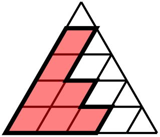
Refined Littlewood-Richardson coefficients and The Saturation Conjecture

Mrigendra Singh Kushwaha

University of Delhi, New Delhi, India



ICTS Bangalore, 26 November 2025

Joint work with K. N. Raghavan and Sankaran Viswanath

Notations

$\mathfrak{g}$	finite dimensional simple Lie algebra over $\mathbb{C}$
P^+	the set of dominant integral weights of $\mathfrak{g}$
$V(\lambda)$	the irreducible integrable $\mathfrak{g}$-module of highest weight $\lambda \in P^+$
W	the Weyl group of $\mathfrak{g}$
Q	the root lattice of $\mathfrak{g}$

Kostant–Kumar (KK) module

These modules were first formulated in the context of a strong form of the PRV conjecture due to Kostant, which Shrawan Kumar and O. Mathieu proved independently.

w	an element of W
v_λ	a highest weight vector of $V(\lambda)$
$v_{w\mu}$	a non-zero weight vector of the one-dimensional weight space of weight $w\mu$ of $V(\mu)$

The cyclic submodule of $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v_{w\mu}$ is called a *KK-module* and is denoted by $K(\lambda, w, \mu)$.

KK Module cont...

Properties:

1. $K(\lambda, 1, \mu) \cong V(\lambda + \mu)$.
2. $K(\lambda, w_0, \mu) = V(\lambda) \otimes V(\mu)$ where w_0 is the longest element of W.
3. $K(\lambda, w, \mu) \subseteq K(\lambda, w', \mu)$ if $w \leq w'$ in the Bruhat order of W.
4. $K(\lambda, w, \mu) \cong K(\mu, w^{-1}, \lambda)$.

Above properties shows that KK modules give a filtration of the tensor product $V(\lambda) \otimes V(\mu)$, with respect to the Bruhat order in W.

Let W_λ and W_μ denote the stabilizers of λ and μ respectively,

then $K(\lambda, w, \mu) = K(\lambda, \sigma, \mu)$ if $W_\lambda w W_\mu = W_\lambda \sigma W_\mu$.

Decomposition of KK module

The decomposition of KK module is:

$$K(\lambda, w, \mu) \cong \bigoplus_{\nu \in P^+} V(\nu)^{\oplus c^{\nu}_{\lambda,\mu}(w)}.$$

- $c^{\nu}_{\lambda,\mu}(w)$ is the multiplicity of $V(\nu)$ in $K(\lambda, w, \mu)$.
- Let $c^{\nu}_{\lambda,\mu}$ denote the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$, then $c^{\nu}_{\lambda,\mu}(w_0) = c^{\nu}_{\lambda,\mu}$.

Decomposition of KK module

The decomposition of KK module is:

$$K(\lambda, w, \mu) \cong \bigoplus_{\nu \in P^+} V(\nu)^{\oplus c^{\nu}_{\lambda,\mu}(w)}.$$

- $c^{\nu}_{\lambda,\mu}(w)$ is the multiplicity of $V(\nu)$ in $K(\lambda, w, \mu)$.
- Let $c^{\nu}_{\lambda,\mu}$ denote the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$, then $c^{\nu}_{\lambda,\mu}(w_0) = c^{\nu}_{\lambda,\mu}$.

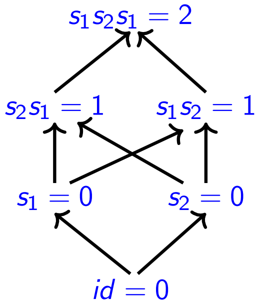
Observe that:

- $c^{\nu}_{\lambda,\mu}(1) = \delta_{\lambda+\mu,\nu}$
- $c^{\nu}_{\lambda,\mu}(w) \leq c^{\nu}_{\lambda,\mu}(w')$ if $w \leq w'$ in the Bruhat order of W
- $c^{\nu}_{\lambda,\mu}(w) = c^{\nu}_{\mu,\lambda}(w^{-1})$ $[c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda}]$
- $c^{\nu}_{\lambda,\mu}(w) = c^{\nu}_{\lambda,\mu}(\sigma)$ if $W_\lambda w W_\mu = W_\lambda \sigma W_\mu$

Example

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $W = S_3$. For $\lambda = (2,1,0) = \mu$ and $\nu = (3,2,1)$, the constant $c_{\lambda,\mu}^{\nu} = 2$ and $c_{\lambda,\mu}^{\nu}(s_1 s_2) = 1$.

The map $w \to c_{\lambda,\mu}^{\nu}(w)$ is an increasing function on the Bruhat poset.



Lakshmibai–Seshadri paths

P. Littelmann (1994) introduced Lakshmibai–Seshadri (LS) paths.

An LS path $\pi : [0,1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$ of shape $\mu \in P^+$ is given by

$$\pi = (w_1 > w_2 > \cdots > w_r \ ; \ 0 < a_1 < a_2 < \cdots < a_r = 1)$$

with "chain conditions", where $w_i \in W/W_\mu$ and $a_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, n$.

$$\pi(t) = \sum_{i=0}^{s-1} (a_i - a_{i-1}) w_i(\mu) \ + \ (t - a_{s-1}) w_s(\mu) \ ; \text{ for } t \in [a_{s-1}, a_s]$$

We say w_1 and w_r is the initial and final direction of π, respectively.

Let P_μ denote the set of LS paths of shape μ.

Path model

Path model of $V(\lambda)\otimes V(\mu)$ [P. Littelmann (1994)]

$$\textbf{Char}\, V(\lambda)\otimes V(\mu) = \sum_{(\pi,\pi')\in P_\lambda\times P_\mu} e^{\pi(1)+\pi'(1)}$$

Path model

Path model of $V(\lambda) \otimes V(\mu)$ [P. Littelmann (1994)]

$$\textbf{Char}\, V(\lambda) \otimes V(\mu) = \sum_{(\pi,\pi') \in P_\lambda \times P_\mu} e^{\pi(1)+\pi'(1)}$$

Path model of $K(\lambda, w, \mu)$ [–, Raghavan, Viswanath (2021)]

For given a $(\pi, \pi') \in P_\lambda \times P_\mu$ we associate a Weyl group element $\mathfrak{w}(\pi, \pi')$.

Let $P(\lambda, w, \mu) = \{(\pi, \pi') \in P_\lambda \times P_\mu \mid \mathfrak{w}(\pi, \pi') \leq w\}$

$$\textbf{Char}\, K(\lambda, w, \mu) = \sum_{(\pi,\pi') \in P(\lambda,w,\mu)} e^{\pi(1)+\pi'(1)}$$

A decomposition rule for KK module

For $\lambda \in P^+$, an LS path $\pi \in P_\mu$ is λ-dominant if $\lambda + \pi(t)$ is in the dominant chamber for all $t \in [0, 1]$.

$$P_\mu^\lambda(w \geq u) = \{\pi \in P_\mu \mid \pi \text{ is } \lambda\text{-dominant; Initial direction of } \pi \leq wW_\mu; \text{ Final direction of } \pi \geq uW_\mu \}$$

A decomposition rule for KK module

For $\lambda \in P^+$, an LS path $\pi \in P_\mu$ is λ-dominant if $\lambda + \pi(t)$ is in the dominant chamber for all $t \in [0,1]$.

$$P_\mu^\lambda(w \geq u) = \{\pi \in P_\mu \mid \pi \text{ is } \lambda\text{-dominant; Initial direction of } \pi \leq wW_\mu; \text{ Final direction of } \pi \geq uW_\mu \}$$

A decomposition rule for KK modules (Joseph [2003]):

$$K(\lambda, w, \mu) \cong \bigoplus_{\pi \in P_\mu^\lambda(w \geq 1)} V(\lambda + \pi(1)) \ ;$$

Let $P_\mu^\lambda(w \geq u;\ \nu) = \{\pi \in P_\mu^\lambda(w \geq u) \mid \lambda + \pi(1) = \nu\}$

Observe that: $c_{\lambda,\mu}^\nu(w) = |P_\mu^\lambda(w \geq 1;\ \nu)|$.

Some properties of $c^{\nu}_{\lambda,\mu}(w)$

Let $\pi = (w_1 > w_2 > \cdots > w_r \ ; \ 0 < a_1 < a_2 < \cdots < a_r = 1) \in P_\mu$ then

$$\pi^\dagger = (w_0 w_r > w_0 w_{r-1} > \cdots > w_0 w_1 \ ; \ 0 < 1 - a_{r-1} < \cdots < 1 - a_1 < 1)$$

is also an LS path of shape μ.

Note that $\dagger : P_\mu \to P_\mu$ is an involution.

Some properties of $c^{\nu}_{\lambda,\mu}(w)$

Let $\pi = (w_1 > w_2 > \cdots > w_r \; ; \; 0 < a_1 < a_2 < \cdots < a_r = 1) \in P_\mu$ then

$$\pi^\dagger = (w_0 w_r > w_0 w_{r-1} > \cdots > w_0 w_1 \; ; \; 0 < 1 - a_{r-1} < \cdots < 1 - a_1 < 1)$$

is also an LS path of shape μ.

Note that $\dagger : P_\mu \to P_\mu$ is an involution.

Proposition

$\pi \in P^{\lambda}_{\mu}(w \geq 1; \; \nu)$ if and only if $\pi^\dagger \in P^{\nu^*}_{\mu}(w_0 \geq w_0 w; \; \lambda^*)$

Observe that $c^{\nu}_{\lambda,\mu}(w) = |P^{\nu^*}_{\mu}(w_0 \geq w_0 w; \; \lambda^*)|$

Also for $w = w_0$ we get $c^{\nu}_{\lambda,\mu} = c^{\lambda^*}_{\nu^*,\mu}$.

properties of $c^{\nu}_{\lambda,\mu}(w)$ cont...

$c^{\nu}_{\lambda,\mu}(w) = |P^{\lambda}_{\mu}(w \geq 1;\ \nu)|$.

Let $\pi = (w_1 > w_2 > \cdots > w_r\ ;\ 0 < a_1 < a_2 < \cdots < a_r = 1) \in P_\mu$ then

$$\pi^* = (w_1^* > w_2^* > \cdots > w_r^*\ ;\ 0 < a_1 < a_2 < \cdots < a_r = 1)$$

is an LS path of shape $\mu^* = -w_0(\mu)$, where $w^* = w_0 w w_0$.

properties of $c^{\nu}_{\lambda,\mu}(w)$ cont...

$c^{\nu}_{\lambda,\mu}(w) = |P^{\lambda}_{\mu}(w \geq 1;\ \nu)|$.

Let $\pi = (w_1 > w_2 > \cdots > w_r\ ;\ 0 < a_1 < a_2 < \cdots < a_r = 1) \in P_{\mu}$ then

$$\pi^* = (w_1^* > w_2^* > \cdots > w_r^*\ ;\ 0 < a_1 < a_2 < \cdots < a_r = 1)$$

is an LS path of shape $\mu^* = -w_0(\mu)$, where $w^* = w_0 w w_0$.

1. $* : P_{\mu} \to P_{\mu^*}$ is a bijection.
2. $* : P^{\lambda}_{\mu}(w \geq 1) \to P^{\lambda^*}_{\mu^*}(w^* \geq 1)$ is a bijection.

From item (2) we conclude that:

$$\boxed{c^{\nu}_{\lambda,\mu}(w) = c^{\nu^*}_{\lambda^*,\mu^*}(w^*).} \qquad [\ c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\lambda^*,\mu^*}\ ]$$

properties of $c^{\nu}_{\lambda,\mu}(w)$ cont...

- Let $I \subset S$ where $S = \{1, 2, \ldots, n\}$ index the nodes of the Dynkin diagram of $\mathfrak{g}$.
- Let $\mathfrak{h}_I := \langle \alpha_i^\vee \mid i \in I \rangle$, for $\mu \in \mathfrak{h}^*$, μ_I denote the restriction μ on $\mathfrak{h}_I$.
- Let $\mathfrak{g}_I := \langle \mathfrak{h}_I, e_i, f_i \mid i \in I \rangle$. (In general $\mathfrak{g}_I$ is semisimple)
- $W_I = \{w \in W \mid \text{supp}(w) \subset I\}$ where $\text{supp}(w)$ denote the set of all $i \in S$ such that s_i occurs in any chosen reduced expression of w.
- $Q_I := \{\alpha \in Q : \text{supp}(\alpha) \subset I\}$ where if $\alpha = \sum_{i \in S} c_i \alpha_i \in \mathfrak{h}^*$, let $\text{supp}(\alpha)$ denote the set of $i \in S$ such that $c_i \neq 0$.

properties of $c^{\nu}_{\lambda,\mu}(w)$ cont...

- Let $I \subset S$ where $S = \{1, 2, \ldots, n\}$ index the nodes of the Dynkin diagram of $\mathfrak{g}$.
- Let $\mathfrak{h}_I := \langle\, \alpha_i^\vee \mid i \in I \,\rangle$, for $\mu \in \mathfrak{h}^*$, μ_I denote the restriction μ on $\mathfrak{h}_I$.
- Let $\mathfrak{g}_I := \langle\, \mathfrak{h}_I, e_i, f_i \mid i \in I \,\rangle$. (In general $\mathfrak{g}_I$ is semisimple)
- $W_I = \{w \in W \mid \mathrm{supp}(w) \subset I\}$ where $\mathrm{supp}(w)$ denote the set of all $i \in S$ such that s_i occurs in any chosen reduced expression of w.
- $Q_I := \{\alpha \in Q : \mathrm{supp}(\alpha) \subset I\}$ where if $\alpha = \sum_{i \in S} c_i \alpha_i \in \mathfrak{h}^*$, let $\mathrm{supp}(\alpha)$ denote the set of $i \in S$ such that $c_i \neq 0$.

Proposition

Let $\lambda, \mu, \nu \in P^+$ and $w \in W$, with $I = \mathrm{supp}(w)$. Then

$$\boxed{c^{\nu}_{\lambda,\mu}(w) \;=\; \delta^{\nu}_{\lambda\mu}(w)\, c^{\nu_I}_{\lambda_I,\,\mu_I}(w;\, \mathfrak{g}_I)}$$

where $\delta^{\nu}_{\lambda\mu}(w) = 1$ if $(\lambda + \mu - \nu) \in Q_I$ and 0 otherwise.

properties of $c_{\lambda,\mu}^{\nu}(w)$ cont...

- Let $w \in W$ and $I = \text{supp}(w)$. Let I_j ; $j = 1, 2, \ldots, r$ be the connected components of I.
- Let $W_j = W_{I_j}$ for $j = 1, 2, \ldots, r$. Then $w = \prod_{j=1}^{r} w_j$ where $w_j \in W_j$; note that w_j and w_k commute for $j \neq k$.
- Let $\mathfrak{g}_j = \mathfrak{g}_{I_j}$ and $\mathfrak{h}_j = \mathfrak{h}_{I_j}$. For the weights λ, μ, ν of $\mathfrak{g}$, let λ_j, μ_j, ν_j denote their restrictions to $\mathfrak{h}_j$.

properties of $c_{\lambda,\mu}^{\nu}(w)$ cont...

- Let $w \in W$ and $I = \text{supp}(w)$. Let $I_j\ ; j = 1, 2, \ldots, r$ be the connected components of I.
- Let $W_j = W_{I_j}$ for $j = 1, 2, \ldots, r$. Then $w = \prod_{j=1}^{r} w_j$ where $w_j \in W_j$; note that w_j and w_k commute for $j \neq k$.
- Let $\mathfrak{g}_j = \mathfrak{g}_{I_j}$ and $\mathfrak{h}_j = \mathfrak{h}_{I_j}$. For the weights λ, μ, ν of $\mathfrak{g}$, let λ_j, μ_j, ν_j denote their restrictions to $\mathfrak{h}_j$.

Proposition

Let notation be as above. Then

$$\boxed{c_{\lambda,\mu}^{\nu}(w) = \delta_{\lambda\mu}^{\nu}(w) \prod_{j=1}^{r} c_{\lambda_j\mu_j}^{\nu_j}(w_j; \mathfrak{g}_j)}$$

where $\delta_{\lambda\mu}^{\nu}(w) = 1$ if $(\lambda + \mu - \nu) \in Q_I$ and 0 otherwise.

Saturation and Semigroup problem

Saturation property

A element $w \in W$ is said to have the *saturation property* if the following holds for all $\lambda, \mu, \nu \in P^+$:

$$c^{k\nu}_{k\lambda,k\mu}(w) > 0 \text{ for some integer } k \geq 1 \text{ , then } \; c^{\nu}_{\lambda,\mu}(w) > 0.$$

Saturation and Semigroup problem

Saturation property

A element $w \in W$ is said to have the *saturation property* if the following holds for all $\lambda, \mu, \nu \in P^+$:

$$c_{k\lambda,k\mu}^{k\nu}(w) > 0 \text{ for some integer } k \geq 1 \text{ , then } \; c_{\lambda,\mu}^{\nu}(w) > 0.$$

Semigroup property

An element $w \in W$ is said to have the *semigroup property* if

$$c_{\lambda,\,\mu}^{\nu}(w) > 0 \text{ and } c_{\lambda',\,\mu'}^{\nu'}(w) > 0, \text{ then } c_{\lambda+\lambda',\,\mu+\mu'}^{\nu+\nu'}(w) > 0$$

for all $\lambda,\ \lambda',\ \mu,\ \mu',\ \nu,\ \nu' \in P^+$.

Saturation and Semigroup cont...

- Identity element has the saturation and semigroup properties since $c^{k\nu}_{k\lambda,k\mu}(1) = \delta_{k\lambda+k\mu,k\nu} = \delta_{\lambda+\mu,\nu} = c^{\nu}_{\lambda,\mu}(1)$.
- Note that, in the case $\mathfrak{g}$ is of type A, the constants $c^{\nu}_{\lambda,\mu}(w_0) = c^{\nu}_{\lambda,\mu}$ are called Littlewood-Richardson (LR) coefficients.
- w_0 have the saturation and semigroup properties proved by Knutson-Tao by using the Honeycomb model for $c^{\nu}_{\lambda,\mu}$.
- In this case ($\mathfrak{g}$ is of type A), we say the constants $c^{\nu}_{\lambda,\mu}(w)$ are refined LR coefficients,
- and we have proved that the saturation and semigroup properties for some special classes of permutations w.

Pattern avoiding permutations

Now we will take the $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ throughout the talk. Note that $W = S_n$ is the symmetric group.

312-avoiding permutations

A permutation $w \in S_n$ contains a 312-pattern if there exist $i < j < k$ such that $w(j) < w(k) < w(i)$. A permutation not containing any 312-pattern is called a 312-*avoiding permutation*.

Permutation $w = 45231$ (written in one-line notation) is not a 312-avoiding permutation. Take $1 < 3 < 4$, then

$$w(3) = 2 < w(4) = 3 < w(1) = 4.$$

Permutation $w' = 34521$ is a 312-avoiding permutations.

Similarly, we can define 231-avoiding permutation.

Saturation and semigroup theorem

Permutations of special form

A permutation $w \in S_n$ is said to be of *special form* if it is one of the following types:

1. w is either 312-avoiding or 231-avoiding.
2. Let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p} \subseteq S_n$ be a Young subgroup and $w = w_1 w_2 \cdots w_p \in H$ be such that each $w_i \in S_{n_i}$ is either 312-avoiding or 231-avoiding.

Saturation and semigroup theorem

Permutations of special form

A permutation $w \in S_n$ is said to be of *special form* if it is one of the following types:

1. w is either 312-avoiding or 231-avoiding.
2. Let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p} \subseteq S_n$ be a Young subgroup and $w = w_1 w_2 \cdots w_p \in H$ be such that each $w_i \in S_{n_i}$ is either 312-avoiding or 231-avoiding.

Theorem [–, Raghavan, Viswanath (2021)]

Let $w \in S_n$ be of special form, then w has the saturation and semigroup property.

Saturation and semigroup theorem

Remarks:

- If we prove the saturation and semigroup for those w given as item (1) above, then both properties follows for any $w = w_1 w_2 \cdots w_p$ given as item (2) by the equation:

$$c^{\nu}_{\lambda,\mu}(w) = \delta^{\nu}_{\lambda\mu}(w) \prod_{i=1}^{p} c^{\nu_i}_{\lambda_i\mu_i}(w_i; \mathfrak{g}_i)$$

  where $\delta^{\nu}_{\lambda\mu}(w) = 1$ if $(\lambda + \mu - \nu) \in Q_I$ and 0 otherwise.

- For $n = 1, 2, 3$, all permutations in S_n are of special form.
- For $n = 4$, our theorem establishes saturation and semigroup properties for all except the following four permutations:

$$2413,\ 3142,\ 3412,\ 4231.$$

Other than type A case

Let $\mathfrak{g}$ is any finite dimensional simple Lie algebra. Recall that, for $\lambda, \mu, \nu \in P^+$ and $w \in W$, with $I = \text{supp}(w)$.

$$\boxed{c^{\nu}_{\lambda,\mu}(w) \;=\; \delta^{\nu}_{\lambda\mu}(w)\, c^{\nu_I}_{\lambda_I,\,\mu_I}(w;\, \mathfrak{g}_I)}$$

where $\delta^{\nu}_{\lambda\mu}(w) = 1$ if $(\lambda + \mu - \nu) \in Q_I$ and 0 otherwise.

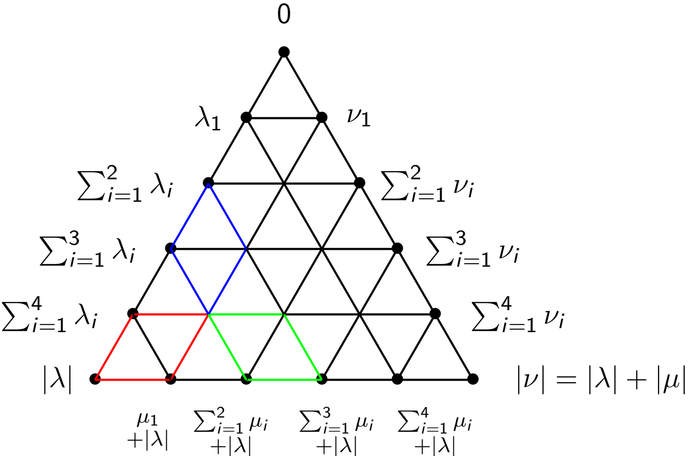
Corollary

w has the saturation (semigroup) property for the ambient Lie algebra $\mathfrak{g}$ if and only if $w \in W(\mathfrak{g}_I)$ has the saturation (semigroup) property for $\mathfrak{g}_I$.

Remark: Let $w \in W$ and $I = \text{supp}(w)$. If I be any type A subdiagram of $\mathfrak{g}$ and w is of special form in $W(\mathfrak{g}_I)$. Then w has saturation and semigroup property for $\mathfrak{g}$.

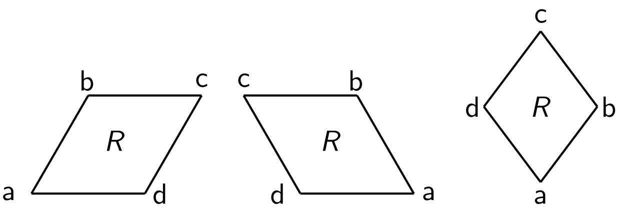
Ideas in the Proof: Hives

Note that, in the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the set of integral dominant weights P^+ corresponds to the integer partitions with at most n-parts denoted by $\mathcal{P}[n]$. Given $\lambda, \mu, \nu \in \mathcal{P}[n]$ such that $|\lambda| + |\mu| = |\nu|$



Rhombus inequalities

Consider the following rhombi R inside the big hive triangle

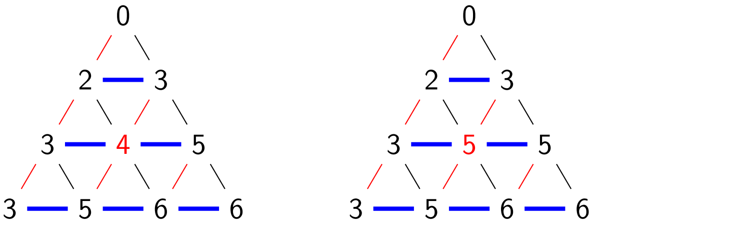


content(R) $:= (b+d) - (a+c)$ (sum of Obtus labels - Acute labels).

Hive(λ, μ, ν) is the set of all labelings of the big hive vertices with real numbers such that the boundary labels are partial sums of λ, μ, ν and all rhombus contents are ≥ 0. This is a polytope.

Example

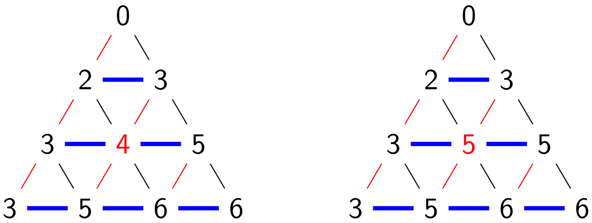
For $\lambda = (2, 1, 0) = \mu$ and $\nu = (3, 2, 1)$, the following are examples of hives in $\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$.



$\mathrm{Hive}(\lambda, \mu, \nu) =$ the closed interval $[4, 5]$

Example

For $\lambda = (2,1,0) = \mu$ and $\nu = (3,2,1)$, the following are examples of hives in $\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)$.



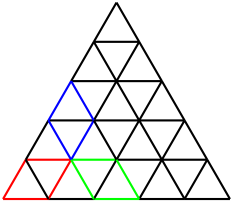
$\text{Hive}(\lambda, \mu, \nu) =$ the closed interval $[4, 5]$

Theorem [Knutson-Tao (1999)]

$|\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu)| = c^{\nu}_{\lambda,\mu}$. (A hive model for LR coefficients)

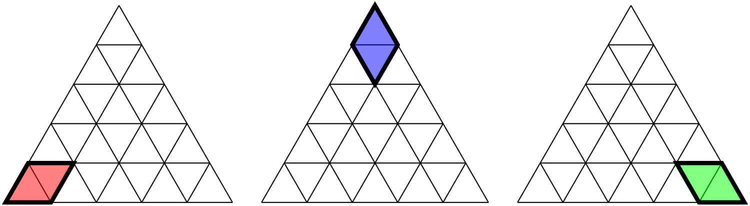
In the above example, we have $c^{\nu}_{\lambda,\mu} = 2$.

Big hive triangle



- Let $\triangle$ denote the above big hive triangle.
- Let $\mathfrak{R} = \{$collection of small rhombi of all three kinds in $\triangle\}$.
- Let $R \in \mathfrak{R}$, then $V(R)$ denotes the set of all four vertices of R, and for any $F \subset \mathfrak{R}$, denote $V(F) = \cup_{R \in F} V(R)$.

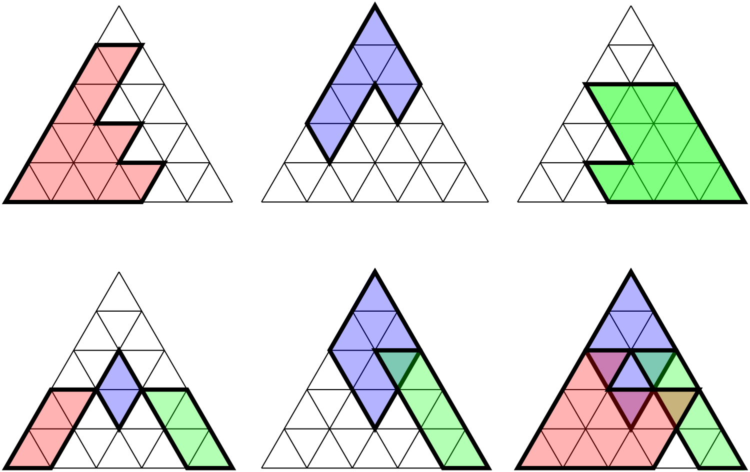
Frozen subset



Frozen subset

A subset $F \subset \mathfrak{R}$ is called *frozen* if $|F| = 1$, then it is one of the above types, and if $|F| > 1$ then there exists an $R \in F$ such that $H = F \backslash \{R\}$ is frozen and the obtuse vertices of R is in $V(H) \cup \partial\triangle$.

Examples



Main Theorem

A rhombus R is called flat if $\text{content}(R) = 0$.

- For a given $h \in \text{Hive}(\lambda, \mu, \nu)$, let $\text{Flats}(h) = \{R \in \mathfrak{R} \mid R \text{ is flat in } h\}$.
- For a given $F \subset \mathfrak{R}$, define a face of the hive polytope $\text{Hive}(\lambda, \mu, \nu)$:

$$\text{Hive}(\lambda, \mu, \nu; F) := \{h \in \text{Hive}(\lambda, \mu, \nu) \mid F \subset \text{Flats}(h)\}.$$

- Let $\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F)$ denote the set of integral points in $\text{Hive}(\lambda, \mu, \nu; F)$.

Main Theorem

A rhombus R is called flat if $\text{content}(R) = 0$.

- For a given $h \in \text{Hive}(\lambda, \mu, \nu)$, let $\text{Flats}(h) = \{R \in \mathfrak{R} \mid R \text{ is flat in } h\}$.
- For a given $F \subset \mathfrak{R}$, define a face of the hive polytope $\text{Hive}(\lambda, \mu, \nu)$:

$$\text{Hive}(\lambda, \mu, \nu; F) := \{h \in \text{Hive}(\lambda, \mu, \nu) \mid F \subset \text{Flats}(h)\}.$$

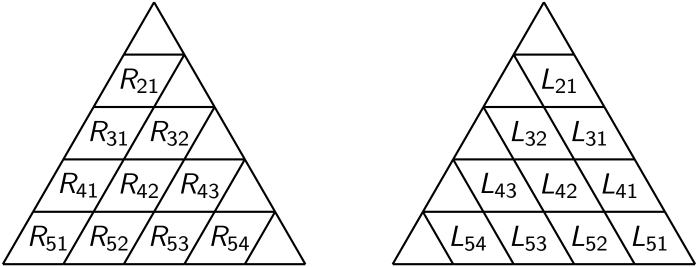
- Let $\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F)$ denote the set of integral points in $\text{Hive}(\lambda, \mu, \nu; F)$.

Main Theorem

Let $F \subset \mathfrak{R}$ be frozen. If $\text{Hive}(\lambda, \mu, \nu; F) \neq \emptyset$ then $\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F) \neq \emptyset$.

Remark: To prove the above theorem, we adopt the Knutson-Tao hive model technique in our case.

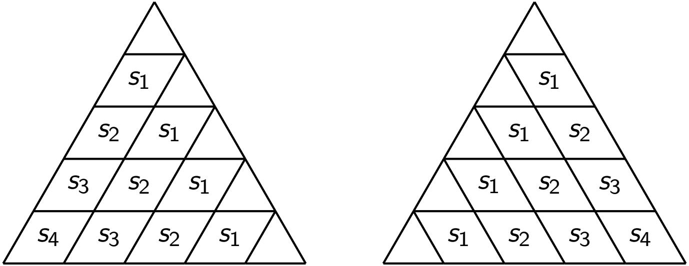
Hive Kogan face



Let $\mathrm{NE}(\mathfrak{R}) = \{R_{ij} \mid n \geq i > j \geq 1\}$ and $\mathrm{SE}(\mathfrak{R}) = \{L_{ij} \mid n \geq i > j \geq 1\}$.

- For $F \subset \mathrm{NE}(\mathfrak{R})$, the face $\mathrm{Hive}(\lambda, \mu, \nu; F)$ is called a Kogan face of hive polytope.
- For $H \subset \mathrm{SE}(\mathfrak{R})$, the face $\mathrm{Hive}(\lambda, \mu, \nu; H)$ is called a dual Kogan face of hive polytope.

Hive Kogan face

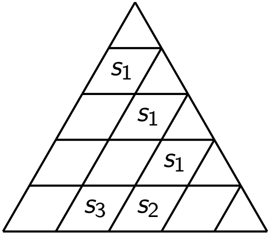


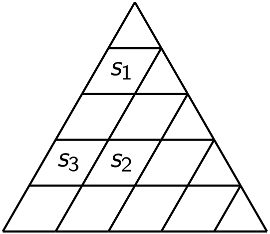
Let North-East and South-East rhombi be labeled by simple transpositions in S_n as in the above figure, i.e:

- R_{ij} and L_{ij} both labeled by s_{i-j} for $n \geq i > j \geq 1$.

Hive Kogan face

Fix a subset F of $\mathrm{NE}(\mathfrak{R})$.





Define $\sigma(F)$ to be the product of s_{i-j} for $R_{ij} \in F$ in the lexicographic order.

Example

Let $F_1 = \{R_{21}, R_{32}, R_{43}, R_{52}, R_{53}\}$ and $F_2 = \{R_{21}, R_{41}, R_{42}\}$, then $\sigma(F_1) = s_1 s_3 s_2 = \sigma(F_2)$.

Hive model for $c_{\lambda,\mu}^{\nu}(w)$

If $len(\sigma(F)) = |F|$, we say that F is *reduced*.

Ex: F_1 is not reduced and F_2 is reduced.

Naoki Fujita [2020]

$$\text{Set: } \varpi(F) = w_0\, \sigma(F)\, w_0.$$

For $w \in S_n$, define $\text{Hive}(\lambda, \mu, \nu; w) := \cup\, \text{Hive}(\lambda, \mu, \nu; F)$,

where the union is taken over all reduced subset $F \subset \text{NE}(\mathfrak{R})$ for which $\varpi(F) = w$.

Theorem [–, Raghavan, Viswanath (2021)]

$c_{\lambda,\mu}^{\nu}(w) = |\,\text{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; w_0 w)|$.

Another hive model for $c^{\nu}_{\lambda,\mu}(w)$

Similarly, for $H \subset \mathrm{SE}(\mathfrak{R})$ we can define $\sigma(H)$.

Fujita [2020]

For $w \in S_n$, define $\overline{\mathrm{Hive}}(\lambda, \mu, \nu; w) := \cup \,\mathrm{Hive}(\lambda, \mu, \nu; H)$,

where the union is taken over all reduced subset $H \subset \mathrm{SE}(\mathfrak{R})$ for which $\sigma(H) = w$.

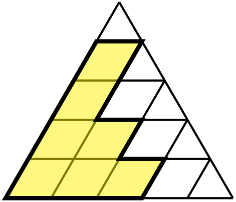
Theorem [–, Raghavan, Viswanath (2021)]

$c^{\nu}_{\lambda,\mu}(w) = |\overline{\mathrm{Hive}}_{\mathbb{Z}}(\nu^*, \mu, \lambda^*; w_0 w)|$.

Observe that above model implies that $c^{\nu}_{\lambda,\mu} = c^{\lambda^*}_{\nu*,\mu}$.

Hive Kogan face for 312-avoiding permutations

Let $w \in S_n$ be 312-avoiding. Then $w_0 w$ is 132-avoiding, and there exists a unique reduced $F_w \subset \mathrm{NE}(\mathfrak{R})$ such that $\varpi(F_w) = w_0 w$. Further, it has the following form:



- Note that F_w is frozen.
- In this case: $\mathrm{Hive}(\lambda, \mu, \nu; w_0 w) = \mathrm{Hive}(\lambda, \mu, \nu; F_w)$,
- and $c^{\nu}_{\lambda,\mu}(w) = |\,\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; w_0 w)| = |\,\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F_w)|$.

Semigroup property for 312-avoiding permutations

- Let w be 312-avoiding and $c^{\nu}_{\lambda,\mu}(w) > 0$ and $c^{\nu'}_{\lambda',\mu'}(w) > 0$.
- There exists a unique $F_w \subset \mathrm{NE}(\mathfrak{R})$,
- such that, $\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F_w) \neq \emptyset$ and $\mathrm{Hive}_{\mathbb{Z}}(\lambda', \mu', \nu'; F_w) \neq \emptyset$.
- $\mathrm{Hive}_{\mathbb{Z}}(\lambda + \lambda', \mu + \mu', \nu + \nu'; F_w) \neq \emptyset \implies c^{\nu+\nu'}_{\lambda+\lambda',\mu+\mu'}(w) > 0$.

Remark:

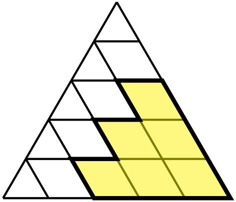
$$c^{\nu+\nu'}_{\lambda+\lambda',\mu+\mu'}(w) \geq \mathbf{max}\ (\ c^{\nu}_{\lambda,\mu}(w),\ c^{\nu'}_{\lambda',\mu'}(w)\ )$$

Sauration property for 312-avoiding permutations

- Let w be 312-avoiding.
- $c^{k\nu}_{k\lambda,k\mu}(w) > 0$ for some positive integer $k \geq 1$.
- There exist a unique $F_w \subset \mathrm{NE}(\mathfrak{R})$ such that $\mathrm{Hive}_{\mathbb{Z}}(k\lambda, k\mu, k\nu; F_w) \neq \emptyset$,
- and after scaling $\mathrm{Hive}(k\lambda, k\mu, k\nu; F_w)$ by $1/k$ we get $\mathrm{Hive}(\lambda, \mu, \nu; F_w) \neq \emptyset$.
- Since F_w is forzen then $\mathrm{Hive}_{\mathbb{Z}}(\lambda, \mu, \nu; F_w) \neq \emptyset \implies c^{\nu}_{\lambda,\mu}(w) > 0$.

Hive dual Kogan face for 231-avoiding permutations

Let $w \in S_n$ be 231-avoiding. Then $w_0 w$ is 213-avoiding and there exists a unique reduced $H_w \subset \mathrm{SE}(\mathfrak{R})$ such that $\sigma(H_w) = w_0 w$. Further, it has the following form:



- Note that H_w is frozen.
- In this case: $\overline{\mathrm{Hive}}(\nu^*, \mu, \lambda^*; w_0 w) = \mathrm{Hive}(\nu^*, \mu, \lambda^*; H_w)$,
- and $c^{\nu}_{\lambda,\mu}(w) = |\,\mathrm{Hive}_{\mathbb{Z}}(\nu^*, \mu, \lambda^*; H_w)|$.

Semigroup property for 231-avoiding permutations

- Let w be 231-avoiding and $c^{\nu}_{\lambda,\mu}(w) > 0$ and $c^{\nu'}_{\lambda',\mu'}(w) > 0$.
- There exist a unique $H_w \in \mathrm{SE}(\mathfrak{R})$,
- such that, $\mathrm{Hive}_{\mathbb{Z}}(\nu^*, \mu, \lambda^*; H_w) \neq \emptyset$ and $\mathrm{Hive}_{\mathbb{Z}}(\nu'^*, \mu', \lambda'^*; H_w) \neq \emptyset$.
- $\mathrm{Hive}_{\mathbb{Z}}(\nu^* + \nu'^*, \mu + \mu', \lambda^* + \lambda'^*; H_w) \neq \emptyset \implies c^{\nu+\nu'}_{\lambda+\lambda',\ \mu+\mu'}(w) > 0$.

Sauration property for 231-avoiding permutations

- Let w be 231-avoiding.
- $c_{k\lambda,k\mu}^{k\nu}(w) > 0$ for some positive integer $k \geq 1$.
- There exist a unique $H_w \subset \mathrm{SE}(\mathfrak{R})$ such that $\mathrm{Hive}_{\mathbb{Z}}(k\nu^*, k\mu, k\lambda^*; H_w) \neq \emptyset$.
- after scaling $\mathrm{Hive}_{\mathbb{Z}}(k\nu^*, k\mu, k\lambda^*; H_w)$ by $1/k$ we get $\mathrm{Hive}(\nu^*, \mu, \lambda^*; H_w) \neq \emptyset$.
- Since H_w is frozen then $\mathrm{Hive}_{\mathbb{Z}}(\nu^*, \mu, \lambda^*; H_w) \neq \emptyset \implies c_{\lambda,\mu}^{\nu}(w) > 0$.

Thank You