ICTS lecture 1 Geodesics

We will cover:

- existence of finite geo. (I)
- infinite geo
  - how many? (I, II, III)
  - do they have directions? (II, III)

Refs

- survey 50 years Ch. 4
- Newman '94 ICM paper
- Hoffman '05, '08
- Damron-Hanson '14

We will not cover

- geodesic wandering
- parallel development in LPP
- uniqueness of geo. in a given direction
- bigeodesics
- midpoint problem
- other variants: Euclidean, LPP, PDE-type models

Finite geodesics

Def. A path Γ from $x \in \mathbb{Z}^d$ to $y \in \mathbb{Z}^d$ is a geodesic if $T(\Gamma) = T(x,y)$.

Check: Every segment of a geodesic is a geodesic.

Q: Do they exist?

Def. (Zhang) Passage time to infinity

Let
$$\rho(x) = \lim_{n\to\infty} T(x, \partial B(n)),$$

where $B(n) = [-n,n]^d$,

$$\partial B(n) = \{x \in B(n) : \exists y \in B(n)^c \text{ s.t. } |x-y| = 1\}.$$

B(n)

- exists by monotonicity.
- Check: Show that

$$\rho(x) = \inf\{T(\gamma) : \gamma \text{ is an infinite, edge-self-avoiding path from } x\}.$$

Prop.

① If $\rho(0) = \infty$, then "geodesics exist". That is, $\forall x, y \in \mathbb{Z}^d$, $\exists$ geo. from

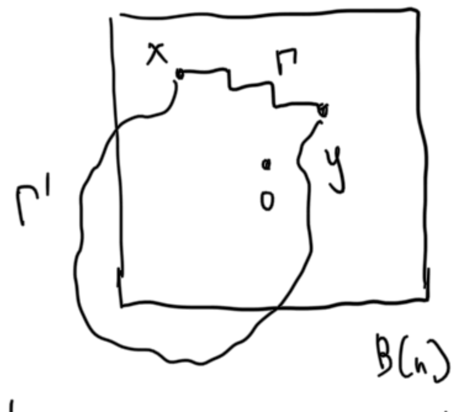
x to y. (deterministic statement)

② If $\underbrace{\mathbb{P}(t_e = 0)}_{F(0)} < p_c$, then $\rho(0) = \infty$ a.s.

③ Check: If (t_e) are iid and continuous, then a.s., $\forall x, y \in \mathbb{Z}^d$, $\exists$ unique geo. from x to y.

Proof.

① If $\rho(0) = \infty$, then for any x,
$|T(x, \partial B(n)) - T(0, \partial B(n))| \le T(0, x) < \infty$,
so $\rho(x) = \infty$ $\forall x$. Fix $x, y \in \mathbb{Z}^d$ and choose a path Γ from x to y. Let n be so large that $x \in B(n)$ and $T(x, \partial B(n)) > T(\Gamma)$.



If $\Gamma' : x \to y$ touches $\partial B(n)$, then
$T(\Gamma') \geq T(x, \partial B(n)) > T(\Gamma)$.
Thus Γ' is not a geo.
$\Rightarrow$ infimum is over a finite set
$\Rightarrow$ $\exists$ minimizer.

② Since $\mathbb{P}(t_e = 0) < p_c$, choose $\delta > 0$ s.t. $\mathbb{P}(t_e \le \delta) < p_c$. A.S., there is no inf. edge-self-avoiding path with weights $\le \delta$. So A.S., each inf. edge-self-avoiding path has inf. many weights $> \delta$. (If not, it would have a final weight $\geq \delta$, and the path beyond that point has all weights $\le \delta$.

)

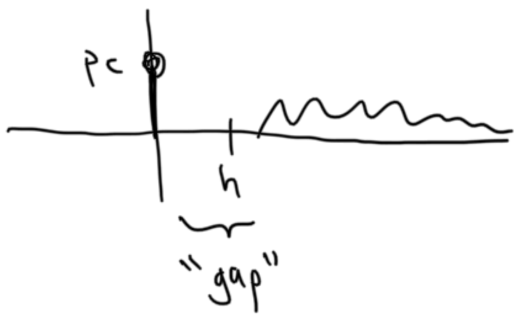
By alternative def. of $\rho(0)$, we get $\rho(0) = \infty$ a.s.

Remarks

① If $F(0) < p_c$, then
$$\mathbb{P}\left(\liminf_{\substack{x \in \mathbb{Z}^d \\ |x| \to \infty}} \frac{T(0,x)}{|x|} > 0 \right) = 1.$$
(Kesten.)
Check: True if $F(0) = 0$.

② Do geodesics exist when $F(0) \geq p_c$?

- Wierman-Reh: yes if $d=2$.
- Zhang: yes if $F(0) > p_c$.
- Bates: yes if $F(0) = p_c$, $d \geq 3$, and $F(h) = p_c$ for some $h > 0$.



(3) If $F(0) < p_c$, $\exists\, C_1, c_2 > 0$ s.t. $\forall x$,
$\mathbb{P}(\exists$ geo. from 0 to x with $\geq C_1|x|$ many edges$)$

$$\leq e^{-c_2 |x|^{1/d}}.$$

(Auffinger-Damron-Hanson)

(4) When is $\rho(0) < \infty$ a.s.?

$\underline{d=2}$ (Damron-Lam-Wang)

$\rho(0) < \infty$ a.s.

$$\Leftrightarrow \sum_k F^{-1}\left(\frac{1}{2} + \frac{1}{2^k}\right) < \infty,$$

where

$F^{-1}(t) = \inf\{x : F(x) \geq t\}$

is the generalized inverse of F.

Infinite geodesics

<u>Def</u>. • An inf. path is an <u>infinite geodesic</u> if each finite segment is a geodesic.

• For an outcome $\omega = (t_e)$, let

$$N(\omega) = \sup\{k \geq 0 : \exists\, k \text{ edge-disjoint inf geo. in } \omega\}.$$

<u>Prop</u>. If (t_e) is translation ergodic, then $N = N(\omega)$ is a.s. constant. If geodesics exist in ω, then $N(\omega) \geq 1$.

<u>Proof</u>

<u>Check</u>: N is measurable.

By ergodicity, N is a.s. constant.

For second claim, let Γ_n be a geo. from 0 to ne_1. Choose f_1 that is 1st edge of inf. many Γ_n (say Γ_{n_k}). Choose f_2 that is second edge of inf. many Γ_{n_k}, and so on. Let Γ be the path that follows $f_1, f_2, \ldots$

Then Γ is an inf. geo.

<u>What is the value of N?</u>

For simplicity, take (t_e) iid, cts.

- Hoffman, Garet-Marchand, Häggström-Pemantle : $N \geq 2$ a.s.

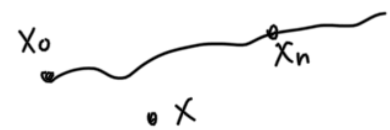
- Hoffman '08 : $N \geq$ # extremal pts of $\mathcal{B}$ $\geq 2d$ a.s.

- Damron-Hochman: given K, $\exists$ iid cts (t_e) in $d=2$ s.t. $N \geq K$ a.s.

<u>Idea of Hoffman's $N \geq 2$:</u>

<u>Def</u>. Let γ be an inf. geo with starting point x_0 and verts $x_0, x_1, \ldots$ in order. For $x \in \mathbb{Z}^d$, set

$$B_\gamma(x) = \lim_{n\to\infty} [T(x,x_n) - T(x_0,x_n)].$$



B_γ is the <u>Busemann function for γ.</u>

- limit exists:

$$
\begin{aligned}
&T(x,x_n) - T(x_0,x_n)\\
&= T(x,x_n) - T(x_0,x_{n-1}) - T(x_{n-1},x_n)\\
&\leq T(x,x_{n-1}) + T(x_{n-1},x_n) - T(x_0,x_{n-1}) - T(x_{n-1},x_n)\\
&= T(x,x_{n-1}) - T(x_0,x_{n-1}).
\end{aligned}
$$

$\Rightarrow$ monotone and bdd in abs. value by $T(x,x_0)$ $\Rightarrow$ limit exists.

<u>Define</u> $B_\gamma(x,y) = B_\gamma(x) - B_\gamma(y)$.

<u>Properties</u>

(A) $|B_\gamma(x,y)| = \left|\lim_{n\to\infty}\left(T(x,x_n) - T(y,x_n)\right)\right| \leq T(x,y)$.

(B) For $m < n$,

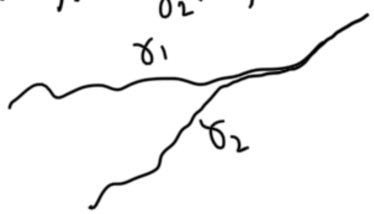
$$
\begin{aligned}
B_\gamma(x_m,x_n) &= \lim_{N\to\infty}\left(T(x_m,x_N) - T(x_n,x_N)\right)\\
&= T(x_m,x_n).
\end{aligned}
$$

(C) $B_\gamma(x,y) = B_\gamma(x,z) + B_\gamma(z,y)$ $\forall x,y,z$.

(D) $B_\gamma(x,y)(\omega) = B_{\theta(\gamma)}(\theta(x),\theta(y))(\theta(\omega))$ if θ is translation by an integer vector.

(E) Check: If γ_1, γ_2 coalesce (have finite symmetric diff.)

then $B_{\gamma_1}(x,y) = B_{\gamma_2}(x,y) \quad \forall x,y.$



Assume for a contradiction that $N=1$ a.s. Then any inf. geo. γ, γ' must share inf. many edges. By uniqueness, they coalesce. Thus if we arbitrarily select an inf. geo. γ, and set

$$f(x,y) = B_\gamma(x,y) \qquad x,y \in \mathbb{Z}^d$$

then f does not depend on the choice of γ.

<u>Check</u>: f is measurable.

Also,

- $\mathbb{E}|f(x,y)| \le \mathbb{E}T(x,y) < \infty$
- $f(x,y)(\omega) = f(\theta(x), \theta(y))(\theta(\omega))$ for integer translations θ.

What is $\mathbb{E}f(0,x)$? First,

$x \mapsto \mathbb{E}f(0,x)$ is additive (<u>Check</u>).

So it is determined by the values of $\mathbb{E}f(0, \pm e_i)$, $i = 1, \dots, d$. These are all equal. For example, for $i \neq j$,

$$\begin{aligned} \mathbb{E}f(0,e_i)(\omega) &= \mathbb{E}B_\gamma(0,e_i)(\omega) \\ &= \mathbb{E}B_{u(\gamma)}(0,e_j)(u(\omega)) \\ &= \mathbb{E}f(0,e_j)(u(\omega)) \\ &= \mathbb{E}f(0,e_j), \end{aligned}$$

where u is an isometry of $\mathbb{Z}^d$ that maps $0 \to 0$, $e_i \to e_j$.

So

$$\begin{aligned} 0 &= \mathbb{E}f(0,e_1) + \mathbb{E}f(e_1,0) \\ &= \mathbb{E}f(0,e_1) + \mathbb{E}f(0,-e_1) \quad \text{(translate)} \\ &= 2\mathbb{E}f(0,e_1). \end{aligned}$$

$\Rightarrow \mathbb{E}f(0,e_1) = 0.$

$\Rightarrow \mathbb{E}f(0,x) = 0 \quad \forall x.$

By the ergodic theorem, $\forall x \in \mathbb{Z}^d$,

$$\begin{aligned} &\frac{1}{n} f(0,nx)(\omega) \\ &= \frac{1}{n} \sum_{k=1}^{n} f((k-1)x, kx)(\omega) \\ &= \frac{1}{n} \sum_{k=1}^{n} f(0,x)(T_x^{k-1}\omega) \end{aligned}$$

trans. by x

$\longrightarrow \mathbb{E}f(0,x) = 0$

a.s. and in L^1 as $n \to \infty$.

This can be upgraded (using a proof similar to that of the shape theorem) to show:

$$\mathbb{P}\left(\lim_{\substack{x \in \mathbb{Z}^d \\ |x| \to \infty}} \frac{f(0,x)}{|x|} = 0 \right) = 1.$$

Let Γ be a subsequential limit of geodesics from 0 to ne_1 with vertices (in order) $0 = x_0, x_1, \ldots$. Then $|x_n| \to \infty$ as $n \to \infty$, so

$$\frac{f(0,x_n)}{|x_n|} \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}$$

But this equals

$$\frac{T(0,x_n)}{|x_n|},$$

This is a contradiction, since the shape theorem gives a.s.

$$\liminf_{\substack{x \in \mathbb{Z}^d \\ |x| \to \infty}} \frac{T(0,x)}{|x|} > 0.$$

lec 2

From now on, take

(A) $\mathbb{E} e^{\alpha t_e} < \infty$ for some $\alpha > 0$,

(B) (t_e) iid, continuous (so unique geodesics)

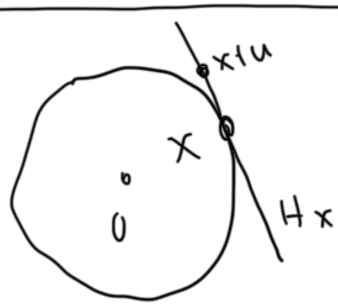
Def. Let Γ be an infinite geo with vertices $x_0, x_1, x_2, \ldots$ in order. Γ has asymptotic direction $\theta \in \mathbb{R}^d$ if

$$\arg x_n := \frac{x_n}{|x_n|} \to \theta \text{ as } n \to \infty.$$

Q: Do inf. geos have asymptotic directions?

To address this, Newman assumes curvature. We use a similar condition by Chatterjee.

(I) $\partial \mathcal{B}$ is differentiable. That is, $\forall x \in \partial\mathcal{B}$, there is a unique supporting hyperplane for $\mathcal{B}$ at x.
[A hyperplane that contains x, but $\mathcal{B}$ intersects only one component of its complement.]
Call this hyperplane H_x.



(II) $\exists C, \delta > 0$ s.t. $\forall x \in \partial \mathcal{B}$,

$$\mu(x+u) - \mu(x) \geq C|u|^2$$

for $|u| < \delta$, $x+u \in H_x$

Thm (Newman, etc.)
Assume A, B, I, II.

① A.S., $\forall \theta$ with $|\theta| = 1$, $\exists$ inf. geo starting at 0 with asymp. direction θ.

② A.S., all inf. geos. have asymp. directions.

Remark In particular, $N = \infty$ a.s.

Damron-Hanson - under I, for deterministic θ, a.s. all inf. geo with asymp. direction θ must coalesce.

Ahlberg-Hoffman - improved condition I.

Check: Under A, B, I, II, a.s. $\exists$ (random) θ s.t. one can find two edge-disjoint inf geos with direction θ.

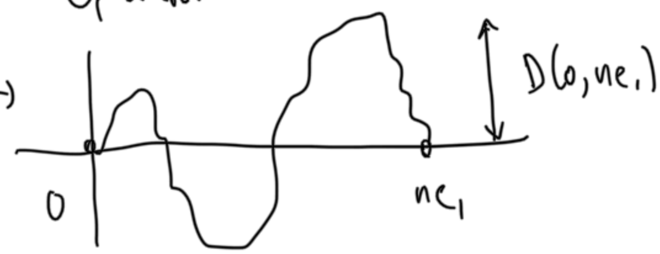
Proof ideas of Newman

Lemma (Geodesic wandering bound, Newman–Piza)

Assume A, B, II for $x = \frac{e_1}{\mu(e_1)}$, and H_x perpendicular to e_1. Then $\forall \varepsilon > 0$, $\exists\, C_1, C_2 > 0$ s.t.

$$\mathbb{P}\left(D(0, ne_1) \geq n^{\frac{3}{4}+\varepsilon}\right) \leq C_1 e^{-n^{C_2}} \quad \forall n \geq 1.$$

Here, $D(0, ne_1)$ = max dist. of geodesic from 0 to ne_1 to the e_1-axis.



In terms of the "wandering exponent" ξ, this states $\xi \leq 3/4$.

Open problem: Show $\mathbb{E}D(0, ne_1) = o(n)$ without II.

Idea of lemma

Note $\mathbb{P}\left(D(0, ne_1) \geq n^{\frac{3}{4}+\varepsilon}\right)$

$$\leq \sum_z \mathbb{P}\left(T(0, ne_1) = T(0, z) + T(z, ne_1)\right)$$

sum over z with dist. $n^{3/4+\varepsilon}$ from e_1-axis.

Claim. For $z = \frac{n}{2} e_1 + n^{3/4+\varepsilon} e_2$,

$$\mathbb{P}\left(T(0, ne_1) = T(0, z) + T(z, ne_1)\right) \leq C_1 e^{-n^{C_2}}.$$

Check: Prove a corresponding bound for other z, sum over z, and obtain the lemma.

Proof of claim. If T were ,

the event in question would be empty:

$$\mu(z) + \mu(ne_1 - z) - \mu(ne_1)$$

$$= \mu(z) - \mu\left(\tfrac{n}{2} e_1\right) + \mu(ne_1 - z) - \mu\left(\tfrac{n}{2} e_1\right)$$

$$= 2\left(\mu\left(\tfrac{n}{2} e_1 + n^{\frac{3}{4}+\varepsilon} e_2\right) - \mu\left(\tfrac{n}{2} e_1\right)\right)$$

$= n\left(\mu(e_1 + \frac{2}{n^{\frac{1}{4}-\varepsilon}} e_2) - \mu(e_1)\right)$

$\geq n c \left(\frac{2}{n^{1/4-\varepsilon}}\right)^2$ (by II)

$= 4cn^{\frac{1}{2}+2\varepsilon}.$

Approx T by μ. If the event in question occurs, then

$0 = T(0,z) + T(z,ne_1) - T(0,ne_1)$

$= [T(0,z) - \mu(z)] + [T(z,ne_1) - \mu(ne_1 - z)]$

$- [T(0,ne_1) - \mu(ne_1)]$

$+ \mu(z) + \mu(ne_1 - z) - \mu(ne_1)$

$\geq [\cdot\cdot] + [\cdot\cdot] - [\cdots]$

$+ 4cn^{1/2+2\varepsilon}.$

$\Rightarrow$ At least one of $|[\cdots]|$ is

$\geq \frac{4}{3} c n^{\frac{1}{2}+2\varepsilon}.$

Fact (Alexander, Kesten, ...)

Under A, $\exists\, C_3, C_4$ s.t.

$\mathbb{P}\left(|T(0,x) - \mu(x)| \geq \lambda\sqrt{|x|}\right)$

$\leq C_3 e^{-C_2\lambda}$

for

$C_3 \log|x| \leq \lambda \leq c_2|x|,\ x \neq 0.$

Taking $x = ne_1$,

$\mathbb{P}\left(|T(0,ne_1) - \mu(ne_1)| \geq \frac{4}{3} c n^{\frac{1}{2}+2\varepsilon}\right)$

$\leq C_3 e^{-C_2' n^{2\varepsilon}},$

and similarly for

$|T(0,z) - \mu(z)|$

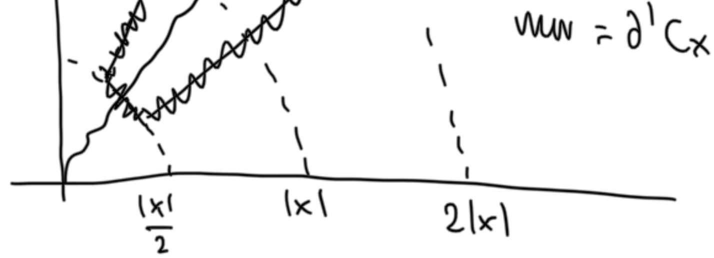
and $|T(z,ne_1) - \mu(ne_1 - z)|.$

A similar argument shows the following useful result of Newman.

Def. (1) For $x \neq 0$, $\varepsilon > 0$, let C_x be the annulus portion

$C_x = \{z \in \mathbb{R}^d : \frac{|x|}{2} \leq |z| \leq 2|x|,$

$|\arg z - \arg x| \leq |x|^{-\frac{1}{4}+\varepsilon}\}.$

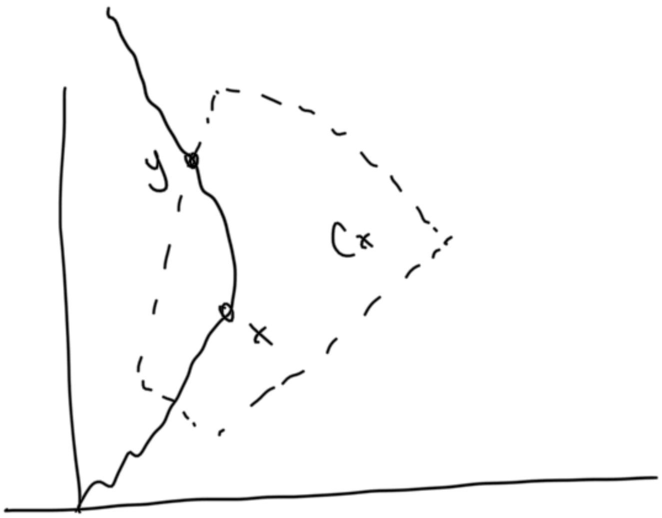


(2) $\partial' C_x = \partial C_x \setminus \{z \in \mathbb{R}^d : |z| = 2|x|\}.$

(3) $out(x) = \{y \in \mathbb{Z}^d : T(0,y) = T(0,x) + T(x,y)\}.$

④ Event

$G_x = \{\mathrm{out}(x) \cap \partial' C_x \neq \phi\}$.



Thm. Assume A, B, I, II. Given $\varepsilon > 0$, $\exists\, C_5, C_6 > 0$ s.t.

$$\mathbb{P}(G_x) \leq C_5\, e^{-|x|^{C_6}} \quad \forall x \neq 0.$$

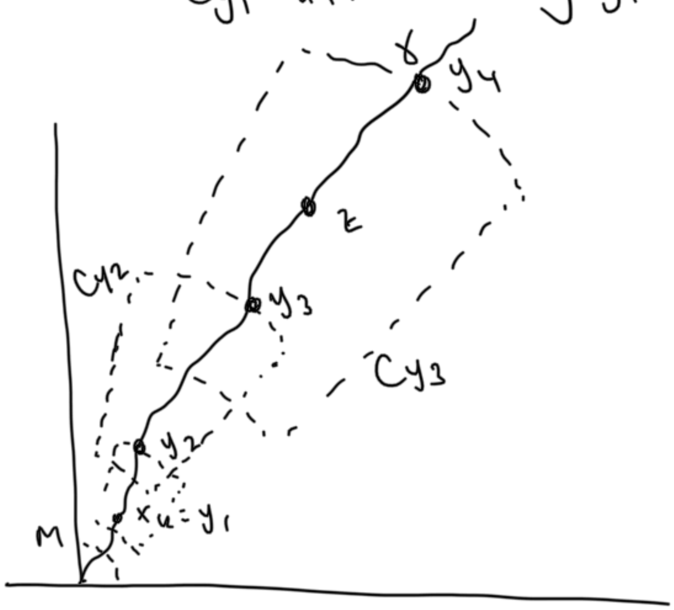
Proof of Newman etc ②

Let γ be an inf. geo. with verts $0 = x_0, x_1, x_2, \dots$ in order and choose $M > 0$ s.t. if $|x| \geq M$, then G_x^c occurs. We will show $(\arg x_n)$ is Cauchy, so choose k s.t. $|x_k| \geq M$ and pick z after x_k on γ.

Inductively, set

$y_1 = x_k$

y_{i+1} = first exit of γ from C_{y_i} after touching y_i.



z is contained in some C_{y_I}, so

$$\begin{aligned}
|\arg z - \arg x_k| &\leq \sum_{i=1}^{I-1} |\arg y_{i+1} - \arg y_i| + |\arg y_I - \arg z| \\
&\leq \sum_{i=1}^{I} |y_i|^{-\frac{1}{4}+\varepsilon} \\
&= \sum_{i=1}^{I} \left(2^{i-1} |y_1|\right)^{-\frac{1}{4}+\varepsilon} \\
&\leq C\, |x_k|^{-\frac{1}{4}+\varepsilon}
\end{aligned}$$

since γ exits far side of C_{y_i} $\forall i$, as $G_{y_i}^c$ occurs

$\Rightarrow (\arg x_n)$ is Cauchy

$\Rightarrow$ converges

Check: Prove 1 of Newman etc. by a similar method.

Lec 3

What can we say without curvature? Want to include dist. of (t_e) that are not iid.

Hoffman assumptions

① (t_e) is ergodic under integer translations and invariant under symmetries of $\mathbb{Z}^d$

② A.S., distinct paths have diff. passage times

($\Rightarrow$ a.s. unique geo)

③ $\mathbb{E}\, t_e^{d+\delta} < \infty$ for some $\delta > 0$.

④ Limit shape is bounded.

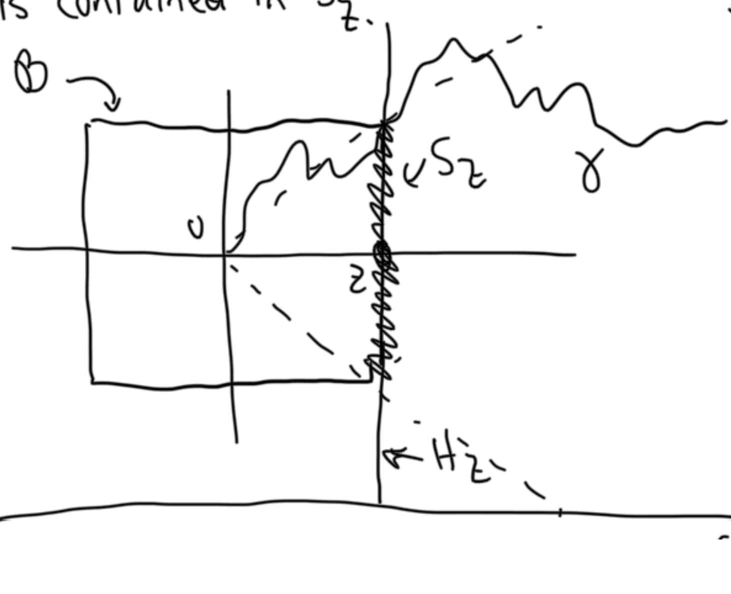
Under these conditions, limit shape can lack curvature.

Thm (Hoffman) Under assumptions, a.s. $N \geq 2d$.

Thm (Damron–Hanson) Under assumptions, let $z \in \partial\mathcal{B}$ be a point of diff. and let H_z be the supp. hyperplane at z. Let $S_z := H_z \cap \partial\mathcal{B}$.

A.S., $\exists$ inf. geo. from 0 that is directed in S_z.

In other words, if γ is the geo. and has verts. $0 = x_0, x_1, x_2, \dots$ in order, then any subseq. limit of $\left(\frac{x_n}{\mu(x_n)}\right)$ is contained in S_z.



Proof ideas for DH

For simplicity, assume that z can be taken as $z = \frac{e_1}{\mu(e_1)}$.

Main assumption

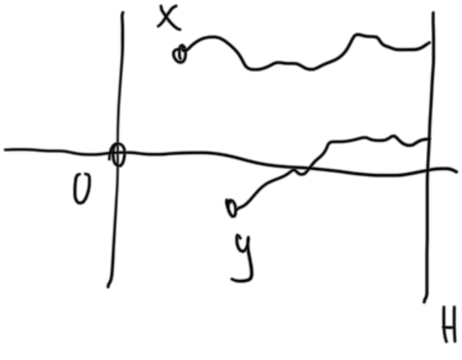
(Not made in Hoff. or DH, but we will make it to greatly simplify the situation.)

For $n \geq 0$, let $H_n := \{w \in \mathbb{R}^d : w \cdot e_1 = n\}$.

Assume a.s.

$$B(x,y) := \lim_{n\to\infty} \left[T(x, H_n) - T(y, H_n)\right]$$

exists $\forall x, y \in \mathbb{Z}^d$.



Open problem to show the assumption is valid.

- Hoffman gets around this by analyzing finite (but large) n.
- Damron-Hanson take weak subseq. limits.

As usual,

$$|B(x,y)| = \left|\lim_{n\to\infty} T(x,H_n) - T(y,H_n)\right| \le T(x,y) < \infty$$

Check: If θ is an integer translation,

$$B(x,y)(\omega) = B(\theta(x),\theta(y))(\theta(\omega)).$$

$$\Rightarrow \mathbb{E}B(x+u,y+u)(\omega) = \mathbb{E}B(x,y)(T_u\omega) = \mathbb{E}B(x,y).$$

So $\mathbb{E}B(x,y)$ exists. What is it?

Claim: $\forall x,y \in \mathbb{Z}^d$,

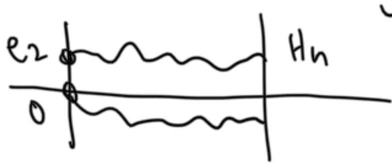
$\mathbb{E}B(x,y) = \rho\cdot(y-x)$, where

$$\rho = \mu(e_1)e_1 = (\mu(e_1),0)$$

Proof of claim

Know $\mathbb{E}B(x,y) = \mathbb{E}B(0,y-x)$. Also, $x \mapsto \mathbb{E}B(0,x)$ is additive. So, must determine $\mathbb{E}B(0,e_1)$, $\mathbb{E}B(0,e_i)$ for $i \ge 2$.

$$\mathbb{E}B(0,e_i) = \mathbb{E}\lim_{n\to\infty}\left[T(0,H_n) - T(e_i,H_n)\right] \overset{DCT}{=} \lim_{n\to\infty}\left[\mathbb{E}T(0,H_n) - \mathbb{E}T(e_i,H_n)\right] = 0 \quad \text{(by symmetry)}$$



For $\mathbb{E}B(0,e_1)$,

$$\mathbb{E}\frac{T(0,H_n)}{n} = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[T((k-1)e_1,H_n) - T(ke_1,H_n)\right]$$

$$\overset{\text{shift}}{=} \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[T(0,H_{n-k+1}) - T(e_1,H_{n-k+1})\right]$$

$$\overset{\text{re-index}}{=} \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\underbrace{\left[T(0,H_k) - T(e_1,H_k)\right]}_{\text{as } k\to\infty \text{, cvg to } B(0,e_1)}$$

$$\xrightarrow[n\to\infty]{DCT} \mathbb{E}B(0,e_1).$$

$$\Rightarrow \lim_{n\to\infty}\frac{\mathbb{E}T(0,H_n)}{n} = \mathbb{E}B(0,e_1)$$

Check: (using shape thm)

$LHS = \mu(e_1)$.

$\Rightarrow \mathbb{E}B(0,x) = \rho \cdot x$.

So, ergodic thm

$\Rightarrow \forall x \in \mathbb{Z}^d$,

$$B(0,nx) = \frac{1}{n}\sum_{k=1}^{n} B((k-1)x, kx)$$

$$= \frac{1}{n}\sum_{k=1}^{n} B(0,x)(T_x^{k-1}\omega)$$

$$\xrightarrow[\substack{a.s.\\ L^1}]{n\to\infty} \mathbb{E}B(0,x).$$

Can upgrade this again:

Thm (Busemann shape thm)

One has

$$\mathbb{P}\left(\lim_{\substack{x\in\mathbb{Z}^d\\ |x|\to\infty}} \frac{B(0,x) - \rho\cdot x}{|x|} = 0\right) = 1.$$

Let $H_B = \{y \in \mathbb{R}^d : y \cdot \rho = 1\}$.

Lem. H_B is a supporting hyperplane for $\mathcal{B}$ at $z = \frac{e_1}{\mu(e_1)}$. If $\mathcal{B}$ is diff. at z, then $H_B = H_z$.

Proof. First, $z \in \partial\mathcal{B}$ since $\mu(z) = 1$.

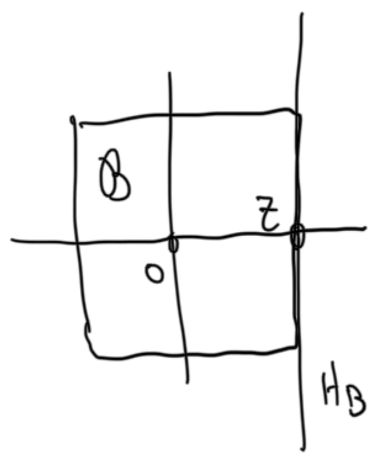
Also $z \in H_B$ since

$$z \cdot \rho = \frac{e_1}{\mu(e_1)} \cdot \mu(e_1)e_1 = 1.$$

$\Rightarrow H_B$ touches $\mathcal{B}$ at z. Want $\mathcal{B}$ to be on one side of H_B. So let $u \in \mathcal{B}$. Then $\mu(u) \le 1$.

$$u\cdot\rho = \lim_{n\to\infty}\frac{B(0,nu)}{n} \le \lim_{n\to\infty}\frac{T(0,nu)}{n} = \mu(u) \le 1.$$

$\Rightarrow H_B$ is supporting at z.



Check: let γ be any subseq. limit of geo from 0 to H_n with verts $0 = x_0, x_1, \ldots$ in order. Then $B(0,x_j) = T(0,x_j)$.

Set $S = H_B \cap \partial \mathcal{B}$.

Thm. Assume Hoffman's conditions and the "main assumption." Then any subseq. limit of geo. from 0 to H_n is directed in S.

Proof. Let γ have verts $0 = x_0, x_1, \ldots$ in order and suppose γ is such a limit. Let (x_{n_k}) be a subseq. s.t. $\dfrac{x_{n_k}}{\mu(x_{n_k})} \xrightarrow[k\to\infty]{}$ some $r \in \partial\mathcal{B}$.

We must show that $r \in H_B$; that is, $\rho \cdot r = 1$.

Have

$$\frac{B(0, x_{n_k}) - \rho \cdot x_{n_k}}{|x_{n_k}|} \to 0$$

$$\Rightarrow \frac{B(0, x_{n_k}) - \rho \cdot x_{n_k}}{\mu(x_{n_k})} \to 0.$$

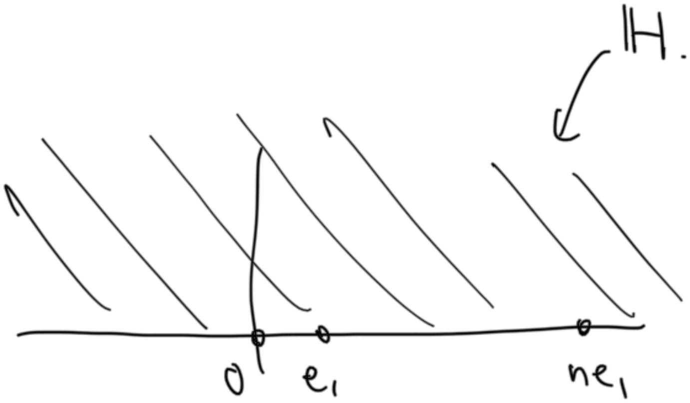
Thus

$$1 = \lim_{k\to\infty} \frac{T(0, x_{n_k})}{\mu(x_{n_k})} = \lim_{k\to\infty} \frac{B(0, x_{n_k})}{\mu(x_{n_k})}$$

$$= \lim_{k\to\infty} \rho \cdot \frac{x_{n_k}}{\mu(x_{n_k})} = \rho \cdot r.$$

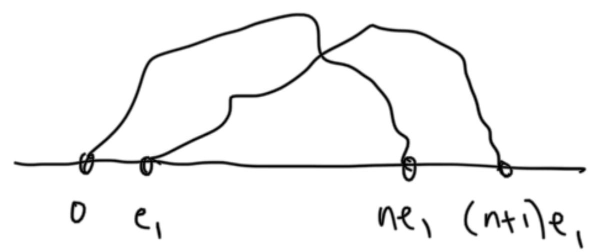
One scenario where Busemann limits are known to exist:

Consider FPP on the upper half-plane $\mathbb{H}$.



Consider

$$\lim_{n\to\infty} \left[T(0, ne_1) - T(e_1, ne_1) \right]$$



Γ_1 : geo from 0 to ne_1

Γ_2 : geo from e_1 to $(n+1)e_1$.

Γ_1 and Γ_2 <u>must</u> intersect.

$$\Rightarrow \quad T(\Gamma_1) + T(\Gamma_2) \geq T(0, (n+1)e_1) + T(e_1, ne_1).$$

$$\underset{\displaystyle T(0, ne_1) + T(e_1, (n+1)e_1)}{\parallel}$$

$$\Rightarrow$$

$$T(0, ne_1) - T(e_1, ne_1) \geq T(0, (n+1)e_1) - T(e_1, (n+1)e_1).$$

⇒ monotone
⇒ limit exists.

"paths crossing" argument

Thm (Auffinger – Damron – Hanson)

① $\forall x, y \in \mathbb{H}$,

$$B(x,y) = \lim_{n\to\infty} \left[T(x, ne_1) - T(y, ne_1) \right]$$

exists surely.

② $\forall x \in \mathbb{H}$, the geodesics Γ_n^x from x to ne_1 have a limit Γ_x a.s.
((t_e) iid, cts)

(A) $\forall x, y \in \mathbb{H}$, Γ_x, Γ_y a.s. coalesce

(B) For $x \in \mathbb{H}$, define

$B_x = \{ y \in \mathbb{H} : \Gamma_y \ni x \}$.

A.S., $\forall x \in \mathbb{H}$, $\# B_x < \infty$.

"finite backward clusters"