Demazure filtration of tensor product modules of current Lie algebra of type A_1

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Set of Notations

- $\mathfrak{g}$ = arbitrary complex Lie algebra;
- $U(\mathfrak{g})$ = universal enveloping algebra of $\mathfrak{g}$;
- $\mathfrak{h}$ = cartan subalgebra of $\mathfrak{g}$;
- R = set of roots of $\mathfrak{g}$ w.r.t $\mathfrak{h}$;
- $I = 1, 2, \cdots, n$ index set;
- $\{\alpha_i : i \in I\}$ = set of simple roots for R;
- $\{\omega_i : i \in I\} \subset \mathfrak{h}^*$ = set of fundamental weights;
- $Q(\text{resp.} Q^+) = \mathbb{Z}$ span($\mathbb{Z}_{\geq 0}$-span) of simple roots;
- $P(\text{resp.} P^+) = \mathbb{Z}$ span($\mathbb{Z}_{\geq 0}$-span) of fundamental weights;
- $R^+ = R \cap Q^+$
- $\{x_\alpha^\pm, h_i : \alpha \in R^+, i \in I\}$ = Chevalley basis of $\mathfrak{g}$

Motivation

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- Our results are inspired by the question asked by Joseph Anthony about whether the tensor product of an integrable module and a Demazure module admits a Demazure flag.
- Since, $W_{loc}(m\omega) \cong D(1, m)$. We have studied $W_{loc}(m\omega) \otimes W_{loc}(n\omega)$ as an $\mathfrak{sl}_2[t]$-module and proved that this module has a filtration by level 2 Demazue modules.
- This work is motivated by the conjecture given by Dr Donna Blanton that the tensor products of Demazure modules of level m and n, respectively, have a filtration by Demazure modules of level $m + n$.

Current Lie algebra and local Weyl module

Let $\mathfrak{g}$ be simple finite-dimensional Lie algebra over $\mathbb{C}$.

Definition: Current algebra

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ where $\mathbb{C}[t]$ is the polynomial ring in one variable.
$\mathfrak{g}[t]$ is a Lie algebra with Lie bracket:
$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t).g(t) \quad x, y \in \mathfrak{g}; \quad f(t), g(t) \in \mathbb{C}[t]$

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Definition: local Weyl module

Let $\lambda \in P^+$ then the local Weyl module, $W_{loc}(\lambda)$, is the cyclic, $\mathfrak{g}[t]$-module generated by w_λ with the following defining relations:
$(x_{\alpha_i}^+ \otimes \mathbb{C}[t])w_\lambda = 0$, $(h_{\alpha_i} \otimes t^r)w_\lambda = \lambda(h_{\alpha_i})\delta_{0,r}w_\lambda$, $(x_{\alpha_i}^- \otimes 1)^{\lambda(h_{\alpha_i})+1}w_\lambda = 0$
where $i \in I$ and $r \in \mathbb{Z}_{\geq 0}$.

Demazure Modules and Demazure flags

Definition: Demazure module of level l

For $\lambda \in P^+$, Demazure modules of level l is denoted by $D(l, \lambda)$. It is a graded quotient of the local Weyl module by the submodule generated by elements of the form

$$\{(y_\alpha \otimes t^s)^{r+1} w_\lambda : s \in \mathbb{Z}_+, r \geq max\{0, \lambda(h_\alpha) - ls\}, \alpha \in R^+\}$$

Definition: Demazure flag of level l

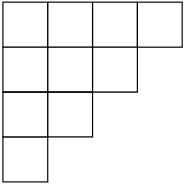
Let V be a graded $\mathfrak{g}[t]$-module. V has a **Demazure flag** of level l if there exists a decreasing sequence of graded $\mathfrak{g}[t]$ submodules of V
$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$ s.t. $\frac{V_i}{V_{i+1}} \cong \tau_{s_i}(D(l, \mu_i)), (\mu_i, s_i) \in P^+ \times \mathbb{Z}_+$.

Chari-Venkatesh modules

Definition: Chari-Venkatesh module

Given, $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_m > 0)$ a partition of $n \in \mathbb{Z}_+$, let $V(\lambda)$ be a cyclic $\mathfrak{sl}_2[t]$-module generated by v_λ satisfying the following relations:
$(x \otimes t^r)v_\lambda = 0, \quad (h \otimes t^r)v_\lambda = |\lambda|\delta_{r,0}v_\lambda, \quad (y \otimes 1)^{|\lambda|+1}v_\lambda = 0$
$(x \otimes t)^s(y \otimes 1)^{r+s}v_\lambda = 0 \quad \forall r + s \geq 1 + rk + \sum_{j \geq k+1} \lambda_j,$ for some $k \in \mathbb{N}$
where $r \in \mathbb{Z}_+$ and $r, s \in \mathbb{N}$.

For Example: $V(4, 3, 2, 1)$



Presentation of $W_{loc}(m\omega) \otimes V(\lambda)$

Define $W(m, \lambda)$ to be an $\mathfrak{sl}_2[t]$-module generated by an element w_λ^m with the following relations:

$$(h \otimes t^r)w_\lambda^m = (|\lambda| - m)\delta_{r,0} w_\lambda^m$$

$$x(r,s)w_\lambda^m = 0, \text{ if } r + s \geq 1 + rk + m - k \text{ for some } k \in \mathbb{N}$$

$$y(r,s)w_\lambda^m = 0, \text{ if } r + s \geq 1 + rk + \sum_{j \geq k+1} \lambda_j \text{ for some } k \in \mathbb{N}$$

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We will be required to prove the following two lemmas to give the presentation of $W_{loc}(m\omega) \otimes V(\lambda)$.

Lemma 1

There exists a surjective, $\mathfrak{sl}_2[t]$-module homomorphism from $W(m, \lambda)$ to $W_{loc}(m\omega) \otimes V(\lambda)$.

Idea of the proof

$W_{loc}(m\omega) \otimes V(\lambda)$ is generated by an element $y_0^m w_m \otimes v_\lambda$ where w_m (resp. v_λ) denotes a generator of $W_{loc}(m\omega)$ (resp. $V(\lambda)$). Define

$$\phi : W(m, \lambda) \to W_{loc}(m\omega) \otimes V(\lambda), \text{ such that,}$$
$$w_\lambda^m \mapsto y_0^m w_m \otimes v_\lambda$$

Therefore, ϕ is surjective $\mathfrak{sl}_2[t]$-module homomorphism because $y_0^m w_m \otimes v_\lambda$ is a generator of $W_{loc}(m\omega) \otimes V(\lambda)$. Now, it is sufficient to prove that ϕ is well-defined to get the result.

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Lemma 2

$dim(W(m, \lambda)) \leq 2^m(\lambda_1 + 1)(\lambda_2 + 1) \cdots (\lambda_n + 1)$

Key ideas of the proof

1. PBW theorem
2. Chari-Loktev basis of the local Weyl module
3. Basis of the $V(\lambda)$-module given by Chari and Venkatesh
4. $dim(W_{loc}(m\omega)) = 2^m$ and $dim(V(\lambda)) = (\lambda_1 + 1)(\lambda_2 + 1) \cdots (\lambda_n + 1)$

Key ideas of the proof

1. PBW theorem
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The above result proves that $dim(W(m, \lambda)) = dim(W_{loc}(m\omega) \otimes V(\lambda))$. Therefore, using the above two lemmas we will get the following result.

Theorem [Khandai, S.]

$W(m, \lambda) \cong W_{loc}(m\omega) \otimes V(\lambda)$

Filtration by $V(\xi)$-modules in $W_{loc}(m\omega) \otimes V(n)$

Definition: Filtration by $V(\xi)$-modules

Let V be a $\mathfrak{g}[t]$-module, then V has a filtration by $V(\xi)$ modules if there exist a decreasing chain of $\mathfrak{g}[t]$-submodules $V = U_r \supset U_{r-1} \supset \cdots \supset U_0 \supset (0)$ such that each $\frac{U_i}{U_{i-1}} \cong V(\xi^i)$, where ξ^i is a partition for each i.

Let $m(V : \xi)$ denote the number of times $V(\xi)$ occurs in the filtration.

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Let $m(V : \xi)$ denote the number of times $V(\xi)$ occurs in the filtration.

Theorem [Khandai, S.]

As $\mathfrak{sl}_2[t]$-modules, $W_{loc}(m\omega) \otimes V(n)$ has a filtration by $V(\xi)$-modules of the form

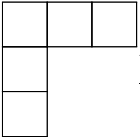
$$V(n+1-i, 1^{m-i-1}); \quad 0 \le i \le n \quad \text{if } m > n$$
$$V(n+1-i, 1^{m-i-1}), V(n-m); \quad 0 \le i \le m-1 \quad \text{if } m \le n$$

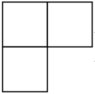
Example

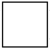
Consider $W_{loc}(3\omega) \otimes V(2)$, then there exist a decreasing chain of submodules

$$W_{loc}(3\omega) \otimes V(2) = U_2 \supset U_1 \supset U_0 \supset (0)$$

where each $U_i = U(\mathfrak{sl}_2[t])(w_3 \otimes y_0^{(i)} v_2), \quad \forall 0 \leq i \leq 2$ such that

$\frac{U_0}{(0)} \cong V($  $)$

$\frac{U_1}{U_0} \cong V($  $)$

$\frac{U_2}{U_1} \cong V($  $)$

Basis of a CV-module

Basis of a CV-module[Chari, Venkatesh (2015)]

Given a partition $\boldsymbol{n} = (n_1 \geq n_2 \geq \cdots \geq n_k)$ of n,

$$\mathbb{B}(\boldsymbol{n}) = \{v_{\boldsymbol{n}}, (y_0)^{i_1}(y_1)^{i_2} \cdots (y_{k-1})^{i_k} v_{\boldsymbol{n}} : (i_1, i_2, \ldots, i_k) \in J(\boldsymbol{n})\}$$

is a basis of $V(\boldsymbol{n})$, where
$J(\boldsymbol{n}) = \{(i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k_{\geq 0} : (ji_{r-1} + (j+1)i_r) + 2\sum_{l=r+1}^{n} i_l \leq \sum_{p=r-j}^{k} n_p, 2 \leq r \leq k+1, 1 \leq j \leq r-1\}$.

Since $W_{loc}(m\omega) \cong V(1^m)$. For $m \in \mathbb{Z}_+$, $W_{loc}(m)$ has a basis $\mathbb{B}(1^m)$ which is indexed by the set $J(1^m) \cup \{\emptyset\}$. Define a function $|.| : J(1^n) \cup \{\emptyset\} \to \mathbb{Z}$ such that $|\emptyset| = 0$,

$$|\boldsymbol{i}| := \sum_{k=1}^{m} i_k, \quad \forall\, \boldsymbol{i} = (i_1, \cdots, i_m) \in J(1^m).$$

Ordering on the basis of $W_{loc}(m\omega)$

Defining in $U(\mathfrak{n}^-[t])$ the elements $y(0,\emptyset) = 1$,

$$y(|\boldsymbol{i}|, \boldsymbol{i}) := (y_0)^{i_1}(y_1)^{i_2}\cdots(y_{m-1})^{i_m}, \quad \forall\, \boldsymbol{i} = (i_1, \cdots, i_m) \in J(1^m),$$

we see that, $\mathbb{B}(1^m) = \{y(|\boldsymbol{i}|, \boldsymbol{i})w_m : \boldsymbol{i} \in J(1^m) \cup \{\emptyset\}\}$. We define an ordering on $J(1^m) \cup \{\emptyset\}$ as follows.

- $\emptyset > \boldsymbol{i}$ for all $\boldsymbol{i} \in J(1^m)$
- Given $\boldsymbol{i}, \boldsymbol{j} \in J(1^m)$, we say, $\boldsymbol{i} > \boldsymbol{j}$, if either $|\boldsymbol{i}| < |\boldsymbol{j}|$ or $|\boldsymbol{i}| = |\boldsymbol{j}|$ and there exists $1 \leq k \leq m$ such that $i_k > j_k$ and $i_s = j_s$ for $k+1 \leq s \leq m$.

This clearly induces an ordering in $\mathbb{B}(1^m)$, thereby making $\mathbb{B}(1^m)$ an ordered basis of $W_{loc}(m)$.

Basis of $W_{loc}(m\omega)$

Given a positive integer m, Set
$F(m) = \{(k, \mathbf{s}) : k \in \mathbb{N}, \mathbf{s} = (s_1, s_2, \ldots s_k) \in \mathbb{Z}^k, 0 \leq s_i \leq m - k \text{ for } 1 \leq i \leq k\}$

Basis of $W_{loc}(m\omega)$[Chari,Loktev (2006)]

The set $\mathbb{B}(m) = \{y(k, \mathbf{s})w_m : (k, \mathbf{s}) \in F(m) \cup \{(0, \phi)\}\}$ is a basis of $W_{loc}(m\omega)$, where $y(k, \mathbf{s}) = (y \otimes t^{s_1})(y \otimes t^{s_2}) \ldots (y \otimes t^{s_k})$ and $y(0, \phi)w_m = w_m$.

Thus using standard q-binomial theory, it follows from the result given above that the number of l-graded elements of weight $m\omega - k\alpha$ in $\mathbb{B}(m)$ and hence in $\mathbb{B}(1^m)$ is equal to the coefficient of q^l in the polynomial $\begin{bmatrix} m \\ k \end{bmatrix}_q$.

Example

Consider $W_{loc}(4\omega)$. Now, we will provide ordering to the basis of $W_{loc}(4\omega)$.

$$\mathbb{B}(1^4) = \{w_4 \geq (y \otimes t^3)w_4 \geq (y \otimes t^2)w_4 \geq (y \otimes t)w_4 \geq (y \otimes 1)w_4 \geq (y \otimes t^1)(y \otimes t^3)w_4 \geq (y \otimes 1)(y \otimes t^3)w_4 \geq (y \otimes t^0)(y \otimes t^2)w_4 \geq (y \otimes t)^2 w_4 \geq (y \otimes 1)(y \otimes t)w_4 \geq (y \otimes 1)^2 w_4 \geq (y \otimes 1)^2(y \otimes t^3)w_4 \geq (y \otimes 1)^2(y \otimes t^2)w_4 \geq (y \otimes 1)^2(y \otimes t)w_4 \geq (y \otimes 1)^3 w_4 \geq (y \otimes 1)^4 w_4\}$$

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Definition : Truncated local Weyl module

Given a pair of integers (m, N), the truncated local Weyl module is a quotient of local Weyl module $W_{loc}(m\omega)$, generated by $w_{m,N}$ satisfying the following relations: $(x \otimes t^r)w_{m,N} = 0, \quad (h \otimes t^r)w_{m,N} = m\delta_{r,0}w_{m,N}$, $(y \otimes t)^{m+1}w_{m,N} = 0, \quad (y \otimes t^s)w_{m,N} = 0, \quad \forall s \geq N$

Important Results

Theorem [Chari, Wand, Schneider, Shereen (2014)]

Let $n \in \mathbb{N}$. Given a partition $\boldsymbol{n} = (n_1 \geq n_2 \geq \cdots \geq n_k)$ of n, the $\mathfrak{sl}_2[t]$-module $V(\boldsymbol{n})$ has Demazure flag of level l if and only if $l \geq n_1$.

Remark

Given non-negative integers a, b, since $2a + b = 1(a + b) + a$, the truncated local Weyl module $W_{loc}(2a + b, a + b) \cong V(2^a, 1^b)$.

Using the above results, we will get the truncated local Weyl module $W_{loc}(2a + b, a + b)$ has a Demazure flag of level 2.

Graded Character

Let $Z[P]$ denote the group ring of P with integer coefficients and basis $e(\lambda), \quad \lambda \in P$. The character of a finite-dimensional $\mathfrak{g}$- module V is the element of $Z[P]$ defined by $Ch_{\mathfrak{g}}(V) = \sum_{\mu \in P} dim V_\mu e(\mu)$ where $V = \oplus_{\mu \in P} V_\mu, \quad V_\mu = \{v \in V : h.v = \mu(h).v \quad \forall h \in \mathfrak{h}\}$

Definition: Graded Character

Given a graded $\mathfrak{g}[t]$-module and an indeterminate q, Let $Ch_{gr}(V) = \sum_{r \geq 0} Ch_{\mathfrak{g}} V[r] q^r \quad \in Z[P][q]$

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Now, we will try to compute the graded character of $W_{loc}(m\omega) \otimes W_{loc}(n\omega)$ in terms of truncated local Weyl modules. This will help us to create filtrations by level 2 Demazure modules in $W_{loc}(m\omega) \otimes W_{loc}(n\omega)$.

Graded Character of $W_{loc}(m\omega) \otimes W_{loc}(n\omega)$

Pieri Formulas

For $m, n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, we have $Ch_{gr}(W_{loc}(m\omega) \otimes W_{loc}(n\omega)) =$
$\sum_{i=0}^{m} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ i \end{bmatrix}_q (1-q)(1-q^2)\dots(1-q^i) Ch_{gr} W_{loc}((n+m-2i)\omega)$.

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Lemma

Given two non-negative integers a, b, we have,
$Ch_{gr} V(2^a, 1^b) =$
$\sum_{k=0}^{a} (-1)^k \begin{bmatrix} a \\ k \end{bmatrix}_q q^{k(a+b)-k(k-1)/2} Ch_{gr} W_{loc}((b+2a-2k)\omega)$.

Using the above formulas together we can prove the following formula

Filtration in $W_{loc}(m\omega) \otimes W_{loc}(n\omega)$

Lemma

For $n, m \in \mathbb{Z}_{\geq 0}$ with $n \geq m$,

$$Ch_{gr} W_{loc}(n\omega) \otimes W_{loc}(m\omega) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q Ch_{gr} V(2^{m-k}, 1^{n-m})$$

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Theorem[Khandai, S.]

Let $n, m \in \mathbb{N}$ and $n \geq m$. The $\mathfrak{sl}_2[t]$-module $W_{loc}(n\omega) \otimes W_{loc}(m\omega)$ admits a filtration whose successive quotients are isomorphic to truncated local Weyl modules

$$\tau_{k_r} W_{loc}(m+n-2r, n-r), \quad 0 \leq r \leq m, 0 \leq k_r \leq r(m-r).$$

Therefore, $W_{loc}(n\omega) \otimes W_{loc}(m\omega)$ has a filtration by level 2 Demazure modules.

Multiplicity of $D(2, l)$ in $W_{loc}(n\omega) \otimes W_{loc}(m\omega)$

Now, we can prove that

$$Ch_{gr} V(2^a, 1^b) = \sum_{k=0}^{\lfloor \frac{b}{2} \rfloor} q^{k(a+\lceil \frac{b}{2} \rceil)} \begin{bmatrix} \lfloor \frac{b}{2} \rfloor \\ k \end{bmatrix}_q Ch_{gr} D(2, 2a+b-2k).$$

Theorem [Khandai, S.]

For $n, m \in \mathbb{Z}_{\geq 0}$, we have $[W_{loc}(n\omega) \otimes W_{loc}(m\omega) : D(2, m+n-2s)]_q =$

$$\begin{cases} \sum_{k=0}^{min\{s,m\}} q^{(s-k)(m-k+\lceil \frac{n-m}{2} \rceil)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{n-m}{2} \rfloor \\ s-k \end{bmatrix}_q, \quad 0 \leq s \leq \lfloor \frac{n-m}{2} \rfloor, \\ \sum_{k=0}^{min\{\lfloor \frac{n-m}{2} \rfloor, m-j\}} q^{(\lfloor \frac{n-m}{2} \rfloor - k)(m-k-j+\lceil \frac{n-m}{2} \rceil)} \begin{bmatrix} m \\ k+j \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{n-m}{2} \rfloor \\ \lfloor \frac{n-m}{2} \rfloor - k \end{bmatrix}_q, \\ s = j + \lfloor \frac{n-m}{2} \rfloor, \quad 1 \leq j \leq m \end{cases}$$

for $m + n - 2s \geq 0$.

Multiplicity of $V(k)$ in $W_{loc}(m\omega) \otimes V(n)$

Theorem [Khandai, S.]

Let $m, n \in \mathbb{N}$, and $i \in \mathbb{Z}_+$ be such that $m + n - 2i \geq 0$.
If $m \leq n$,

$$[W(m,n) : V(m+n-2i)]_q = \begin{bmatrix} m \\ i \end{bmatrix}_q, \quad 0 \leq i \leq m,$$

If $m > n$,

$$[W(m,n) : V(m+n-2i)]_q = \begin{cases} \begin{bmatrix} m \\ i \end{bmatrix}_q, & 0 \leq i \leq n, \\ \begin{bmatrix} m \\ i \end{bmatrix}_q - \begin{bmatrix} m \\ i-n-1 \end{bmatrix}_q, & n+1 \leq i \leq \lfloor \frac{n+m}{2} \rfloor \end{cases}$$

Thank You

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