Sequential information theoretic protocols in continuous variable systems

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Introduction

Quantum communication protocol

Uses quantum mechanics for secure and efficient information transfer.

Exploits quantum states, superposition, and entanglement.

Entanglement: Essential for protocols like teleportation, telecloning, dense coding

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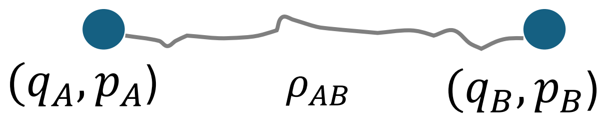

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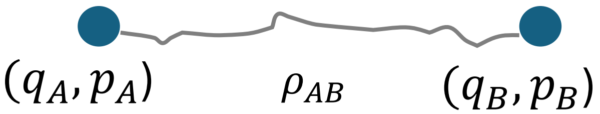

For a two-mode state, can we detect its entanglement?

$$\xi_{AB} = \langle \Delta^2(\hat{q}_A - \hat{q}_B) \rangle + \langle \Delta^2(\hat{p}_A + \hat{p}_B) \rangle < 2$$

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(q_A, p_A) ρ_{AB} (q_B, p_B)

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If the protocol fails, can we reuse the same resource in the next attempt?

Outline

1. CV quantum system
2. Resource reusability based on unsharp measurement
    A. Motivation
    B. Sequential detection of non-classicality
3. Reusability of resource via resource-splitting and Sequential teleportation
    A. Motivation
    B. Sequential teleportation using reusable resource
4. Conclusions

CV quantum system

Quantum Information

Discrete Variable

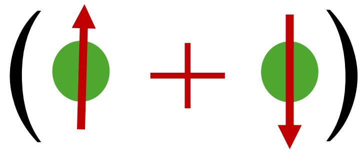
Continuous Variable

CV quantum system

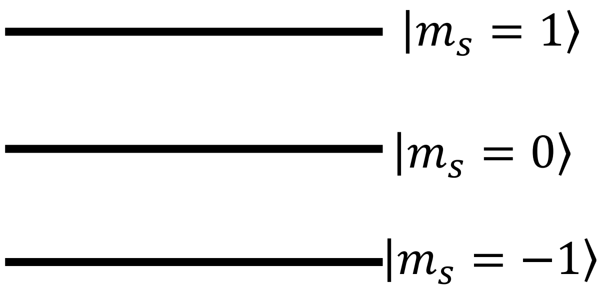
Quantum Information

Discrete Variable | **Continuous Variable**

- Information is encoded in distinct, separate levels (Finite dimension Hilbert space). Ex.: qubit, qudit

$|\psi\rangle = \frac{1}{\sqrt{2}}$ 

Qubit (Spin ½ Particle)



Spin-1 Particle: $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |0\rangle + |-1\rangle)$

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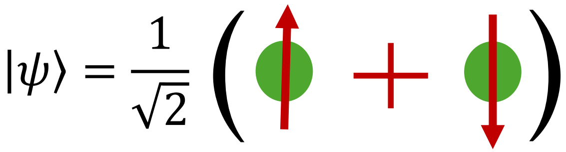

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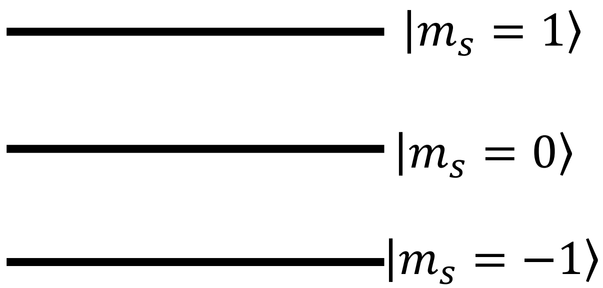

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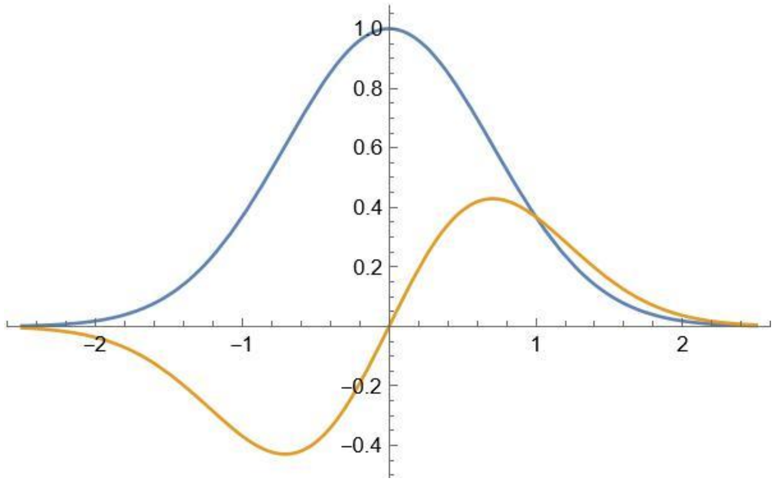
- Information is encoded in continuous spectra, such as the position and momentum (Infinite dimension Hilbert space). Ex.: Coherent state, TMSV



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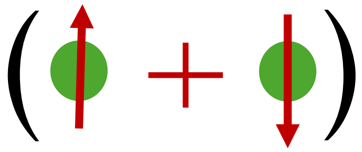
$|\psi\rangle = \frac{1}{\sqrt{2}}$ 

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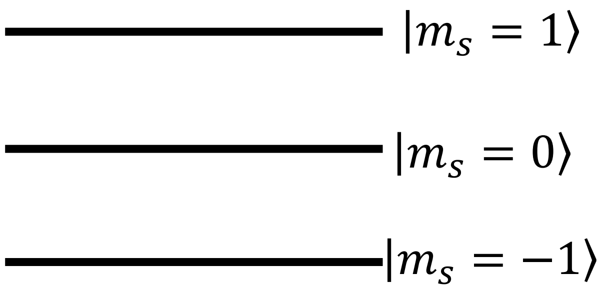

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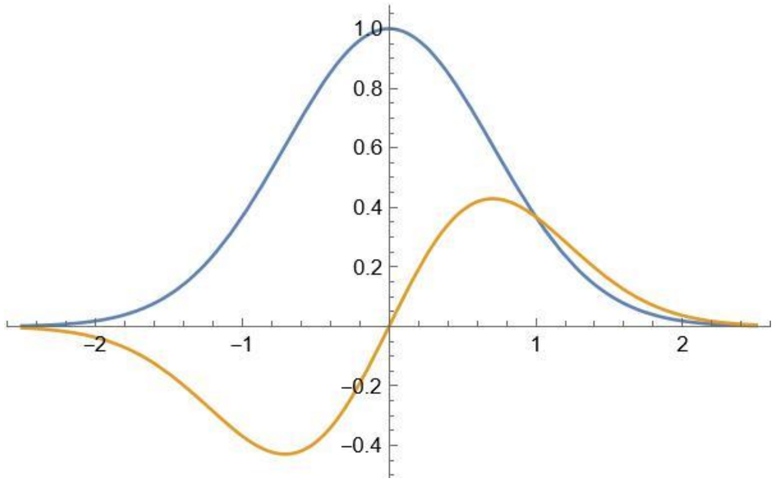

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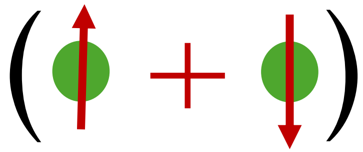

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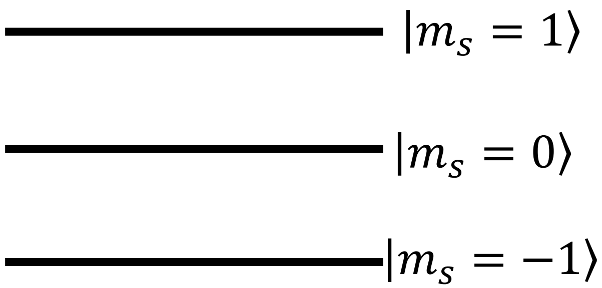

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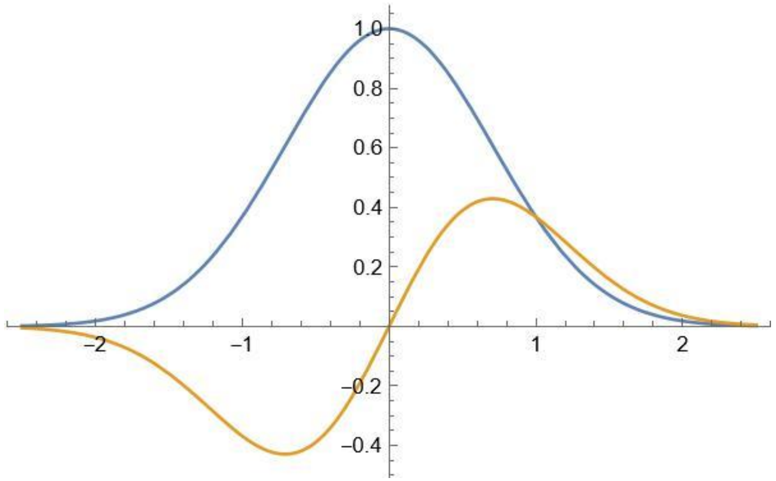


- Homodyne and Heterodyne measurement. Measurements yield a continuum of possible outcomes.

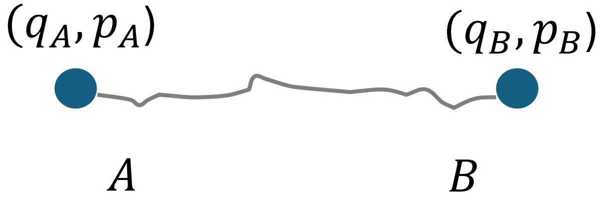
Ex.: Outcome of $\hat{x}$ on $|\alpha\rangle \Rightarrow x$

Resource reusability based on unsharp measurement

❖ Necessary and sufficient condition for quantum teleportation and telecloning*:

$$\xi_{AB} = \langle \Delta^2(\hat{q}_A - \hat{q}_B)\rangle + \langle \Delta^2(\hat{p}_A + \hat{p}_B)\rangle < 2$$

Duan Criterion$



*S Das, R Gupta, HSD, ASD Phys. Rev. A **110**, 012410

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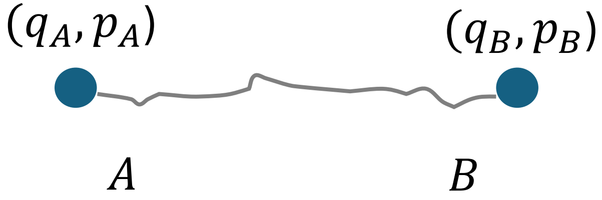

Aim

- To determine whether a given resource is entangled.
- If the resource is entangled, how many times it can be sequentially verified for entanglement?

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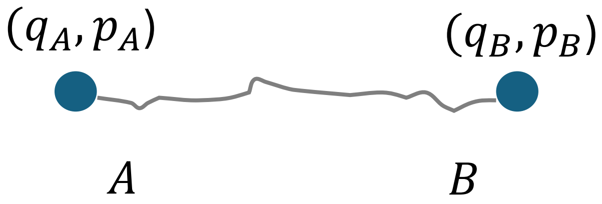

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To do this

- Projective measurement → Destroys the resource, making it unusable for further applications.
- Unsharp measurement ✓

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Combine squeezed state (auxiliary/ meter state) $\Longrightarrow (mA, mB)$

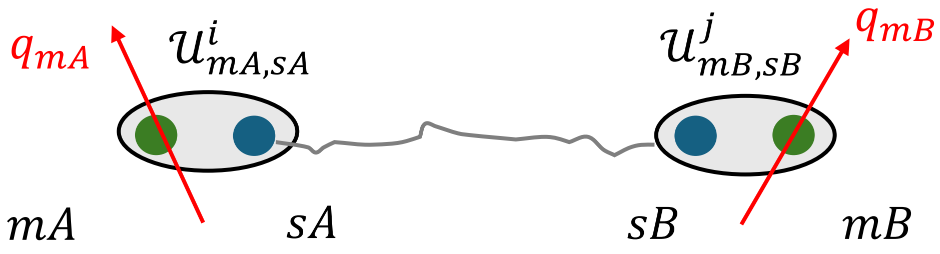
Local unitary evolution on auxiliary-system pair*:

$$(\rho_{ms})_{i,j}{}' = \left(\mathcal{U}_{ms}^{ij}\right)\rho_{ms}\left(\mathcal{U}_{ms}^{ij}\right)^{\dagger}$$

Local projector for auxiliary/ meter:

$$E_{AB} = |q_{mA}\rangle\langle q_{mA}| \otimes I_{sA,sB} \otimes |q_{mB}\rangle\langle q_{mB}|;$$

$$\int E_{AB}\, \mathcal{D}[q_{mA}]\mathcal{D}[q_{mB}] = \mathrm{I}$$



m= meter and s= system (resource)

$$\mathcal{U}_{ms}^{ij} = \mathcal{U}_{mA,sA}^{i} \otimes \mathcal{U}_{mB,sB}^{j}$$

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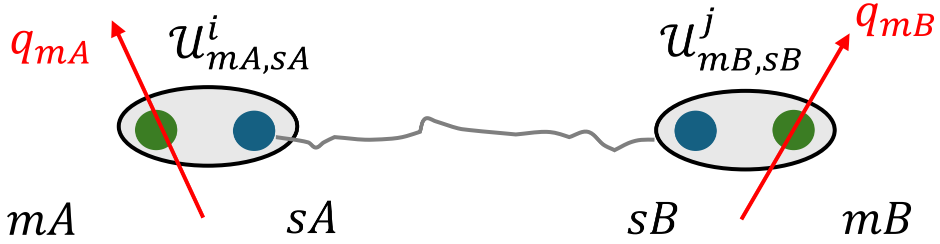

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If $\mathrm{i} = q_{sA}$:

$$\mathcal{H}_{mA,sA}^{i} = \delta(t - t_0) q_{sA} P_{mB}$$

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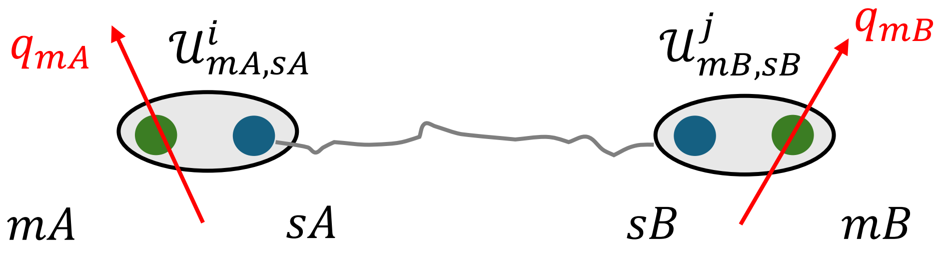

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If $\mathrm{i} = q_{sA}; \mathrm{j} = p_{sB}$:

$E_{AB} \Rightarrow$ Partial information about q_{sA} and p_{sB}

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Probability of getting (q_{mA}, q_{mB}):

$$P(q_{mA}, q_{mB})_{i,j} = Tr\left[(\rho_{ms})_{i,j}{}' . \mathrm{E}_{AB}\right]$$

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Non-classicality measure:

$$\langle\Delta^2(q_{sA}-q_{sB})\rangle_{measure}=\langle\Delta^2(q_{mA}-q_{mB})\rangle$$

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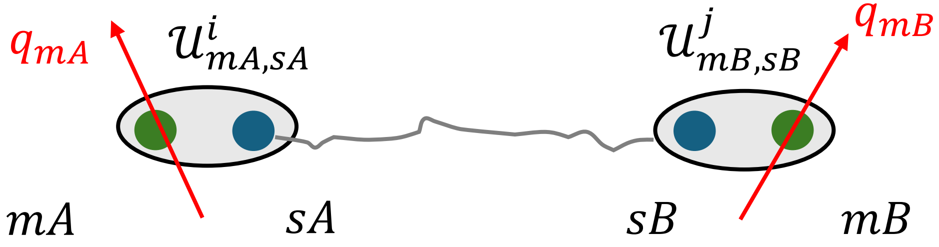

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$$\zeta(sA, sB)_{measure} = 2e^{-2r} + \langle \Delta^2 \hat{q}_{mA} \rangle + \langle \Delta^2 \hat{q}_{mB} \rangle$$

If $\langle \Delta^2 \hat{q}_{mA} \rangle = \langle \Delta^2 \hat{q}_{mB} \rangle = 0$ (Projective measurement)

(q_{mA}, q_{mB}) ≡ quadrature of the system

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r: squeezing of resource

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Local unitary evolution on meter-system pair:

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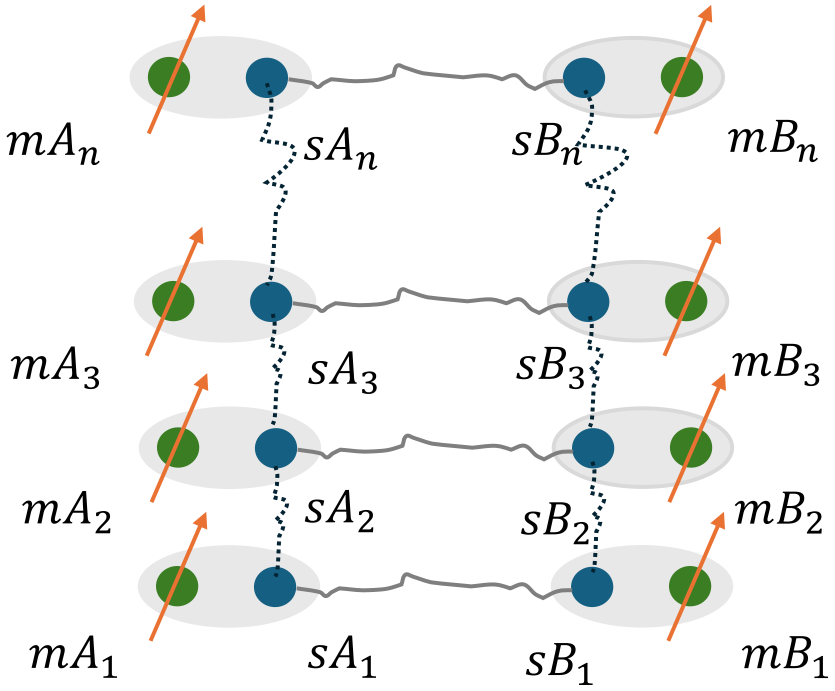
Average Post measurement state of the system:

$$\rho_{A_1,B_1} = \frac{1}{4} \sum_{i,j} (\rho_s)'_{i,j}$$

Sequential detection of non-classicality

❖ Non-classicality of n_{th} pair (A_n, B_n)

$$\zeta(sA_n, sB_n) = \frac{1}{2}\sum_{i=1}^{n-1}\left[\left[\langle\Delta^2\hat{p}_{mA_i}\rangle + \langle\Delta^2\hat{p}_{mB_i}\rangle\right]\right] + 2e^{-2r} + \langle\Delta^2\hat{q}_{mA_n}\rangle + \langle\Delta^2\hat{q}_{mB_n}\rangle$$



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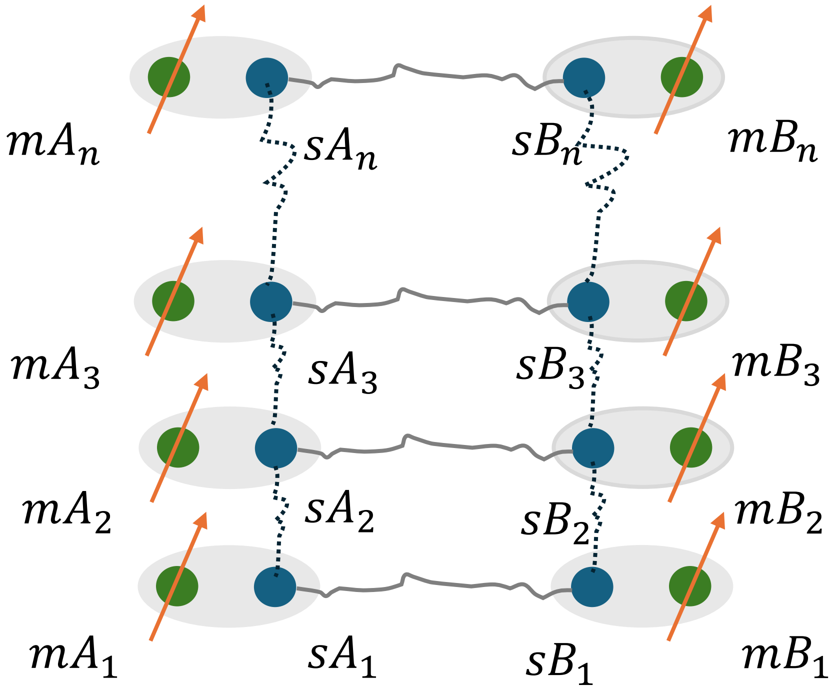


➢ For equal weakness parameter:

$$\langle\Delta^2\hat{q}_{A_i}\rangle = \omega^2 \ \& \ \langle\Delta^2\hat{p}_{A_i}\rangle = \frac{1}{4\,\omega^2};\ \forall i \in n \qquad \Longrightarrow n_{max} = 3 - 2e^{-2r}(2 - e^{-2r}) \ \text{ at } \ \omega = \sqrt{e^{-r}\sinh r}$$

1. For $\frac{1}{2}\log[2+\sqrt{2}] \leq r < \infty$; $n_{max} = 2$
2. $0 < r < \frac{1}{2}\log[2+\sqrt{2}]$; $n_{max} = 1$

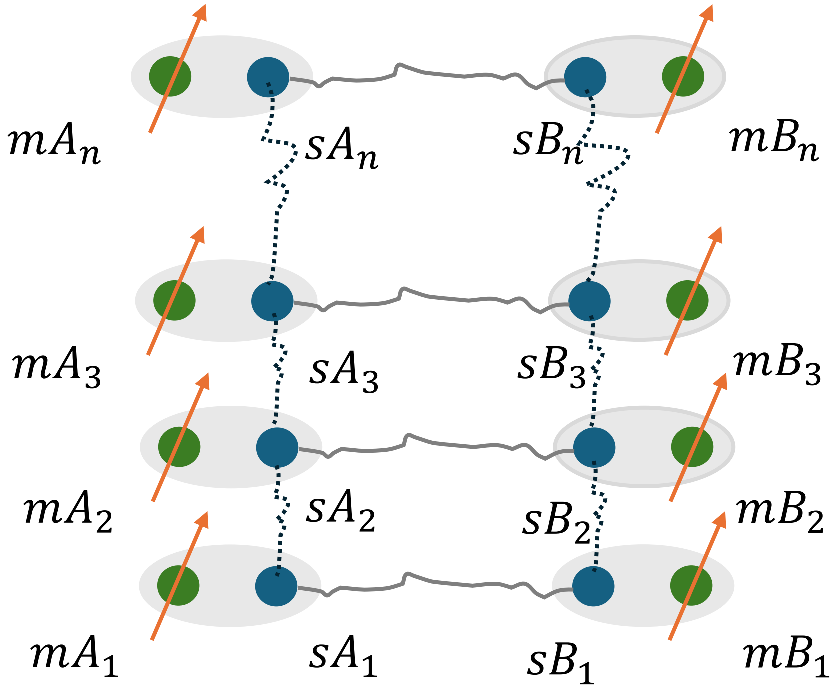
Sequential detection of non-classicality

- For different weakness parameter in the last round:

$$\begin{cases} \langle \Delta^2 \hat{q}_{A_i} \rangle = \omega^2 \ \& \ \langle \Delta^2 \hat{p}_{A_i} \rangle = \frac{1}{4\,\omega^2} \quad \text{for } i \le n-1 \\ \langle \Delta^2 \hat{q}_n \rangle = \omega_n^2 \ \& \ \langle \Delta^2 \hat{p}_{A_n} \rangle = \frac{1}{4\,\omega_n^2} \end{cases}$$

$$\zeta(A_n, B_n) < 2$$

$$\left. \begin{array}{l} r \ge r_{th} = \frac{1}{2} \log \left[\frac{8\omega^2}{1 - n + 8\omega^2(1-\omega_n^2)} \right] \\ 0 < \omega_n \le 1 \ \text{ and } \omega = \frac{\sqrt{n-1}}{2\sqrt{2(1-\omega_n^2)}} \end{array} \right\} n_{max}$$



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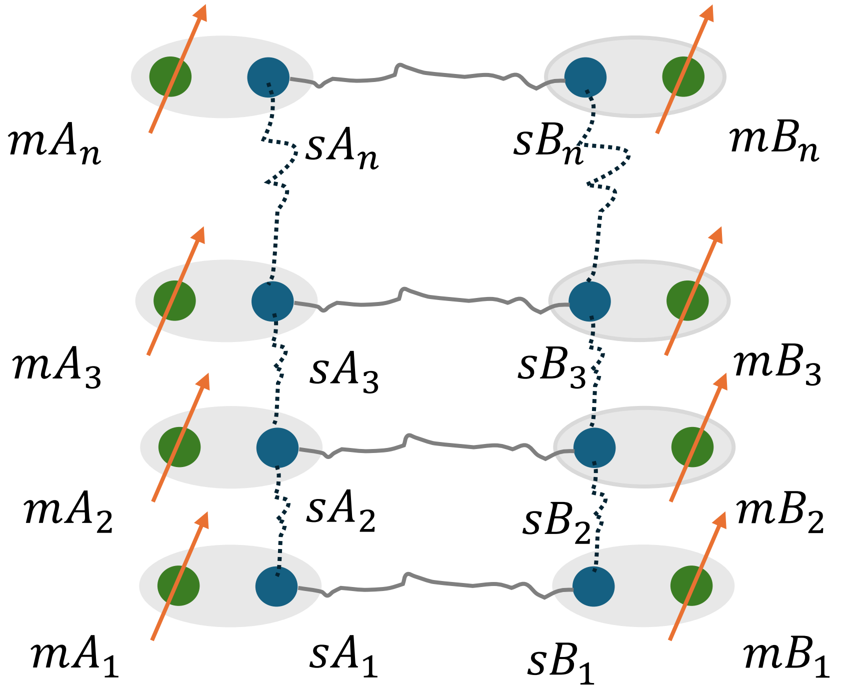
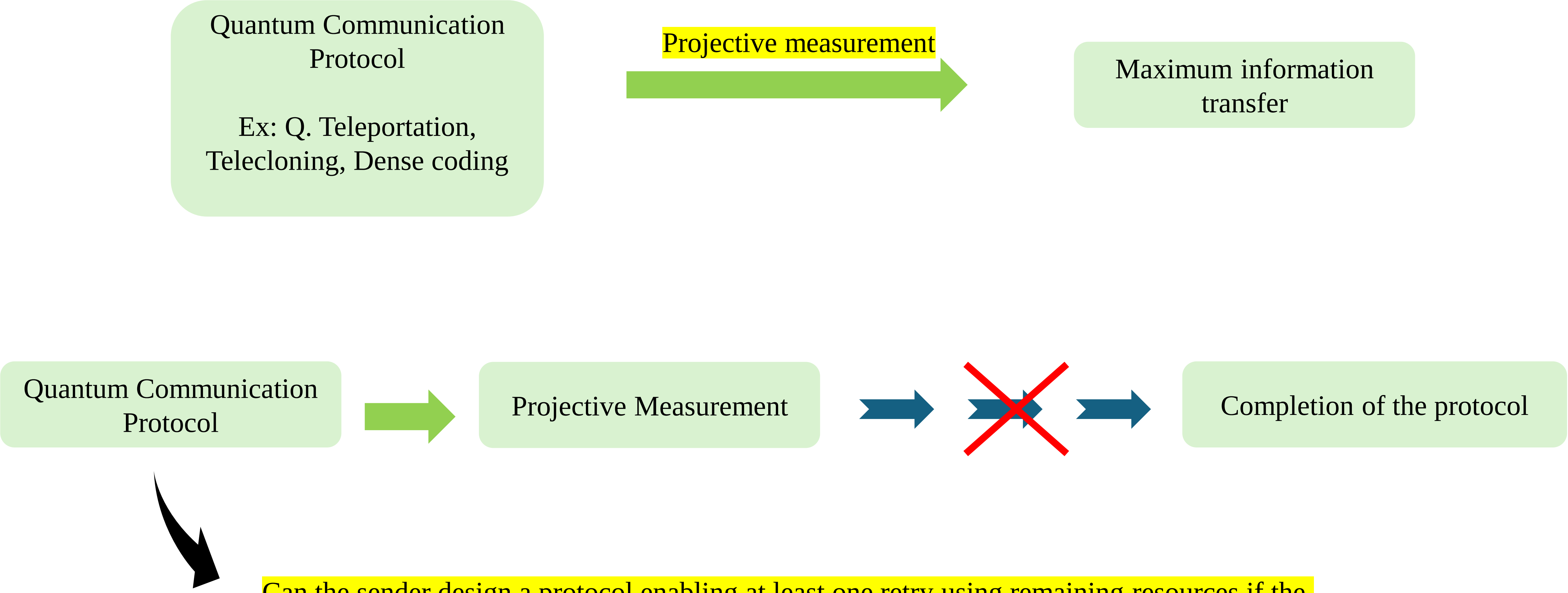
$\Rightarrow n_{max}$



Ex.: If $r_{th} = 0.509285$; $\omega_n = 0.5$ & $\omega = 1.08012$

$n_{max} \le 8$

Reusability of resource via resource-splitting



Can the sender design a protocol enabling at least one retry using remaining resources if the initial attempt fails?

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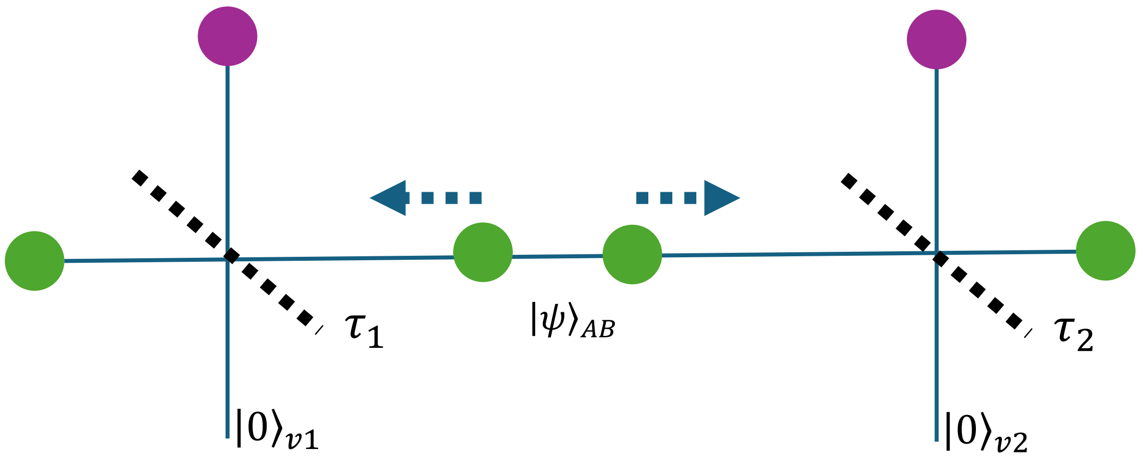

- Initial state: $\rho_{in} = \rho_{v1} \otimes \rho_{AB} \otimes \rho_{v2}$
- Final state: $\rho_{out} = \mathcal{U}(\tau_A, \tau_B)\rho_{in}\mathcal{U}(\tau_A, \tau_B)^\dagger$

- Transmitted state: $\rho_{A_T B_T} = Tr_{A_R B_R}[\rho_{out}]$
- Reflected state: $\rho_{A_R B_R} = Tr_{A_T B_T}[\rho_{out}]$

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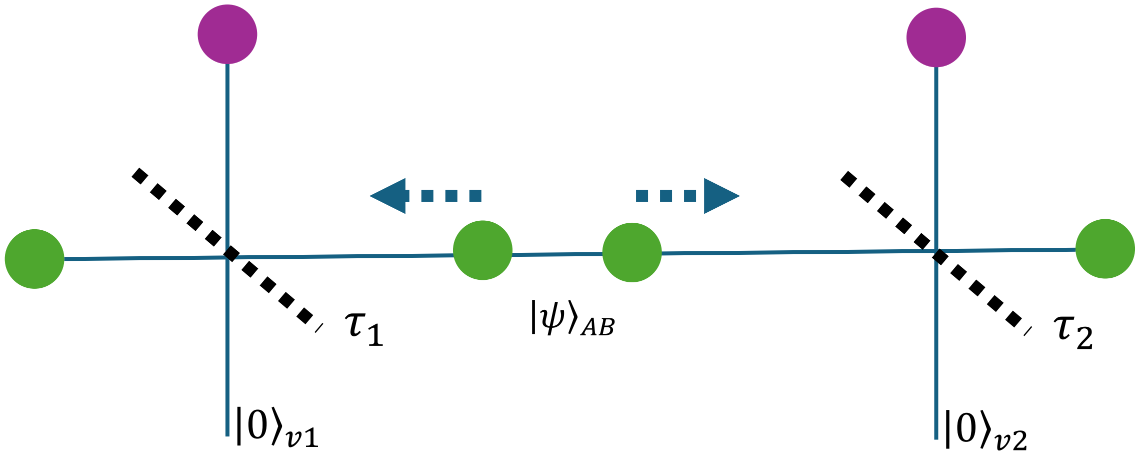

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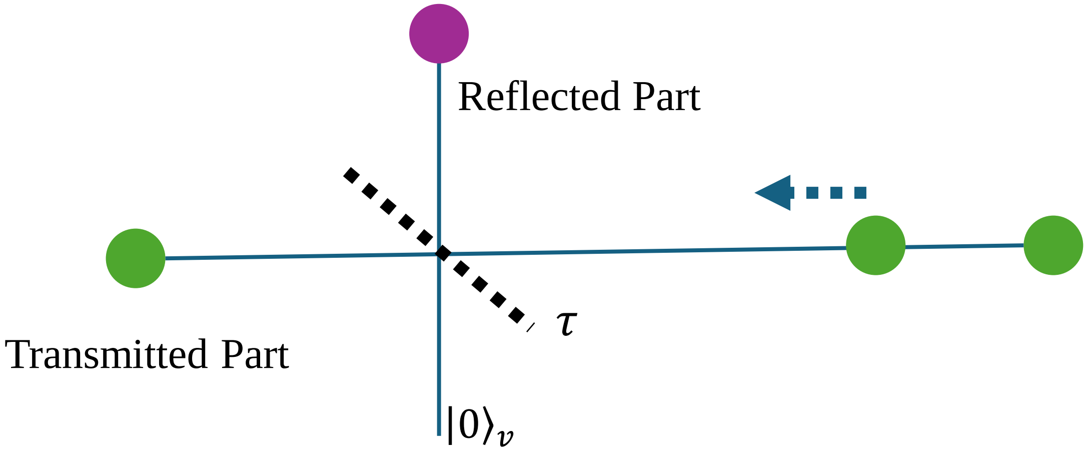
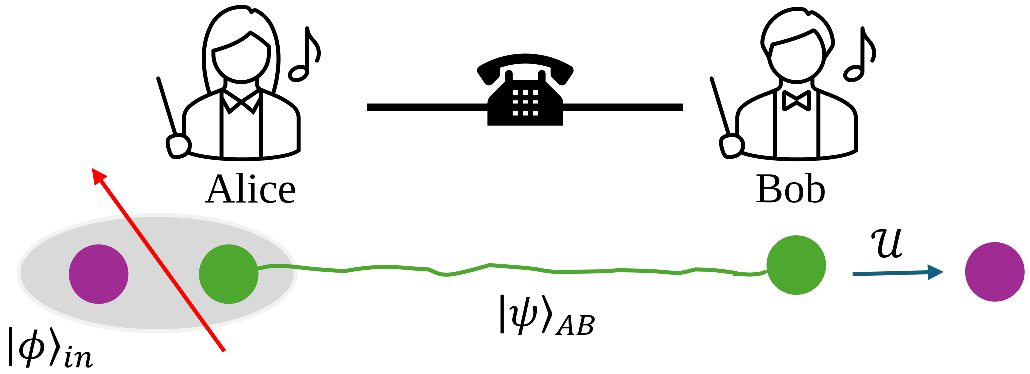
- Reflected state: $\rho_{A_R B_R} = Tr_{A_T B_T}[\rho_{out}]$

- Transmitted state: $\rho_{A_T B}$
- Reflected state: $\rho_{A_R B}$



Sequential teleportation using reusable resource

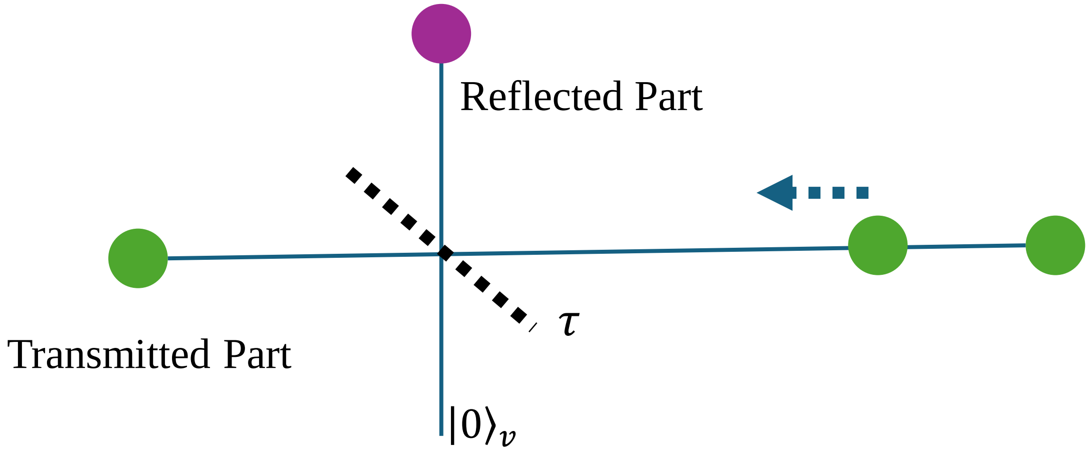


$|\psi\rangle_{AB}$: Two-mode Gaussian state (Ex: TMSV)

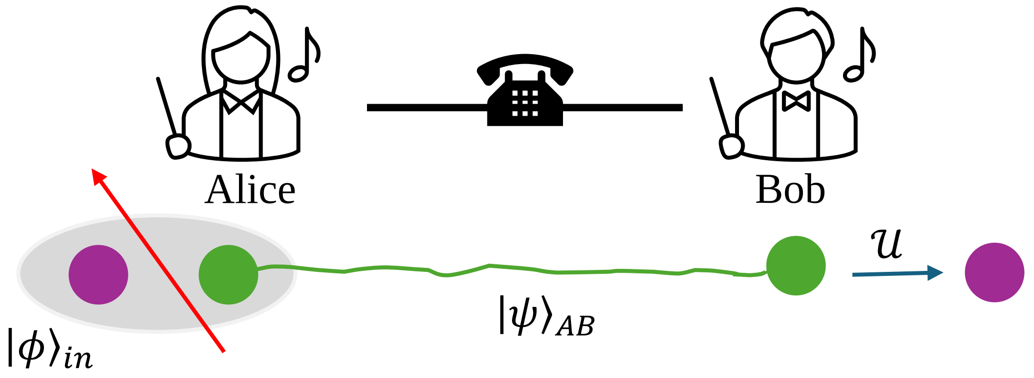
- The fidelity of the teleported state, achieved when Bob successfully completes the protocol, is:

$$\mathcal{F} = Tr[\rho_{in}\rho_{out}]$$

$$= \frac{2}{3 - \tau + (1+\tau)cosh(2r) - 2\sqrt{\tau}sinh(2r)}$$



Sequential teleportation using reusable resource



$|\psi\rangle_{AB}$: Two-mode Gaussian state (Ex: TMSV)

❖ The fidelity of the teleported state, achieved when Bob successfully completes the protocol, is:

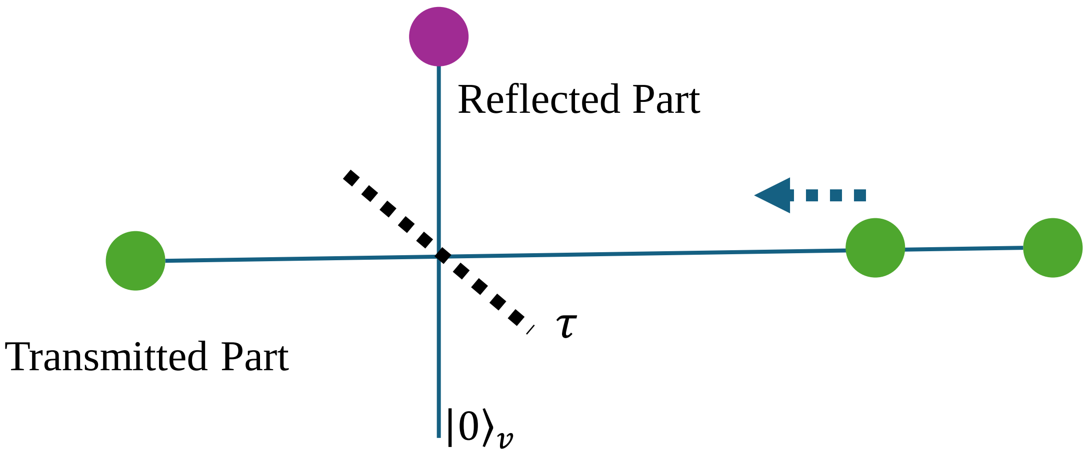
$$\mathcal{F} = Tr[\rho_{in}\rho_{out}]$$

$$= \frac{2}{3 - \tau + (1+\tau)cosh(2r) - 2\sqrt{\tau}sinh(2r)}$$

❖ If Bob fails to complete the protocol, n-1 times, the fidelity of nth attempt is:

$$\mathcal{F}_n = \frac{2}{3 - \tau^{(n)} + (1+\tau^{(n)})cosh(2r) - 2\sqrt{\tau^{(n)}}sinh(2r)}$$

$$\tau^{(n)} = (1-\tau_1)(1-\tau_2) \quad \ldots\ldots(1-\tau_{n-1})\tau_n$$

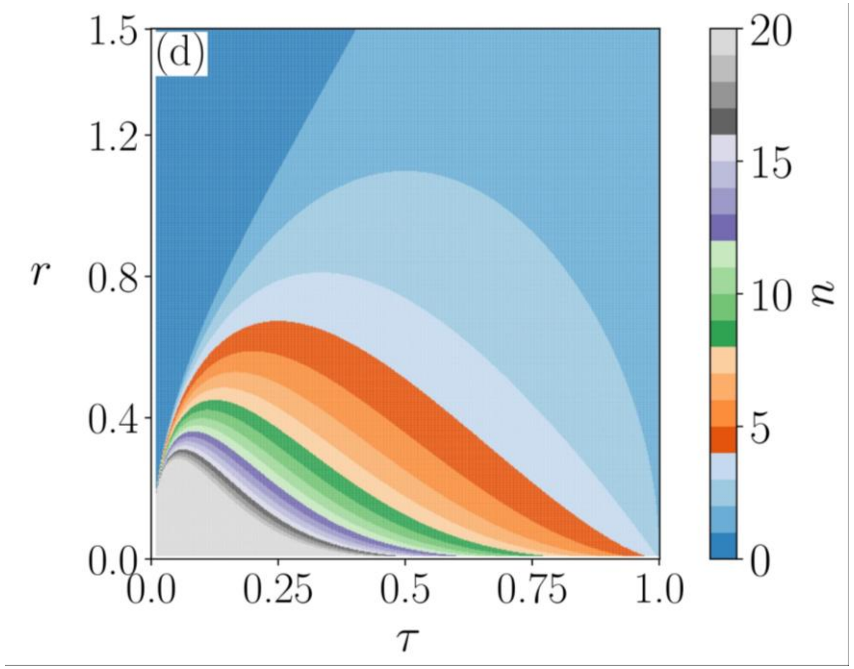


Sequential teleportation using reusable resource

❖ Equal Transmissivity: $\tau_1 = \tau_2 = \tau_3 = \cdots = \tau_n = \tau$

$\tau^{(n)} = (1-\tau)^{n-1}\tau$

$$\mathcal{F}_n = \frac{2}{3 - \tau^{(n)} + (1+\tau^{(n)})cosh(2r) - 2\sqrt{\tau^{(n)}}sinh(2r)}$$

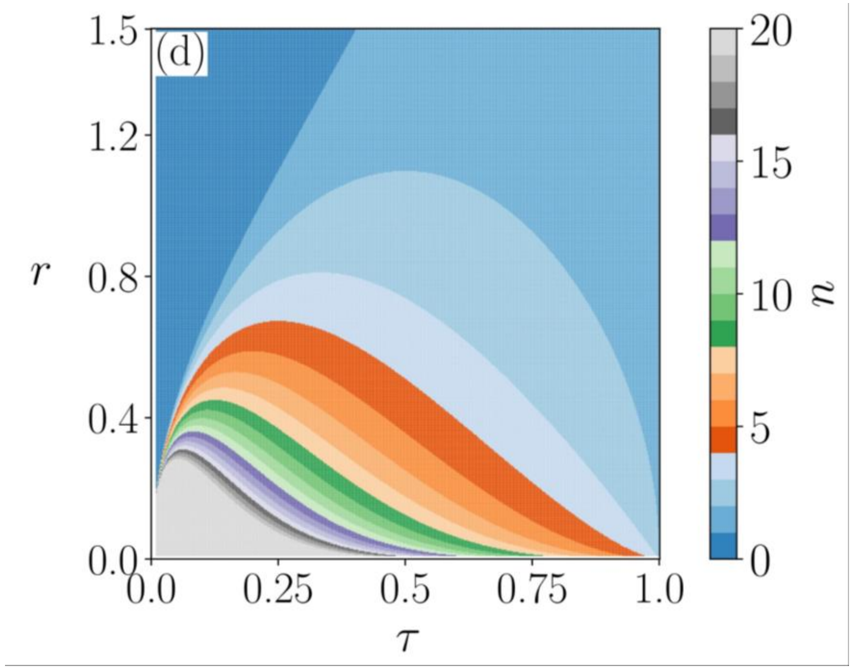


Sequential teleportation using reusable resource

- Equal Transmissivity: $\tau_1 = \tau_2 = \tau_3 = \cdots = \tau_n = \tau$

$$\tau^{(n)} = (1-\tau)^{n-1}\tau$$

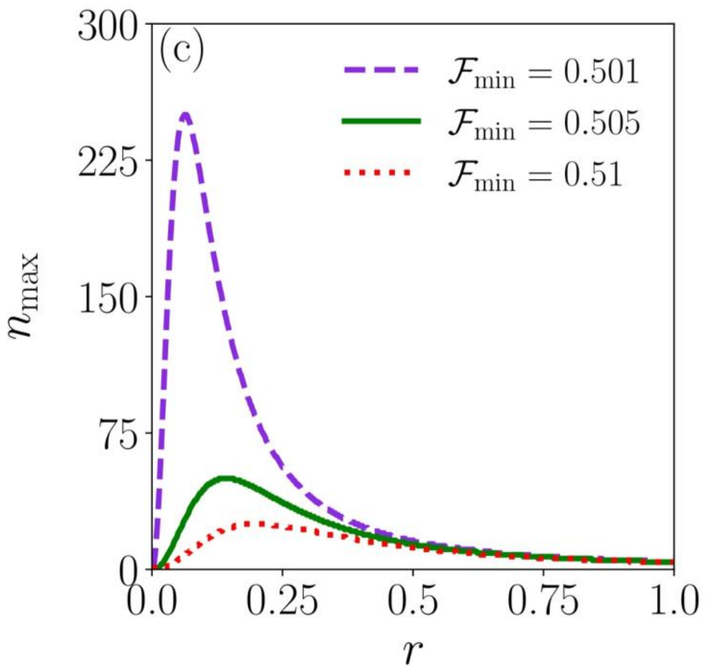
$$\mathcal{F}_n = \frac{2}{3 - \tau^{(n)} + (1+\tau^{(n)})cosh(2r) - 2\sqrt{\tau^{(n)}}sinh(2r)}$$



- Equal Fidelity: $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \cdots = \mathcal{F}_n = \mathcal{F}_{min} > 1/2$

$$\tau_i = \frac{\tau_{i-1}}{1-\tau_{i-1}} \longrightarrow \tau_1 = \frac{1}{n}$$

$$\mathcal{F}_{min} = \frac{2n}{3n - 1 + (1+n)cosh(2r) - 2\sqrt{n}sinh(2r)}$$



Conclusions

1. We introduced two strategies—resource splitting and unsharp quadrature measurements—to enable the reusability of continuous variable (CV) quantum resources in sequential quantum information protocols.
2. The resource-splitting scheme allows an initial resource state to be divided into multiple lower-resource copies, enabling multiple rounds of quantum protocols while maintaining quantum advantage.
3. We applied this scheme to CV teleportation and analyzed the trade-offs between fidelity, resource splitting, and initial squeezing, revealing constraints on the maximum number of successful sequential teleportation.
4. Our study on unsharp quadrature measurements demonstrated that entanglement detection can be sequentially carried out multiple times, with an intriguing possibility of unbounded detection under specific conditions.
5. This work highlights the potential for reusing expensive CV quantum resources and lays the groundwork for future applications in quantum communication, cryptography, and computation.

Thank You

Acknowledgement: *Prof. Himadri Shekhar Dhar*, *Prof. Aditi Sen De*, Ayan Patra, *Rivu Gupta*