Lattice Study of the Thermal Potential and Its Corrections at Nonzero Density

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The Deconfined Media

The QCD partition function:

$$Z_{\rm QCD}(T,V) = {\rm Tr}(e^{-\frac{H_{\rm QCD}}{T}})$$

Thermodynamic quantities:

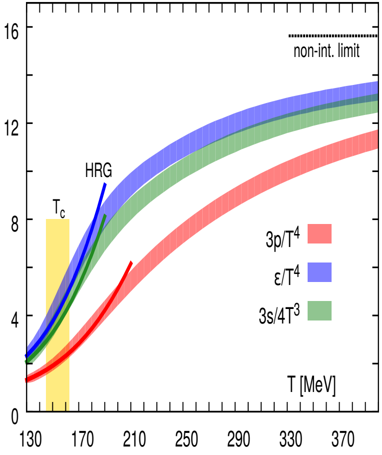
$$P = T\frac{\partial \log Z}{\partial V}, \quad E = T^2\frac{\partial \log Z}{\partial T}$$

At low T: Gas of Hadrons and resonances

At very high T: Quasi Free gas of Quarks and Gluons

$$\epsilon(T_{pc}) \sim 1 \text{ GeV/fm}^3$$

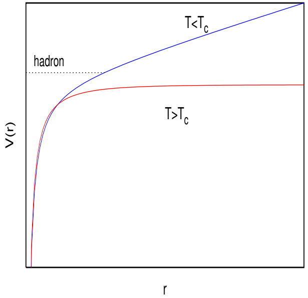
QGP is non-perturbative.
Need Lattice Calculation.

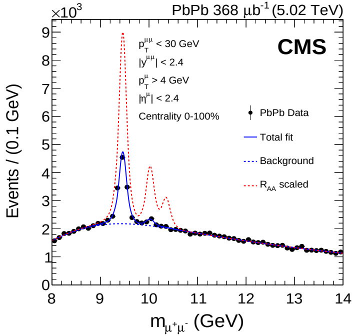


PRD 90, 094503(HotQCD Collaboration)

Understanding this phase

- **Modification of in-medium quarkonia**
  - Color screening





CMS Collaboration, PLB 790 (2019) 270)

- **Thermal photons and dileptons**
- **Jets**
- **Screening masses**
- **Many other observables**

Content

How can one extract in-medium potentials from lattice QCD?

Static Potentials

- Singlet potential
- Quarkonia spectral function.
- Octet potential
- Finite density corrections to the thermal potential
  (Work in progress with Jishnu Goswami, Olaf Kaczmarek)

Important additional corrections due to spin will be discussed in Swagatam's talk.

Correlation function

$$O(r,x;t) = \bar{\psi}\left(x-\frac{r}{2},t\right)\,\Gamma\,U\left(x-\frac{r}{2},x+\frac{r}{2}\right)\,\psi\left(x+\frac{r}{2},t\right)$$

$$U(x_1,x_2) = P\exp\left(ig\int_{x_1}^{x_2} A_\mu(x)\,dx^\mu\right)$$

$$C_\Gamma^{>}(r,r',\vec{k},t) = \int d^3\vec{x}\,e^{i\vec{k}\cdot\vec{x}}\sum_n \frac{e^{-\beta E_n}}{Z_{QCD}}\,\langle n|O(r,x;t)\,O(r',0;0)|n\rangle$$

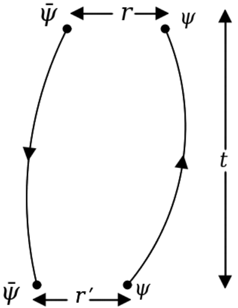
Γ specifies the quantum channel:

$\Gamma = \gamma_\mu \quad \Rightarrow$ vector channel $(J/\psi,\ \Upsilon)$,

$\Gamma = \gamma_5 \quad \Rightarrow$ pseudoscalar channel $(\eta_c,\ \eta_b)$.

It can also specify the color channel:

$$\Gamma = \{\,1\ \text{(singlet)},\ \ T^a\ \text{(octet)}\,\}.$$



Non-relativsitc expansion

Heavy quark mass $M \gg T, \Lambda_{QCD}$,

$$\mathcal{L}_{\rm QCD}^{Q\bar{Q}} = \theta^\dagger \left(iD_0 - M + \frac{c_2 \mathrm{D}^2 + c_B\, \boldsymbol{\sigma}\cdot g\mathrm{B}}{2M} \right) \theta + \phi^\dagger \left(iD_0 + M - \frac{c_2 \mathrm{D}^2 + c_B\, \boldsymbol{\sigma}\cdot g\mathrm{B}}{2M} \right) \phi$$

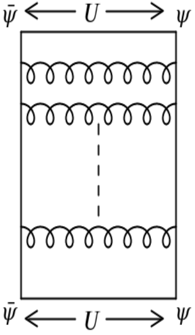
In the static limit,

$$\lim_{M\to\infty} C_>(r,t) = \lim_{M\to\infty} \exp(-2iMt) W_T(r,t)$$

At large time t,

$$\lim_{t\to\infty} W_T(r,t) \sim e^{-iV(r)t}$$

$$V(r) = i \lim_{t\to\infty} \frac{\partial \log W(r,t)}{\partial t}$$



M.Laine et al, JHEP 0703:054

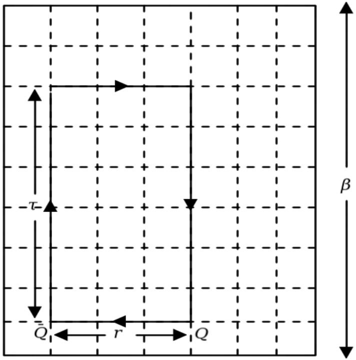
On the lattice

$$W_E(r,\tau) = \int_{-\infty}^{\infty} d\omega\, \rho(r,\omega)\, e^{-\omega\tau}$$

$$W(r,t) = \int_{-\infty}^{\infty} d\omega\, \rho(r,\omega)\, e^{-i\omega t}$$

Numerically Ill-posed Problem:

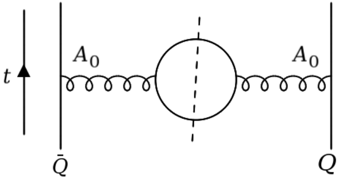
- Difference in the number of degrees of freedom.
- Small errors in W^E can lead to large errors in ρ.
- Requires further physics information from e.g pQCD, EFT



$$W_E(r,\tau) \overset{\text{danger}}{\rightarrow} W_E(r,\tau \rightarrow it)$$

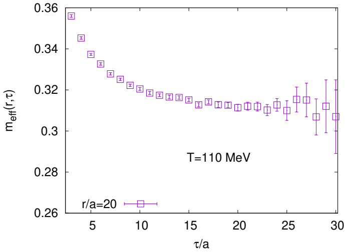
- From first principle at large τ: $W(r,\tau) \sim \exp(-V\tau)$
- $i \lim_{t\to\infty} \frac{\partial \log W^M(r,t)}{\partial t}$ exists trivially,
- In LO at finite T,

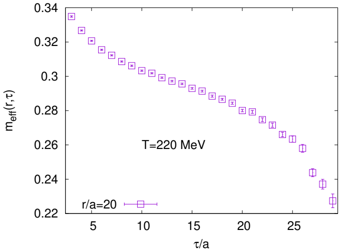
$$V(r) = i \lim_{t\to\infty} \frac{\partial}{\partial t} \log W(r,\tau \to it)$$
$$= V_{re}(r) - i\, V_{im}(r)$$



V_{re}=Screening in the plasma
V_{im}=Landau damping

$$a\, m_{\rm eff}(r,\tau) = \log\left(\frac{W(r,\tau)}{W(r,\tau+a)}\right)$$





- Leading order results:

$$\log W(r,\tau) = g^2 \left[\tau V_{re}(r) + \int_{-\infty}^{\infty} \sigma(q_0, r) \left(e^{\tau q_0} + e^{(\beta-\tau)q_0} \right) dq_0 \right],$$

M. Laine *et al.*, JHEP 03 (2007) 054

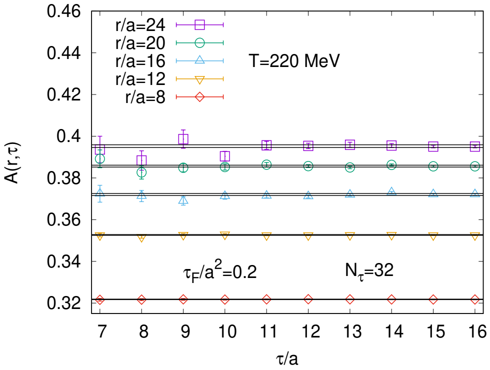
$$\sigma(q_0) \sim \frac{1}{q_0^2} \qquad \Rightarrow \qquad V(r) = i \lim_{t\to\infty} \frac{\partial \log W(r,it)}{\partial t} = V_{re}(r) - iV_{im}(r).$$

$$\frac{W(r,\tau)}{W(r,\beta-\tau)} = \exp(-V_{re}(2\tau-\beta))$$

$$A(r,\tau) = -\frac{1}{2}\frac{\partial}{\partial\tau} \log\left(\frac{W_E(r,\tau)}{W_E(r,\beta-\tau)} \right)$$

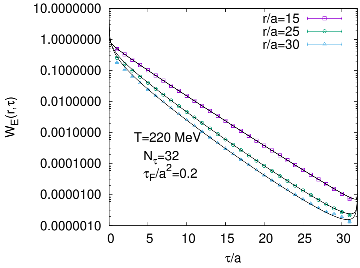
DB and S.Datta, PRD 101, 034507

DB, O. Kaczmarek et al, PRD 112, 054510



$$\log W(r,\tau) = -V_{\text{re}}(r)\tau - \int_{-\infty}^{\infty} du\, \sigma(r,u)\left[e^{u\tau} + e^{u(\beta-\tau)}\right] + \dots$$

- $i \lim_{t\to\infty} \frac{\partial \log W(r,t)}{\partial t}$ finite
  $\Rightarrow \lim_{u\to 0} \sigma(r,u) \sim \frac{1}{u^2}$
- $\sigma(r,u) =$
  $n_B(u)\left[\frac{V_{\text{im}}}{u} + c_1 u + c_3 u^3 + \dots\right]$

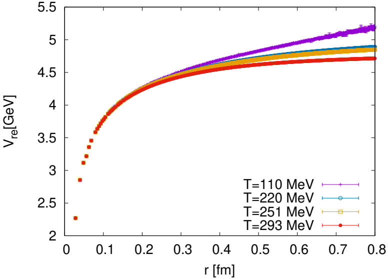


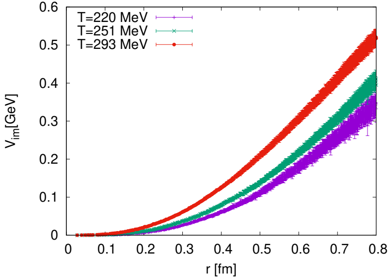
$$W(r,\tau) = A \exp\left[-V_{\text{re}}(r)\tau - \frac{\beta V_{\text{im}}(r)}{\pi}\log\left(\sin\left(\frac{\pi\tau}{\beta}\right)\right) + \dots\right]$$

$\chi^2/\text{ndf} \sim 1$ for all distances

D. Bala and S. Datta, PRD 101, 034507
DB, O. Kaczmarek et al, PRD 112, 054510

Thermal Potential

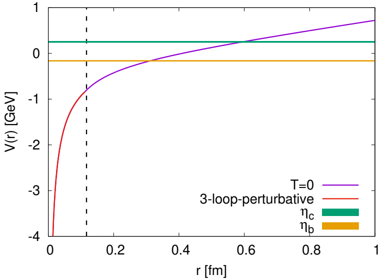


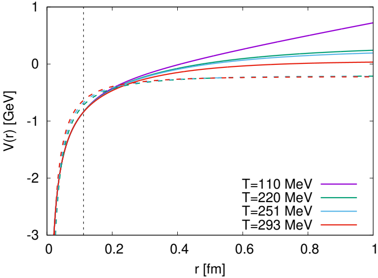


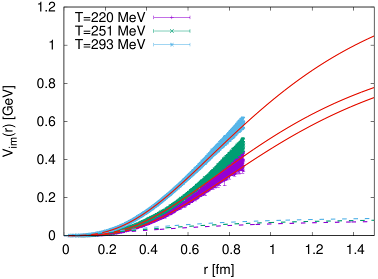
- $2+1$ flavor dynamical lattice, $m_\pi = 320\,\text{MeV}$.
- Lattice spacing: $a = 0.028\,\text{fm}$.
- Temporal extent: $N_\tau =$ $24\,(293\,\text{MeV}),\ 28\,(251\,\text{MeV}),$ $32\ (220\ \text{MeV})$

- Color screening of potential.
- Large imaginary part.

DB, O. Kaczmarek et al, PRD 112, 054510







- Matching short distance part with perturbative vacuum potential.

$$V(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{\alpha_V(q^2)}{q^2} \exp(i\vec{q}.\vec{r})$$

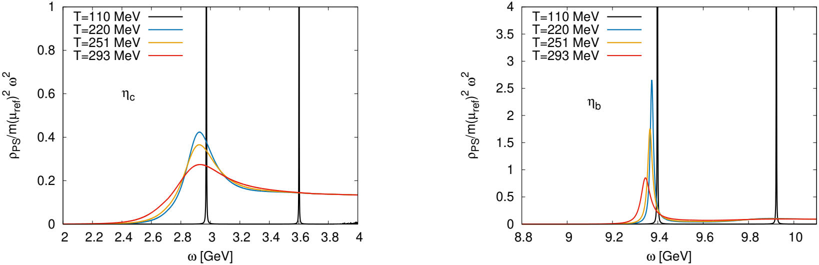
DB, O. Kaczmarek et al, PRD 112, 054510

Quarkonia Spectral function

$$\left\{ i\partial_t - \left[2M + V_T(r) - \frac{\nabla^2_{\vec{r}}}{M} \right] \right\} C_>(t; \vec{r}, \vec{r}\,') = 0$$

$$C_>(t; \vec{r}, \vec{r}\,') \sim \delta^3(\vec{r} - \vec{r}\,')$$

$$\rho_\Gamma(\omega, k) = \lim_{r\to 0,\, r'\to 0} \mathrm{FT}\left[C_>(t; \vec{r}, \vec{r}\,') \right]$$

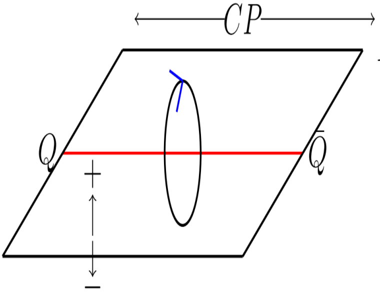


*** Significant thermal effects for charmonium.**
*** Charmonium melts around $T \sim 300\,\mathrm{MeV}$.**

DB, O. Kaczmarek et al., Phys. Rev. D 112 (2025) 054510

Hybrid Potential

$$|\phi_n^{\mathrm{H}}(\vec{r},t)\rangle = \underbrace{\bar{\psi}(-\vec{r}/2,t)\,U(-\vec{r}/2,\vec{0})\,H\,U(\vec{0},\vec{r}/2)\,\psi(\vec{r}/2,t)}_{O_H(r,t)}|n\rangle$$



$$L = 0, 1, 2\,(\Sigma, \Pi, \Delta)$$

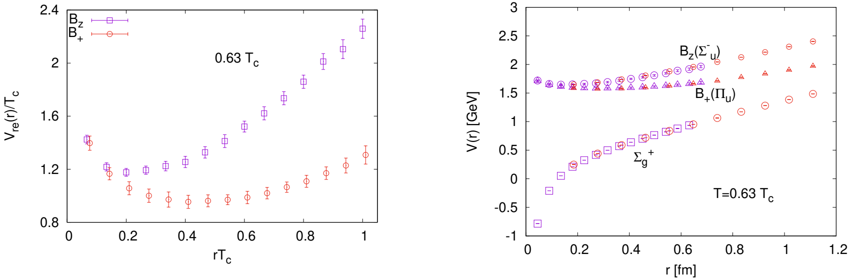
$$CP = +, -\,(g, u)$$

$$P_x = +, -$$

$H = 1 \quad (\Sigma_g^+)$ — Standard Cornell Potential

$H = B_z \quad (\Sigma_u^-)$
$H = B_+ = B_x + iB_y \quad (\Pi_u)$ } Hybrid Potential

Hybrid Potential: Zero temperature



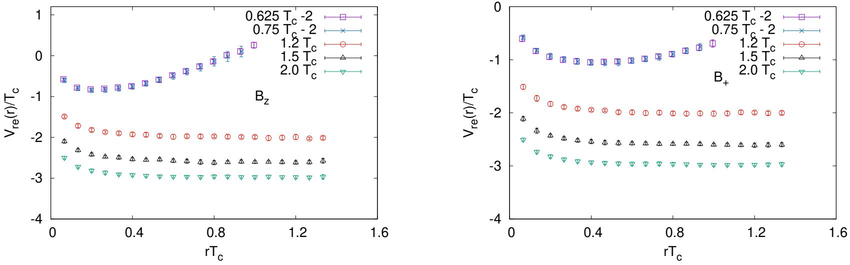
$$V_H(r) = \underbrace{\frac{1}{6}\,\frac{g^2}{4\pi r}}_{V_o} \qquad \text{as } r \to 0$$

Octet information is possible only at short distance.

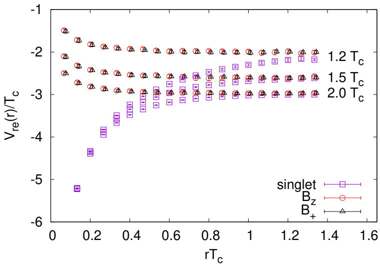
DB and S. Datta, PRD 103, 014512

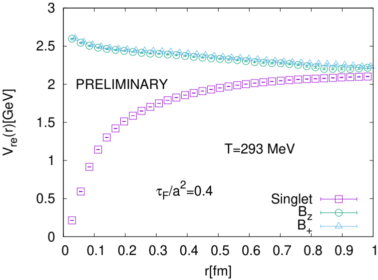
S. Capitani et al, PRD 99, 034502

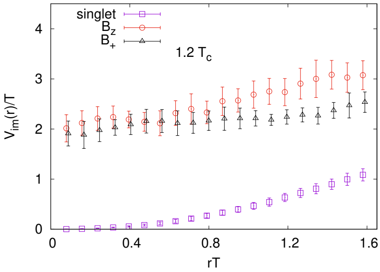
Finite temperature

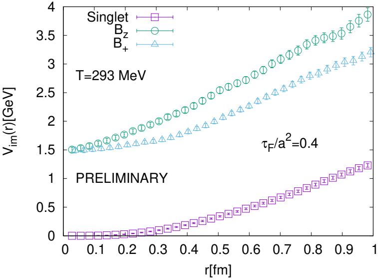


- Both channels does not have any attractive part.









$N_f = 0$

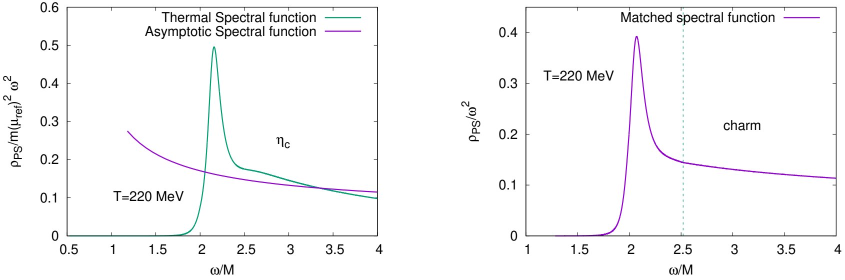
DB and S. Datta, PRD 103, 014512

$N_f = 2 + 1$

(work in progress)

$$\left\{ i\,\partial_t - \left[2M + V_T(r) - \frac{\nabla^2_{\vec{r}}}{M} \right] \right\} C_>\left(t;\vec{r},\vec{r'}\right) = 0$$

$$\rho_{PS}(\omega) = A_0\,\rho^T_{PS}(\omega)\theta(\omega_0-\omega) + \rho^{T=0}_{PS}(\omega)\theta(\omega-\omega_0)$$



- Matching at $T = 220 MeV$
- $A_0 \sim 1.2 \qquad \omega_0 \sim 2.7\,M$

Similar spectral function using perturbative potential.

Moving towards finite density

At nonzero baryon chemical potential, the QCD partition function is

$$Z = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\; e^{-S_g - S_F} = \int \mathcal{D}U\; \det(D(\mu) + m)\, e^{-S_g}.$$

For $\mu \neq 0$, the Dirac operator is

$$M \equiv D(\mu) + m, \qquad \det M \in \mathbb{C}.$$

- The fermion determinant becomes complex.
- Therefore, the Boltzmann weight is no longer positive definite.
- Standard Monte Carlo importance sampling is not applicable.

Taylor expansion of the Wilson line correlator

We perform a similar Taylor expansion for the Wilson line correlator,

$$W(r,\tau,\mu) = \frac{\displaystyle\int \mathcal{D}U\; W(r,\tau)\; \det M(\mu)\, e^{-S_g}}{\displaystyle\int \mathcal{D}U\; \det D(\mu)\, e^{-S_g}}.$$

Expanding around $\mu = 0$, and keeping terms up to second order, we obtain

$$W(r,\tau,\mu) = W(r,\tau,0)\left[1 + \frac{1}{2}\,\hat{\mu}^T\, W_2(r,\tau)\,\hat{\mu} + \mathcal{O}(\hat{\mu}^4)\right].$$

Here $W_2(r,\tau)$ is the second-order response matrix, given by

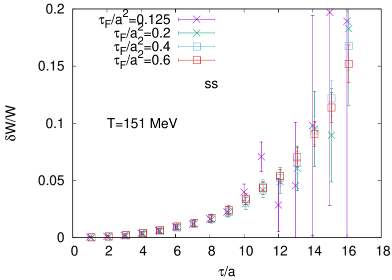
$$[W_2(r,\tau)]_{ij} = \frac{\langle W(r,\tau)\,\chi_{ij}\rangle}{\langle W(r,\tau)\rangle} - \langle\chi_{ij}\rangle\,, \qquad i,j \in \{u,d,s\},$$

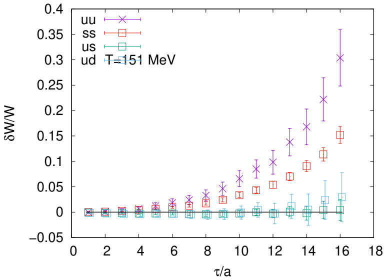
$$\chi_{ud} = \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu_u}\right)\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu_d}\right)$$

$$\chi_{uu} = \mathrm{Tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu_u^2}\right) - \mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu_u}M^{-1}\frac{\partial M}{\partial\mu_u}\right) + \left(\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial\mu_u}\right)\right)^2$$

Correlation of W with quark susceptibilities

Gradient flow is used to reduce the statistical uncertainty.





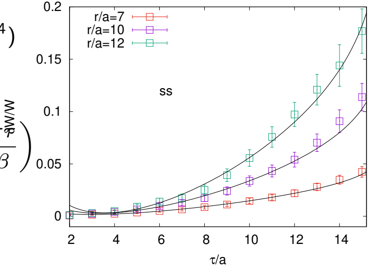
Potential in uds

Generalizing the parametrization to finite chemical potential we get,

$$W(r,\tau,\mu) = A(\mu)\ \exp\left[-V_{\rm re}(r,\mu)\tau - \frac{\beta V_{\rm im}(r,\mu)}{\pi}\log\left(\sin\frac{\pi\tau}{\beta}\right)\right]$$

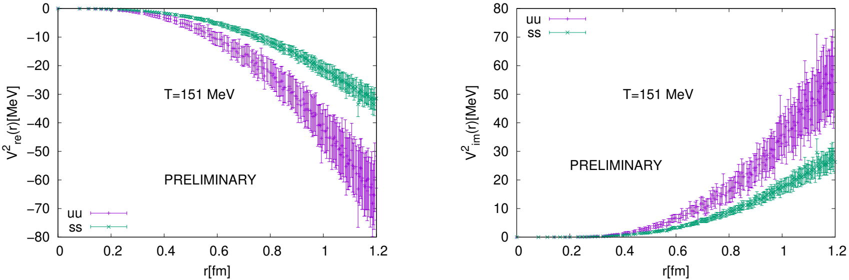
$$V_{\rm re,im}(r,\mu) = V_{\rm re,im}(r,0)+\frac{1}{2}\vec{\mu}^T V^{(2)}_{\rm re,im}(r)\vec{\mu}+\mathcal{O}(\mu^4)$$

$$(W_2(r,\tau))_{ij} = C - V^{(2)}_{\rm re}(r)_{ij}\,\tau - \frac{\beta V^{(2)}_{\rm im}(r)_{ij}}{\pi}\log\left(\sin\frac{\pi\tau}{\beta}\right)$$



- Fit form for extracting $V^{(2)}_{\rm re}$ and $V^{(2)}_{\rm im}$

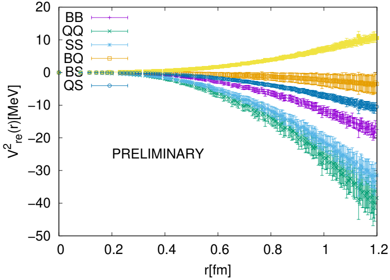
Correlation of W with quark susceptibilities

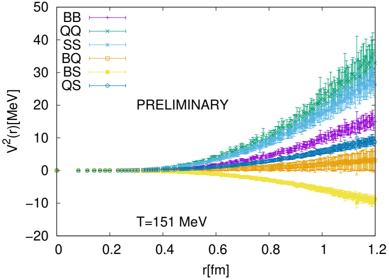


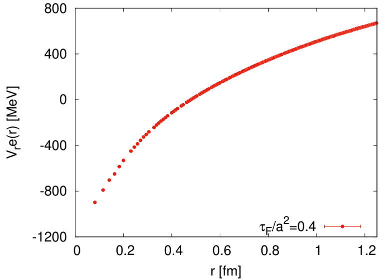
- More screening due to light quark than heavier quark.

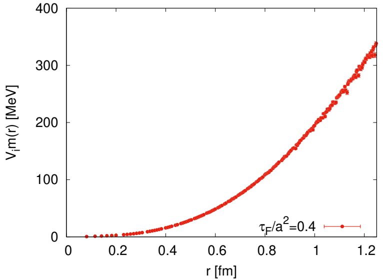
Potential in BQS

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \qquad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \qquad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$$









- Most of the correations comes from QQ sector.

Summary

- In-medium modification of the thermal potential in the deconfined phase
- Strong non-perturbative effects from the QGP
- Large thermal decay width for charmonium
- Second-order Taylor expansion of the potential at finite density

Lattice results show that color screening is an intrinsic property of the QGP

Color Octet Potential

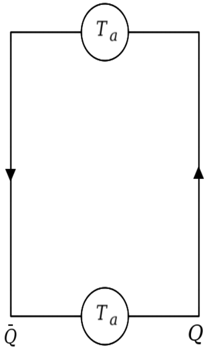
- In the medium, quarkonia can absorb a thermal gluon and transition to an octet state (no QED analog):
  N. Brambilla et al, PRD 78, 014017

$$(\bar{Q}Q)_1 + g \;\rightarrow\; (Q\bar{Q})_8$$

- An octet potential is needed to describe the evolution of this state in the QGP.

$$V^O = i \lim_{t\to\infty} \frac{\partial}{\partial t} \log W^O(r,t)$$

- This state is not gauge invariant. How can it be calculated on the lattice?



$$O = \bar{\psi} T^a \psi$$

- In Coulomb gauge:

$$V(r) = \frac{1}{6}\frac{g^2}{4\pi r}$$

Non-perturbative problem

- **Lorenz Gauge:** $\partial_\mu A^\mu(x) = 0$
  Hamiltonian does not exist.

$$W(r,\tau) \neq \sum c_n \exp(-E_n(r)\tau)$$

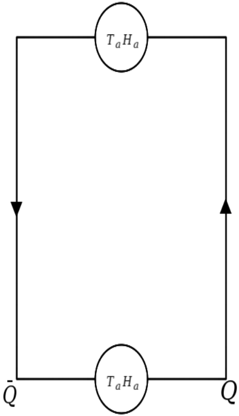
- **Coulomb Gauge:** $\nabla.\vec{A} = 0$

$$W(r,\tau) = \sum c_n \exp(-E_n(r)\tau)$$

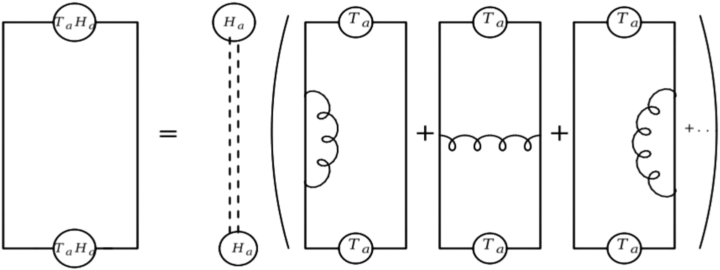
  Does not fix the gauge completely, and the expectation values are zero.

- **Temporal Gauge**: $A_0 = 0$, Describe the potential QQQ^{adj} system.
- No non-perturbative method is known that can calculate the octet potential.

Philipsen and Wagner, PRD 89, 014509



$$O = \bar{\psi}\, T^a\, H_a\, \psi$$



$$V_H(r) = \underbrace{\frac{1}{6}\,\frac{g^2}{4\pi r}}_{V_o} + \Lambda_H \quad \text{as } r \to 0$$

$$\Lambda_H \equiv \lim_{t\to\infty} \; i\frac{\partial}{\partial t} \ln \left(H^a(t)\, \phi^{\text{adj}}_{ab}(t,0)\, H^b(0) \right)$$

Gluelump mass

Color Octet Potential

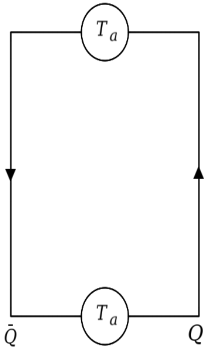
- In the medium, quarkonia can absorb a thermal gluon and transition to an octet state (no QED analog):
N. Brambilla et al, PRD 78, 014017

$$(\bar{Q}Q)_1 + g \;\rightarrow\; (Q\bar{Q})_8$$

- An octet potential is needed to describe the evolution of this state in the QGP.

$$V^O = i \lim_{t\rightarrow\infty} \frac{\partial}{\partial t} \log W(r,t)$$

- This state is not gauge invariant. How can it be calculated on the lattice?



$$O = \bar{\psi} T^a \psi$$

- In Coulomb gauge:

$$V(r) = \frac{1}{6}\frac{g^2}{4\pi r}$$

Non-perturbative problem

- **Lorenz Gauge:** $\partial_\mu A^\mu(x) = 0$
  Hamiltonian does not exist.

$$W(r,\tau) \neq \sum c_n \exp(-E_n(r)\tau)$$

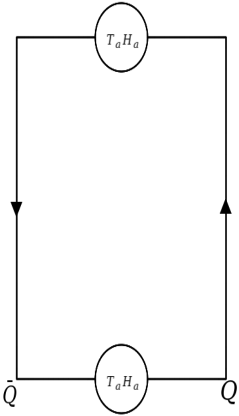
- **Coulomb Gauge:** $\nabla.\vec{A} = 0$

$$W(r,\tau) = \sum c_n \exp(-E_n(r)\tau)$$

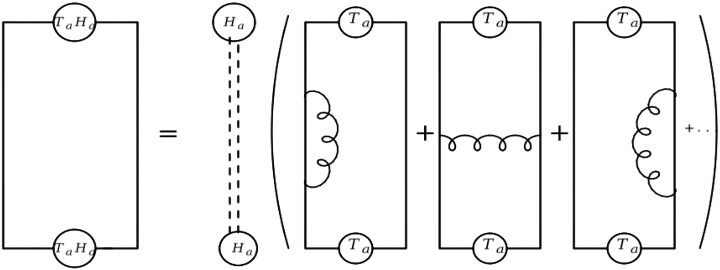
  Does not fix the gauge completely, and the expectation values are zero.

- **Temporal Gauge**: $A_0 = 0$, Describe the potential QQQ^{adj} system.
- No non-perturbative method is known that can calculate the octet potential.

Philipsen and Wagner, PRD 89, 014509



$$O = \bar{\psi}\, T^a\, H_a\, \psi$$

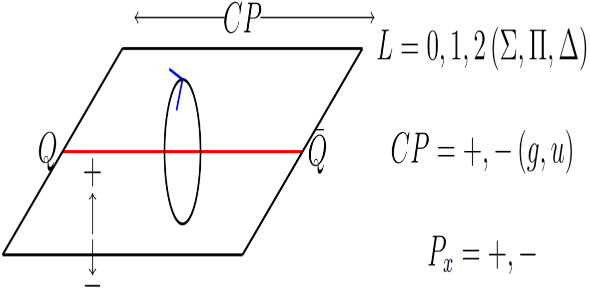


$$V_H(r) = \underbrace{\frac{1}{6}\,\frac{g^2}{4\pi r}}_{V_o} + \Lambda_H \quad \text{as } r \to 0$$

$$\Lambda_H \equiv \lim_{t\to\infty} i\frac{\partial}{\partial t} \ln\left(H^a(t)\,\phi^{\text{adj}}_{ab}(t,0)\,H^b(0)\right)$$

Gluelump mass

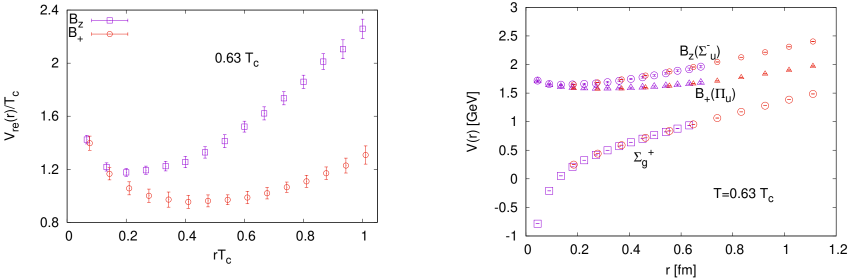
$$|\phi_n^H(\vec{r},t)\rangle = \underbrace{\bar{\psi}(-\vec{r}/2,t)\,U(-\vec{r}/2,\vec{0})\,H\,U(\vec{0},\vec{r}/2)\,\psi(\vec{r}/2,t)}_{O_H(r,t)}|n\rangle$$



$H = 1 \quad (\Sigma_g^+)$ — Standard Cornell Potential

$$\left.\begin{array}{l} H = B_z \quad (\Sigma_u^-) \\ H = B_+ = B_x + iB_y \quad (\Pi_u) \end{array}\right\} \text{Hybrid Potential}$$

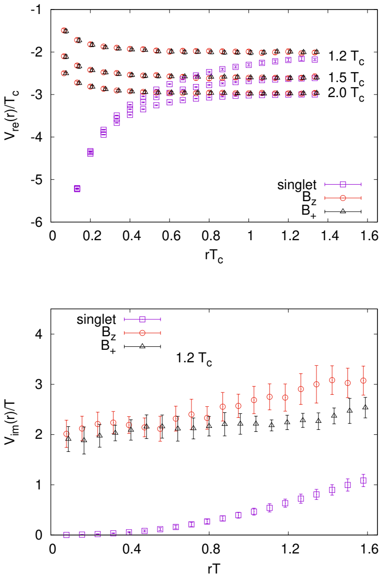
Hybrid Potential



Octet information is possible only at short distance.

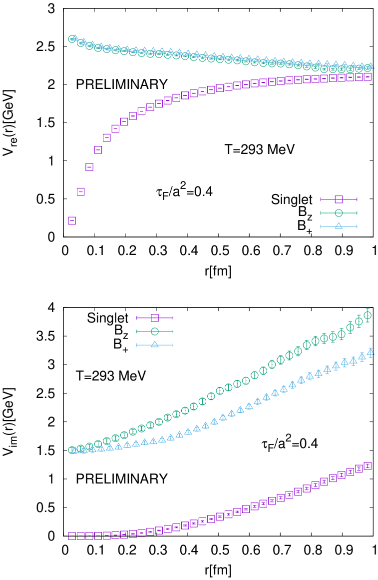
DB and S. Datta, PRD 103, 014512

S. Capitani et al, PRD 99, 034502



$N_f = 0$

DB and S. Datta, PRD 103, 014512

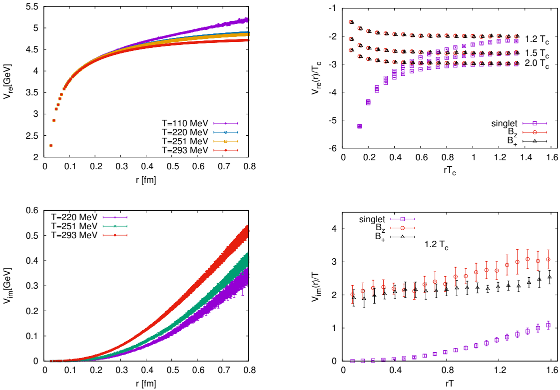


$N_f = 2 + 1$

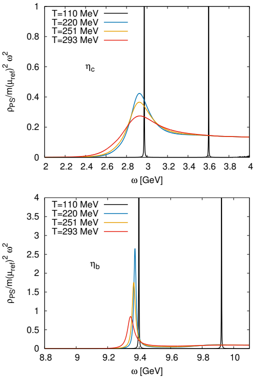
work in progress

Summary

Thermal Potential



Spectral Function



Strong evidence of Color screening from Lattice QCD data

Hybrid/Octet quarkonia bound state are unlikely to form

Large medium modification of charm states