Representation Theory of Finite Groups: An Overview

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Abstract

In this talk, we will briefly cover a survey of the representation theory of finite groups, mostly in positive characteristic. We will be discussing some properties of the group algebra and will focus on irreducible and indecomposable modules over the group algebra. Since it is only a survey most of the proofs will be omitted.

Contents

1 Introduction

As we start our journey, let us understand what is the representation theory of finite groups and what is the objective of this subject. The algebraic structure of groups is rather abstract, yet useful. On the other hand, linear algebra, i.e., the study of modules over rings or in the special case the theory of vector spaces over fields is much enriched theoretically. Thus this branch of mathematics observes elements of groups as linear maps and then tries to get group theoretic information such as, to distinguish and classify them or to determine simplicity or solvability etc., out of it.

Throughout the talk, G will denote a group (often we will be interested in the case where G is finite). k is a field and V is a vector space over k.

Definition 1 (Representation)**.** Given a finite group G, and a field k, a *representation of* G *over* k is a map $\rho : G \to Aut_k(V)$ where V is a vector space over k. In such a case, we will call (V, ρ) a representation of G over k. The dimension of V is called the *dimension* of the representation.

We will only consider finite dimensional vector spaces V over a field k.

Example 2. (1) For any vector space V over any field k, the representation

$$\rho : G \to Aut_k(V)$$

which sends everything to $1 = id_V$ is called a *trivial representation* of G.

(2) Let, X be a set and $G \curvearrowright X$. Define, $V = \oplus_{x \in X} kx$. Setting $\rho_g(x) = gx, \forall x \in X$, we get a representation $\rho : G \to Aut_k(V)$, called the *permutation representation associated to the action of G on X.*

(3) In the special case, when $G \curvearrowright G$ by left multiplication V is denoted by kG and the permutation representation is called the *regular representation* of G over k. Some authors use the notation $k[G]$ for kG.

kG will play a central role in developing the theory, as we shall shortly see. Hence, we will have a closer look at it.

2 Group Algebra

Note that, kG can be given a structure of an augmented k-algebra. We get a k-algebra structure by defining multiplication

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{g,h} a_g b_h gh$$

and obtaining the ring homomorphism

$$\eta : k \to kG, \text{ sending } \lambda \mapsto \lambda 1.$$

We get the augmentation map by setting

$$\epsilon : kG \to k \text{ which sends } g \mapsto 1 \text{ for each } g \in G.$$

With these structures, kG is called the *group algebra* associated with G over k. This was introduced by German mathematician Ferdinand Georg Frobenius in 1897. We study some properties of kG in this section. Most of the materials here are taken from [[2], §1].

2.1 Relationship Between Representations and kG-modules

Before proceeding further, let us understand why kG is important.

Suppose, (V, ρ) is a representation of a G. Then the map ρ extends uniquely to a map $\rho' : kG \to End_k(V)$. Thus V becomes a kG-module with respect to the action of ρ'.

Conversely, given a kG-module M, we get a natural k-vector space structure on M equipped with a map from $kG \to End_k(M)$ sending the elements of kG to the action of the left multiplication on M. Restricting this ring homomorphism to the group of units (which contains G) we get the desired $\rho : G \to Aut_k(M)$.

Definition 3 (Intertwining Maps)**.** If $(V, \rho), (V', \rho')$ are two representations of G over k, then an *intertwining map* is a linear map $T : V \to V'$ such that the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ \downarrow{\rho_g} & & \downarrow{\rho'_g} \\ V & \xrightarrow{T} & V' \end{array}$$

commutes for all $g \in G$.

Note that, *between two representations* $(V, \rho), (V', \rho')$ *a* k*-linear map* $T : V \to V'$ *is* kG*-linear if and only if* T *is an intertwining map.*

Thus, this association given an equivalence of categories $Rep_k(G)$ consisting of all finite-dimensional k-representations of G and intertwining maps and f.g. $kG - mod$ consisting of all left finitely generated kG-modules and kG-linear maps.

Summarizing the above discussion, we can conclude that *considering representations of* G *over* k *is the same as considering left* kG*-modules.*

Example 4 (Revisited)**.** (1) k considered as a kG-module via the augmentation map ϵ gives the one dimensional *trivial representation* of G over k.

(2) kG considered as a left kG-module gives the *regular representation* of G over k.

2.2 Properties of The Group Algebra

(1) $\ker \epsilon = I(G)$ is a two-sided ideal of kG, called the augmentation ideal of kG. The set $\{g - 1 : g \neq 1 \in G\}$ forms a k-basis of $I(G)$. In view of the relations

$$g^{-1} - 1 = g^{-1}(1 - g), gh - 1 = g(h - 1) + g - 1 = (g - 1)h + h - 1,$$

one obtains that if G is generated by the set $\{g_\lambda : \lambda \in \Lambda\}$ then $I(G)$ is generated both as a left and a right ideal by the elements $g_\lambda - 1$.

(2) kG is commutative if and only if G is abelian (with G embedded in $(kG)^*$ and is finite dimensional over k if and only if G is a finite group).

(3) More generally, if $C_1, \ldots, C_r$ are all the distinct conjugacy classes of G and

$$z_i = \textstyle\sum_{x \in C_i} x, \text{ then } Z(kG) = \oplus_{i=1}^{r} kz_i.$$

To prove this, just compute using the description of the basis of kG.

(4) If A is a k-algebra, then $Hom_{gp}(G, U(A)) \cong Hom_{k-alg}(kG, A)$.

(5) kG is functorial, in the sense that, if $\phi : G_1 \to G_2$ is a group homomorphism then it can be uniquely extended to an augmented k-algebra homomorphism $k\phi : kG_1 \to kG_2$ whose kernel $\ker k\phi = \langle g - 1 | g \in \ker \phi \rangle$ both as a left and a right ideal.

In particular, if N is a normal subgroup of G then the natural projection

$$\pi : G \to G/N \text{ induces } k\pi : kG \to k(G/N)$$

with the kernel being generated by $\{n-1 : n \in N\}$ as both a left and a right ideal, i.e., the kernel is the extension of the augmentation ideal $I(N)$ of kN in kG as a left ideal as well as a right ideal. This gives the isomorphism,

$$k(G/N) \cong kG/I(N)kG \cong k \otimes_{kN} kG.$$

As an illustration of the above, we explicitly find out the group algebra of the cyclic groups below.

First note that, $k\mathbb{Z} = k[x, x^{-1}]$, the algebra of the Laurent polynomials over k is the variable x with augmentation ideal $I(\mathbb{Z})$ generated by $x-1$.

From the above discussion, it follows that the group ring

$$k(\mathbb{Z}/d\mathbb{Z}) = k[x, x^{-1}]/\langle x^d - 1\rangle \cong k[x]/\langle x^d - 1\rangle$$

with augmentation ideal $I(\mathbb{Z}/d\mathbb{Z}) = \langle x-1\rangle$.

(6) The natural inclusions of $G_j \overset{i_j}{\hookrightarrow} G_1 \times G_2$ for $j = 1, 2$ induces

$$ki_j : KG_j \to k(G_1 \times G_2).$$

Since elements of $Im(ki_1)$ commute with elements of $Im(ki_2)$, one obtains an augmented k-algebra homomorphism

$$kG_1 \otimes_k kG_2 \to k(G_1 \times G_2)$$

sending $g_1 \otimes g_2 \mapsto (g_1, g_2)$. As it gives a bijection on the natural bases of the k-vector spaces involved, it turns out to be an isomorphism. Hence, we can identify $k(G_1 \times G_2)$ with $kG_1 \otimes_k kG_2$. In view of this, we can describe the group algebra of any finitely generated abelian group.

If G is a finitely generated abelian group then by the structure theorem,

$$G \cong \mathbb{Z}^n \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_m\mathbb{Z}$$

for some unique $n \geq 0, d_i \geq 2$ with $d_{i+1}|d_i$ for all i. Thus we obtain

$$kG = \frac{k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, y_1, \ldots, y_m]}{\langle y_1^{d_1} - 1, \ldots, y_m^{d_m} - 1\rangle}$$

with $I(G) = \langle x_1 - 1, \ldots, x_n - 1, y_1 - 1, \ldots, y_m - 1\rangle$.

(7) kG is an (left and right) Artinian ring, being a finite dimensional k-vector space (since G is a finite group). This property of kG will be a saviour in the positive characteristic case, in view of the Krull-Schmidt property, as we will see shortly.

Recall that, for a ring R (not necessarily commutative) the *radical* of R is defined as the intersection of all left maximal ideals of R, which turns out to be the intersection of annihilators of all simple left R-modules and hence, a two-sided ideal of R. To see this, look at the equation

$$\bigcap_M ann(M) = \bigcap_M \bigcap_{x \neq 0} ann(x),$$

where the intersection is taken over all simple left R-modules. Now it is easy to see that the right-hand side is the intersection of all the left maximal ideals. This will be denoted by $J(R)$. One can show that,

$$\text{for an element } x \in R, x \in J(R) \iff 1 - yx \in R^*, \forall y \in R.$$

Also, it coincides with the intersection of all right maximal ideals of R. This follows from the fact that

$$J(R) \text{ is the largest two-sided ideal } I \text{ such that } 1 - x \in R^* \text{ for all } x \in I.$$

A ring R will be called *local* if it has a unique left (or right or two-sided) maximal ideal.

Recall that, if R is an (left or/and right) Artinian ring then $J(R)$ is the largest nilpotent (left or right or two-sided) ideal of R. This gives,

(8) If $Ch(k)$= characteristic of the field $k = p \geq 2$ (and G is finite), then kG is a local ring with maximal ideal $I(G)$ if and only if G is a p-group.

Proof. $\Longrightarrow$: Since kG is Artinian, $J(kG)$ is nilpotent, hence $I(G)$ is nilpotent. So, for all $x \in G$, $x - 1 \in I(G)$. Choosing l large enough such that $I(G)^{p^l} = 0$, we get $(x-1)^{p^l} = x^{p^l} - 1 = 0$, hence $|x| \mid p^l$, proving that G is a p-group.

$\Longleftarrow$: It is sufficient to show that $I(G)$ is nilpotent. Since G is a p-group,

$$|Z(G)| > 1 \text{ and } kG/I(Z(G))kG \cong k[G/Z(G)].$$

Assume by induction that $I(G/Z(G))$ is nilpotent. Thus, from the above isomorphism, it follows that $I(G)^n \subset I(Z(G))kG$ for some $n > 0$. Since, $Z(G)$ is an abelian group, by the above characterisation, $I(Z(G))$ is nilpotent and $I(Z(G)) \subset Z(kG)$, we deduce the nilpotency of $I(G)$ (see [2],§1.5 and §1.6). □

3 Semisimplicity

To enable us with the language of modern representation theory, i.e., to focus on the kG-modules, we first describe some preliminaries from (non-commutative) ring theory. In what follows we will always be concerned about rings with unity. For an elaborated discussion, see [3], Chapter XVII.

Definition 5 (Simple and Semisimple Modules)**.** A nonzero R-module M is called *simple* if it contains no nontrivial proper submodules. An R-module M is called *semisimple* if it is expressible as a sum of a family of simple submodules.

We have the following characterisation:

Theorem 6. For a ring R and a module M over R, the following conditions are equivalent:

(A) M is Semisimple.

(B) M is a direct sum of simple modules.

(C) Every submodule of M is a direct summand of M.

Proof. See [3], Chapter XVII, §2, for a discussion. □

Lemma 7 (Schur)**.** If M, N are simple R-modules then any morphism $f : M \to N$ is either 0 or an isomorphism. Thus, if $M \ncong N$ then $Hom_R(M, N) = 0$ and $End_R(M)$ is always a division algebra over R.

Proof. Consider $\ker f, Im f$ and use the definition of simple R-modules. □

The simple-looking Schur's lemma has far-stretched consequences in representation theory. We derive some corollaries.

Corollary 8. (1) Submodules and quotient modules of a semi-simple module are semi-simple (see [3], Chapter XVII, §2, Proposition 2.2).

(2) If $M = \sum S_i$ with S_i simple for each i, then any simple submodule $S \leq M$ is isomorphic to one of the S_i's. Note that in view of the above theorem, there is a projection from $M \to S$, which, when restricted on S_i, is nonzero, for some i. Schur's lemma takes care of the rest of the statement.

(3) A semisimple module M is expressible as a direct sum of finitely many simple submodules if and only if M has finite length. Such a decomposition is unique up to permutation and number of simple summands, in view of the Jordan-Hölder theorem.

Examples of semisimple modules include vector spaces of arbitrary dimension over an arbitrary field.

The module $\mathbb{Z}/n\mathbb{Z}$ is semisimple $\mathbb{Z}$-module if and only if n is a product of distinct primes with multiplicity 1 (exercise).

Definition 9 (Semisimple Rings)**.** A ring R is called *semisimple* if R is semisimple left R-module. A ring R is called *right semisimple* if it is semisimple as a right R-module, i.e., R^{op} is semisimple.

In fact, one can show that the notion of semisimplicity of a ring R is equivalent to R being right semisimple, with the help of the Artin-Wedderburn structure theorem. So we will only talk about semisimplicity (as a left module) in our discussion.

The semisimplicity of rings is in general stronger than the semisimplicity of modules. We also have the notion of simple rings as follows.

Definition 10 (Simple Rings)**.** A ring R is called *simple* if it is a nonzero semisimple ring with no proper nontrivial two-sided ideal.

Note that, considering the map η, its kernel $I(G)$ and comparing k-dimensions one can easily obtain that

$$kG \text{ is simple if and only if } G \text{ is trivial.}$$

Proposition 11 ([3], Chapter XVII, §4)**.** (1) For a ring R the following are equivalent

(A) R is semisimple.

(B) Each R-module is semisimple.

(C) Every left or right R-module is projective.

Proof. In view of the fact that any R-module is a quotient of a free R-module, the equivalence of (A) and (B) is clear.

In view of condition (C) in Theorem 6, the equivalence of (B) and (C) is clear. □

(2) If R is a semisimple ring then R is the direct sum of a finitely many simple left ideals. Moreover, any simple left R-module is isomorphic to one of the ideals mentioned above. In particular, there exist only a finitely many simple left R-modules up to isomorphism.

Proof. Write

$$1 = \sum_{i=1}^{n} x_i, \text{ where } R = \oplus_{i\in I} S_i$$

and $x_i \in S_{\lambda_i}$ for $i = 1, \ldots, n$ and S_i are simple left R-ideals. Then $R = \oplus_{i=1}^{n} S_{\lambda_i}$. The second part follows from Schur's lemma by proceeding similarly as in part (2) of Corollary 8. □

(3) For a semisimple ring R, R is simple if and only if there exists exactly one simple left R module up to isomorphism.

(4) For a ring R the following are equivalent. (Use Jordan-Hölder theorem and Chinese remainder theorem)

(A) R is semisimple.

(B) R is (left and right) Artinian and $J(R) = 0$.

(C) R is Artinian and it has no non-zero nilpotent left ideal.

(5) From the last equivalence, it follows that a ring R is simple if and only if R is Artinian and has no nontrivial two-sided ideal. Thus, if R is Artinian then $R/J(R)$ is semisimple.

One can in fact describe all the semisimple and simple rings thanks to the Artin-Wedderburn structure theorem.

Theorem 12 (Artin-Wedderburn; [1], Chapter 1, §1.3)**.** A ring R is semisimple if and only if R is isomorphic to $M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for some uniquely determined k, n_i's and division rings D_i's (up to isomorphism).

In fact, a ring R is simple if and only if it is isomorphic to $M_n(D)$ for some uniquely determined n and a unique (up to isomorphism) division ring D.

Now we discuss some facts about representations of finite groups and point out some fundamental differences between the zero characteristic and the positive characteristic cases.

4 Representation Theory of Finite Groups

Definition 13 (Irreducible Representations)**.** A simple kG-module, i.e., a kG-module which has no nontrivial submodule (submodules of a kG-module are called *subrepresentations* of G) is called an *irreducible representation* of G over k.

Definition 14 (Indecomposable Representations)**.** An *indecomposable representation* of G over k is an indecomposable kG-module, i.e., an kG-module which has no nontrivial direct summands.

In the sequel, every indecomposable module is assumed to be finitely generated, unless otherwise stated.[1]

In general irreducible representations are indecomposable, by definition. Converse may not hold always.

As an illustration, consider

$$\mathbb{F}_p(\mathbb{Z}/p\mathbb{Z}) = \mathbb{F}_p[x]/\langle (x-1)^p \rangle.$$

It is not irreducible since the trivial subrepresentation $\mathbb{F}_p(1 + x + \cdots + x^{p-1})$ is a $\mathbb{F}_p(\mathbb{Z}/p\mathbb{Z})$-submodule of the group algebra. However it is indecomposable since if

$$\langle f(x) \rangle / \langle (x-1)^p \rangle \oplus \langle g(x) \rangle / \langle (x-1)^p \rangle = \mathbb{F}_p[x]/\langle (x-1)^p \rangle$$

[1]Note that, simple modules are, by definition, finitely generated. However, for indecomposable modules, this is not the case. Consider the indecomposable $\mathbb{Z}$-module $\mathbb{Q}$, which is not finitely generated.

then notice that $f(x), g(x) | (x-1)^p$ and expressing the image of 1 as $a(x)f(x)+b(x)g(x)$ and then raising the power p yields $1 = 0$, giving a contradiction to the nontrivial cases.

However, in zero characteristic and in some special cases in positive characteristic the notions of irreducible and indecomposable representations coincide.

Theorem 15 (Maschke)**.** For a finite group G, each short exact sequence of kG-modules split if and only if $|G| \neq 0$ in k.

Proof. Consider a short exact sequence of kG-modules

$$0 \to L \to M \xrightarrow{\pi} N \to 0.$$

Since k is a field π has a k-linear section $\sigma : N \to M$. Define $\tilde{\sigma} : N \to M$ by setting

$$\tilde{\sigma}(n) = \tfrac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}n)$$

which turns out to be a kG-linear section of the short exact sequence. This construction of $\tilde{\sigma}$ is known as the "averaging trick", which is of fundamental importance in the cases where $|G| \neq 0$ in k.

Conversely, one has a split exact sequence

$$0 \longrightarrow I(G) \longrightarrow kG \underset{\sigma}{\overset{\epsilon}{\rightleftarrows}} k \longrightarrow 0.$$

Now, $\sigma(1) = \sum_g a_g g$. For a fixed $h \in G, \sigma(1) = \sigma(h.1) = h.\sigma(1)$. Comparing coefficients, thus one obtains $a_g = a_1, \forall g \in G$. Now

$$1 = \epsilon(\sigma(1)) = a_1 \sum \epsilon(g) = a_1 \sum_g 1 = a_1|G|,$$

proving the result (see [2], §2.6). □

As a consequence (this is just a restatement) one gets the traditional form of Maschke's theorem:

Theorem 16 (Maschke)**.** For a finite group G, the following are equivalent:

(A) kG is a semisimple ring.

(B) k, as a kG-module via ϵ, is projective.

(C) $|G| \neq 0$ in k.

Proof. $(A) \implies (B)$: Obvious.

$(B) \implies (C)$: follows from the second part of the previous theorem.

$(C) \implies (A)$: also follows from the first part of the previous theorem showing that every submodule of a kG-module M is a direct summand of M (see [2], §3.1). □

Thus, for any representation of a finite group G over a field k such that $ch(k) = 0$ or $ch(k) \nmid |G|$, every subrepresentation is a direct summand and every representation is completely reduced to irreducible subrepresentations. Moreover, if we assume k is algebraically closed, we get the following results.

Theorem 17. (1) The number of non-isomorphic simple kG-modules (i.e., irreducible representations of G over k) is the same as the number of conjugacy classes of G. If $n_1, \ldots, n_r$ are the k-dimensions of the irreducible representations then $\sum n_i^2 = |G|$. In particular, G is abelian if and only if each irreducible representation of G is one-dimensional.

[Note that, here the hypothesis that k is algebraically closed is important, consider the $\mathbb{Q}$-representation of $\mathbb{Z}/3\mathbb{Z}, V = \mathbb{Q}[x]/\langle x^2 + x + 1\rangle$. Then V is an irreducible representation of dimension 2.]

(2) If V be a simple kG-module then $End_{kG}(V) = k$. (This is nothing but Schur's lemma).

In fact, one can generalise (1) to

(3) Brauer's Theorem: If k is algebraically closed of characteristic $p > 0$, then the number of isomorphism classes of simple kG-modules is the same as the number of p-regular conjugacy classes of G (see [5], the whole notes is devoted to illustrating a proof of this statement). (By a *p-regular element* of G we mean an element whose order is not divisible by p. A *p-regular conjugacy class* is a conjugacy class of a p-regular element.)

(4) From the decomposition of kG into simple ideals one can obtain all the irreducible representations of G over k up to isomorphisms.

In fact in the general case, one can always claim that for any finite group G over an arbitrary field k, there are only finitely many irreducible representations (up to isomorphism).

For any field k and a finite group G, we have previously remarked that kG is Artinian. Thus any finite-dimensional representation M of G over k is both Noetherian and Artinian. Hence it has a composition series, i.e., a sequence of submodules

$$0 = M_l \subset M_{l-1} \subset \cdots \subset M_0 = M$$

such that each of the successive quotients is simple. Jordan-Hölder theorem hence tells that any two such composition series have the same length (l) and the simple factors are uniquely determined up to permutation and isomorphism (see [1], Chapter 1, §1.1).

Since every simple kG-module appear as a quotient of kG, they appear in a composition series and thus *there are only finitely many irreducible representations of G over k*.

In view of Maschke's theorem, if kG is semisimple, so is every module over kG. Thus, every representation of G over k can be written as a direct sum of irreducible representations of G over k. However, this is not true in general. For example, take

$$k[\mathbb{Z}/2\mathbb{Z}] = k[x]/\langle x^2 - 1\rangle \text{ with } ch(k) = 2.$$

Note that, $k(1+x)$ is the unique 1-dimensional subrepresentation of this regular representation. Hence, it can not be decomposed as a direct sum of irreducibles. This forces us to shift our attention to indecomposable representations in view of the Krull-Schmidt theorem.

Definition 18. A ring R is said to have the *Krull-Schimdt* property if each finitely generated R-module can be written as a finite direct sum of indecomposable modules and such a decomposition is unique in the sense that, if

$$\oplus_{i=1}^{n} M_i \cong \oplus_{j=1}^{m} N_j$$

with M_i, N_j's being indecomposable, then $m = n, M_i \cong N_i$ for all i (possibly after a permutation).

Theorem 19 (Krull-Schimdt; [1], Chapter 1, §1.4)**.** Artinian rings have the Krull-Schmidt property.

In particular, one obtains that kG, being Artinian, has the Krull-Schmidt property.

In view of the Krull-Schmidt property, thus if one can characterise all the indecomposable representations of G over k, then one can obtain all the possible representations of G up to isomorphism. This motivates us to study the classification of representation types (see [1], Chapter 4, §4.4).

Definition 20. Suppose k is an infinite field. Λ is a finite-dimensional k-algebra.

(1) Λ is of *finite representation type* if there are only finitely many isomorphism classes of indecomposable Λ-modules.

(2) Λ is of *tame representation type* if it is not of finite representation type, and for any dimension n, there is a finite set of $\Lambda - k[T]$-bimodules M_i which are free as right $k[T]$-modules, with the property that all but a finite number of indecomposable Λ-modules of dimension n are of the form $M_i \otimes_{k[T]} M$ for some i, and for some indecomposable $k[T]$-module M.

(3) Λ is of *domestic representation type* if the M_i may be chosen independently of n.

(4) Λ is of *wild representation type* if there is a finitely generated $\Lambda - k[X,Y]$-bimodule M which is free as a right $k[X,Y]$-module, such that the functor $M \otimes_{k[X,Y]} -$ from finite dimensional $k[X,Y]$-modules to finite dimensional Λ-modules preserves indecomposability and isomorphism class.

A complete classification for a field of positive characteristic is as follows.

Theorem 21 (Trichotomy)**.** Let G be a finite group and k an infinite field of characteristic p.

(i) kG has finite representation type if and only if G has cyclic Sylow p-subgroups.

(ii) kG has domestic representation type if and only if $p = 2$ and the Sylow 2-subgroups of G are isomorphic to the Klein's four group.

(iii) kG has tame representation type if and only if $p = 2$ and the Sylow 2-subgroups are dihedral, semi-dihedral ($SD_{2^n} = \langle a, x | a^{2^{n-1}} = x^2 = 1, xax = a^{2^{n-2}-1} \rangle$) or generalised quaternions ($Q_{4n} = \langle a, b | a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1} \rangle$).

(iv) In all the other cases kG has wild representation type.

There is a nice connection between indecomposable modules and irreducible modules over a finite-dimensional k-algebra (in particular, for kG).

Theorem 22 (see [2], §4.2. Also see, [4])**.** For a finite-dimensional k-algebra R, the association $P \mapsto P/J(R)P$ gives a one-one correspondence between the set of all isomorphism classes of indecomposable projective R-modules and the set of all isomorphism classes of simple R-modules. (In fact, every projective indecomposable R-module is finitely generated.)

Thus the isomorphism classes of projective indecomposable representations of G over k are in one-one correspondence with the irreducible representations and hence there are only finitely many of them.

Note that, *all the finitely generated indecomposable modules may not be projective.* Thus, in particular, there may exist more indecomposable modules than irreducibles (even when kG is of finite representation type).

As an illustration consider, G to be a finite cyclic group. (i.e., kG is of finite representation type for any infinite field k). Let p denote 1 if $ch(k) = 0$ and the characteristic of k otherwise. Suppose, $|G| = p^n q$, with $n \in \mathbb{N} \cup \{0\}$ and $(p, q) = 1$. Then

$$R = kG = k[x]/\langle x^{p^n q} - 1 \rangle.$$

With the conventions of p,

$$x^{p^n q} - 1 = (x^q - 1)^{p^n} = (\Pi_{i=1}^d f_i(x))^{p^n},$$

where f_i's are irreducibles and $\sum_{i=1}^d deg(f_i) = q$. By comaximality of the ideals $\langle f_i(x)^{p^n} \rangle$, applying Chinese Remainder theorem, one obtains that

$$R \cong \Pi_{i=1}^d R_i, \text{ with } R_i = k[x]/\langle f_i^{p^n} \rangle.$$

Thus each R-module M uniquely decomposes as $M = \bigoplus_{i=1}^d M_i$ where each M_i is an R_i-module. Observe that *M is an indecomposable R-module if and only if exactly one M_i is indecomposable as an R_i-module and the other M_j's are* 0. Thus it is sufficient to find out all the finitely generated indecomposable R_i-modules.

Now, note that, *for any commutative ring R and ideal I, R/I is decomposable as an R-module (equivalently, as an R/I-module) if and only if there are ideals J_1, J_2 of R containing I whose sum is R and the intersection is I.* This shows that $R_i/\langle f_i^s \rangle$ are indecomposable modules for $1 \leq s \leq p^n$ (working out you will see that $J_1 + J_2 \subset \langle f_i(X) \rangle$, in case of a nontrivial decomposition). Moreover, no two of them are isomorphic which can be seen by looking at their annihilators. We claim that they exhaust the list of finitely generated indecomposable modules over R. This follows from the structure theorem of finitely generated modules over PID, by noting that a $k[x]$-module is an R_i-module if and only if it is annihilated by $f_i^{p^n}$.

Next, we find out all the projective indecomposable modules over R. We claim that the R_i's are precisely all the projective indecomposable R-modules. That they are projective is clear and indecomposability follows from the previous paragraph. Also, no two of them are isomorphic, as before. Note that, the simple kG modules up to isomorphism, in this case, are precisely

$$k[x]/\langle f_i(x) \rangle, i = 1, \dots, d.$$

From Theorem 22, thus it follows that the list mentioned above is precisely all the projective indecomposable R-modules up to isomorphism. This list is way smaller than the list of indecomposable modules in positive characteristic.

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For the theory of non-commutative rings and representation theory in zero characteristic covered here, the reader can refer to my handwritten notes available at my homepage, where I have uploaded the notes I prepared from various resources while studying these things.