\rho) = 1$$

- The **separable** matrices have the following decomposition,

$$\rho = \sum_i p_i \left|v_i\right\rangle \left\langle v_i\right| \otimes \left|w_i\right\rangle \left\langle w_i\right|$$

  where $p_i \geq 0$ and $\sum_i p_i = 1$
- Entangled state := Any state that is not separable.
- Some intuition : Separable matrices are *all* the matrices that are prepared starting with a product state and LOCC.

$$\left|0\right\rangle \otimes \left|0\right\rangle \xrightarrow{\text{LOCC}} \mathrm{SEP}$$

Peres-Horodecki Criterion

The Peres-Horodecki criterion provides a useful but limited tool for detecting entanglement:

- Determining whether a given matrix is separable or entangled is **NP-hard** in general.
- Possibly no general efficiently computable criterion exists.
- The **PPT criterion** :

$$\rho^{T_1} \not\geq 0 \implies \rho \text{ is entangled}$$

- This criterion completely characterizes entanglement for systems where $d_A d_B \leq 6$, but fails in higher dimensions.
- Some entangled states satisfy $\rho^{T_1} \geq 0$, known as **PPT entangled states**.

PPT entanglement

PPT Entangled States

PPT entangled states exhibit unique properties:

- Provide **no advantage** for quantum teleportation.
- Cannot **distill** any entanglement using LOCC.
- Have a **non-zero entanglement cost**—*unlike* separable states.
- Can provide an **advantage in tasks** when used alongside non-PPT entangled states.
- Many open questions remain:

PPT^2 conjecture, NPT distillability, and more.

PPT entangled states: Black Holes in Entanglement Theory

PHYSICS TODAY

LATEST CURRENT ISSUE COLLECTIONS WEBINARS

Volume 78, Issue 1
1 January 2025

Quantum entanglement: A modern perspective

It's not your grandfather's quantum mechanics. Today, researchers treat entanglement as a physical resource: Quantum information can now be measured, mixed, distilled, concentrated, and diluted.

Special Collection: Quantum archive

Barbara M. Terhal; Michael M. Wolf; Andrew C. Doherty

"Black holes" of quantum information

Because the modern theory of entanglement treats quantum states as physical resources for processing information, one might consider them hierarchically. A simple and ideal world would have only two classes of quantum states: unentangled, classically correlated states that are useless as a resource in quantum teleportation and don't violate any Bell inequalities, and entangled states whose distillation rate D measures their usefulness in quantum teleportation. If the distillation rate D is nonzero, one can distill from such states some EPR pairs, known to violate Bell inequalities.

Bound entanglement tells us that life is not so simple. Bound entangled states are costly ($E > 0$), but useless in various quantum-information-processing protocols like teleportation. Furthermore, there is evidence that bound entangled states do not violate any Bell inequalities.

In those two senses, bound entangled states are the "black holes" of quantum information theory. Entanglement goes in but is impossible to recover. And like black holes in the theory of gravitation, bound entangled states test the limits of our understanding and puzzle us by their intrinsic irreversibility.

How to construct PPT entangled states?

- Realignment criterion detects some PPT entangled states.

$$\rho \text{ is separable} \implies ||\rho^{\mathrm{Re}}||_1 \leq 1$$

- Need a framework to systematically identify such states - we look at symmetric states.
- Symmetries are physically interesting.
- Leads to reduction in number of parameters, provides better tests for entanglement.

Locally symmetric states

Local symmetries

- A bipartite state is locally G-symmetric if

$$(U \otimes U)\rho(U \otimes U)^\dagger = \rho \text{ for all } U \in G \subseteq U(d)$$

  or locally (conj.) G-symmetric if

$$(U \otimes \bar{U})\rho(U \otimes \bar{U})^\dagger = \rho \text{ for all } U \in G \subseteq U(d)$$

- For the group $G = \mathrm{U}(d)$, we get the well-known Werner states, or the isotropic states (in the conj. case)
- For Werner (iso) states, there is no PPT entanglement.

A new class of symmetric states : Local Cyclic Sign Invariance

Cyclic Sign Symmetries

We consider states that exhibit the following symmetries:

- **Diagonal Orthogonal Group:** The state ρ remains invariant under the transformation:

$$(O \otimes O)\rho(O \otimes O) = \rho,$$

  where O is an orthogonal matrix of the form:

$$O = \begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{bmatrix}.$$

- **Cyclic Group $\mathbb{Z}_d$:** The state is **locally invariant** under cyclic level permutations:

$$|ij\rangle \langle kl| \to |i \oplus q, j \oplus q\rangle \langle k \oplus q, l \oplus q| \quad \forall q \in [d].$$

Properties of Local Cyclic Sign Invariant States

The states are parametrized by three vectors:

$$(a, b, c) \in (\mathbb{C}^d)^{\times 3}, \quad \text{with } a_0 = b_0 = c_0$$

This results in $3d - 2$ parameters.

- Parametrization

$$\rho_{(a,b,c)} = \sum_{j,k} a_k |j \oplus k, j\rangle\langle j \oplus k, j| + \sum_{j,k \geq 1} b_k |j \oplus k, j \oplus k\rangle\langle j, j| + \sum_{j,k \geq 1} c_k |j \oplus k, j\rangle\langle j, j \oplus k|$$

- Positivity Condition: [1]

$$\rho_{a,b,c} \geq 0 \iff a \geq 0, \quad \mathcal{F}b \geq 0, \quad \bar{c} = c^{\mathrm{R}} \quad \forall i; a_i a_{d-i} \geq |c_i|^2$$

- Normalization Condition:

$$\mathrm{Tr}(\rho) = 1 \iff \sum_i a_i = \frac{1}{d}$$

[1] $(x_i^{\mathrm{R}}) = x_{d-i}$ and $\mathcal{F}b$ is the discrete fourier transform

PPT and Separability

- $\rho_{a,b,c}$ is PPT $\iff a \geq 0, \mathcal{F}b \geq 0, \mathcal{F}c \geq 0 \quad \forall i; a_i a_{d-i} \geq |b_i|^2, |c_i|^2$
- $\rho_{a,b,c}$ is a separable state if and only if [2]

$$a = \sum_k (v_k \odot \bar{v_k}) * (w_k \odot \bar{w_k})^{\mathrm{R}}$$

$$b = \sum_k (\bar{v_k} \odot w_k) * (\bar{v_k} \odot w_k)^{\mathrm{R}}$$

$$c = \sum_k (\bar{v_k} \odot \bar{w_k}) * (v_k \odot w_k)^{\mathrm{R}}$$

- Deciding separability is still non-trivial

[2] $(x_i^{\mathrm{R}}) = x_{d-i}$ and $\odot$ is the entrywise product of vectors

Simple Constructions of PPT Entangled States

We construct PPT entangled states using the following conditions:

- Define the vector $b = (1, \omega, \omega^2, \dots, \omega^{d-1})$, where ω is the d-th root of unity. Choose c such that $\mathcal{F}c \geq 0$ and $c_0 = 1$.
- The state ρ is **PPT** if the vector a satisfies:

$$\forall i, \quad a_i a_{d-i} \geq 1.$$

- The state is **separable** if and only if:

$$\forall i, \quad a_i \geq 1.$$

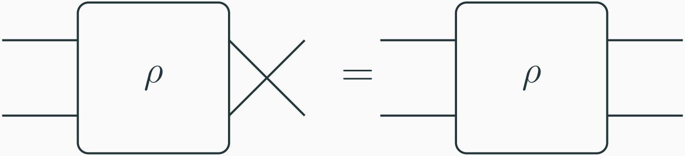
- Choosing:

$$a_1 = \frac{1}{\mu}, \quad a_{d-1} = \mu, \quad \mu \neq 1 \quad \text{and } a_k = 1 \text{ for all other } k$$

  results in a **PPT entangled state**.

(Cyclic) Mixtures of Dicke States

We consider states with the following additional symmetry:



- In this case, $a = c$ and $b \sim |0\rangle = (1, 0, \ldots 0)$.
- The parametrized state is given by:

$$\rho_a = \sum_{i,k} a_k \left|D_{i,k}\right\rangle \left\langle D_{i,k}\right|$$

where

$$\left|D_{i,k}\right\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |i \oplus k\rangle), \quad \left|D_{i,0}\right\rangle = |i\rangle \, .$$

Mixtures of Dicke States (cont)

The state ρ_a satisfies the following conditions:

- ρ_a is PPT if and only if

$$a \geq 0 \quad \text{and} \quad \mathcal{F}a \geq 0.$$

  This defines a polyhedral cone.

- ρ_a is separable if and only if

$$a = \sum_k |v_k\rangle * |v_k\rangle^{\mathrm{R}}.$$

- If the local dimension $d \leq 4$, then:

$$\rho_a \text{ is PPT} \iff \rho_a \text{ is SEP.}$$

The system is now described by $\lfloor d/2 \rfloor + 1$ parameters.

$\lfloor 5/2 \rfloor + 1 = 3$ parameters, normalising $a_0 = 2$.

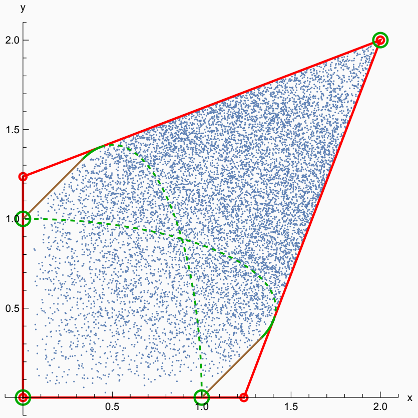


Figure 1: The PPT and SEP states in the slice $a = (2, x, y, y, x)$ in $5 \otimes 5$ cyclic mixtures of Dicke states

Beyond local dimension 5

$\lfloor 7/2 \rfloor + 1 = 4$ parameters, normalising $a_0 = 1$.

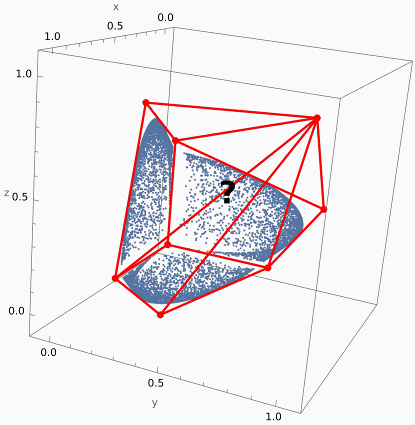


Figure 2: PPT and SEP states in the $7 \otimes 7$ cyclic mixtures of Dicke states, bulk of geometry is still an open problem

A Review of symmetric states

Group	Inv. Q. States	Dim. Inv.	Abelian	$\exists$ PPT ent.
$\{\mathrm{id}\}$	all states	d^4	Y	Y
$\mathcal{DO}(d)$	LDOI	$3d^2 - 2d$	Y	Y
$\mathcal{DU}(d)$	(C)LDUI	$2d^2 - d$	Y	Y
Cyc(d) $\ltimes \mathcal{DO}(d)$	LCSI	$3d - 2$	N	Y
Sym(d) $\ltimes \mathcal{DO}(d)$	hyperoctahedral	4	N	N
$\mathcal{O}(d)$	Brauer	3	N	N
$\mathcal{U}(d)$	Werner (UU) isotropic ($U\bar{U}$)	2	N	N

Conclusions

- Introduced another class of symmetric quantum states, with $3d-2$ parameters. The separability problem in mixed states is still non-trivial, PPT $\neq$ SEP.
- Some systematic constructions of PPT entangled states. New results for mixtures of Dicke States in small dimensions.
- Can act as a testbed for important conjectures surrounding PPT entangled states.
- Important question - What *minimal* local symmetry in $d \geq 3$ that leads to the collapse of PPT = SEP?