Algebraic construction of optimal recovery for arbitrary quantum noise channels

Debjyoti Biswas, Prabha Mandayam

On arXiv soon
January 21, 2025

Noise

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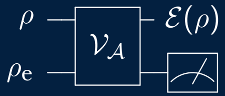

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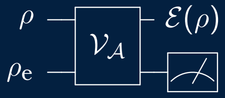

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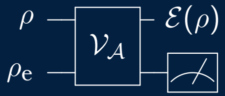

- One of the dominant noise processes in several physical realization of qubits (Superconducting qubits) is the Amplitude damping (AD) noise.
- Single qubit AD channel: $D_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$

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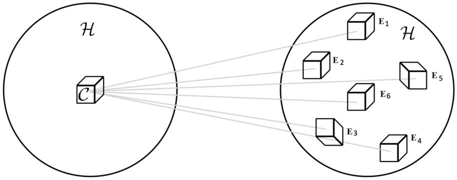
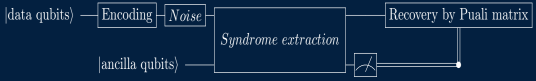
Quantum error correction: Perfect Error Correction





- Knill-Lafllame condition: $PA_i^\dagger A_j P = \lambda_{ij} P$
- At least five qubits are necessary to correct arbitrary single qubit noise.
- There exists a recovery $\mathcal{R} \sim \{PU_i^\dagger\}$[a].

- First to detect which qubit get affected by the noise process.
- Apply the recovery accordingly.

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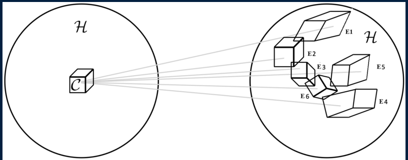
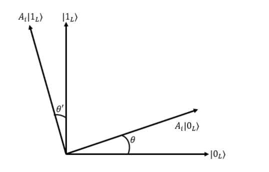
Approximate Quantum Error Correction (AQEC)





- Beny- Oreskov condition:
  $\underbrace{PA_i^\dagger A_j P}_{M} = \lambda_{ij} P + PB_{ij}P.$
  (*C. Bény et al. PRL. 104, 120501*)
- The error subspaces are not orthogonal to each other. The unitarity (or deformability) condition gets violated.
- For a $t-$ error correcting code the deformation $\sim \langle m_L | B_{ij}^{mn} | n_L \rangle$ should be small $\sim \epsilon^{t+1}$. (ϵ is the noise strength)
- $F^2 = \langle \psi | \mathcal{R} \circ \mathcal{E}(|\psi\rangle\langle\psi|) | \psi \rangle \sim 1 - \mathcal{O}(\epsilon^{t+1})$

What is the recovery ?

Example: [4,1]-Leung code [a] and AD noise

[a] Debbie Leung *et al.* Phys. Rev. A 56, 2567 (1997)

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \qquad |1_L\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)$$

- Stabilizer generator : $\langle XXXX, IIZZ, ZZII \rangle$.
- These operators are not sufficient to detect which qubit has faced the damping [2].

Errors	ZZII	IIZZ
D_{1000}	-1	+1
D_{0100}	-1	+1
D_{0010}	+1	-1
D_{0001}	+1	-1

Table: Syndrome Table 1

[2] Andrew Fletcher *et al.* arXiv:0710.1052v1 (2007)

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Table: Syndrome Table 2.

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The Algorithm

- Given the actual noise operators $\{A_k\}$

$$PA_k^\dagger A_l P = \lambda_{kl} P + PB_{kl} P \longrightarrow \langle m_L | E_k^\dagger E_l | n_L \rangle = \delta_{kl}\delta_{mn}\beta_k^m \tag{1}$$

- We start with the choice $E_1 = A_1$.
- Construct $E_2 = A_2 - U_1 P_1 U_1^\dagger A_2 \leftarrow U_1 \leftarrow E_1 P = U_1 \sqrt{PE_1^\dagger E_1 P}$. P_1 is projector onto the non-null space of $\sqrt{PE_1^\dagger E_1 P}$.
- By construction $PE_1^\dagger E_2 P = 0$.
- We generate the k^{th} operator $E_k = A_k - \sum_{i=1}^{k-1} U_i P_i U_i^\dagger A_k$.

  The operator P_i is the projector onto the non-null space of $E_i P$, i.e, a space spanned by the eigenvectors of $E_i P$ with non-zero eigenvalues. U_is are the polar decomposition unitary of $E_i P$.
- $U_i^\dagger U_i = \delta_{ij} \implies \sum_i E_i^\dagger E_i \leq I$.

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Recovery ?

Recovery

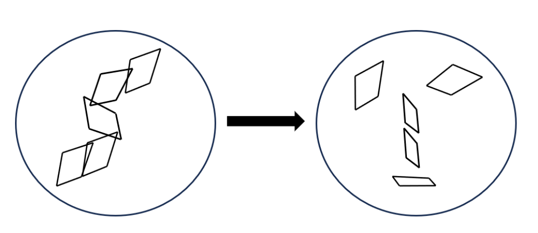

Figure: Subspaces after the orthogonalisation.

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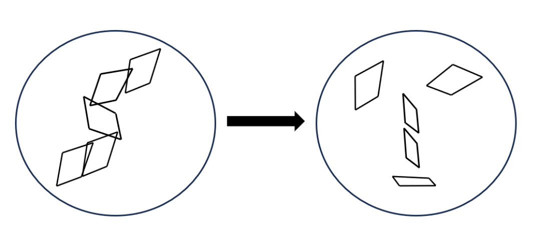

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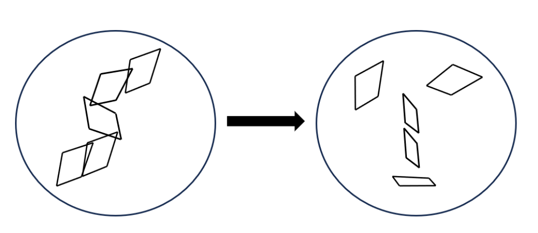

Figure: Subspaces after the orthogonalisation.

- We have $\langle m_L | E_k^\dagger E_l | n_L \rangle = \delta_{kl} \delta_{mn} \beta_k^m$.
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- One can easily verify that $P U_k^\dagger U_l P = \delta_{kl} P$.

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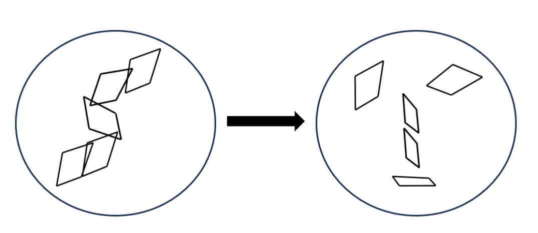

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- We have $\langle m_L | E_k^\dagger E_l | n_L \rangle = \delta_{kl}\delta_{mn}\beta_k^m$.
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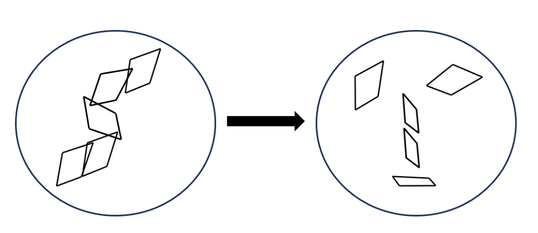

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- After the orthogonalisation the deformation is still there.
- One can easily verify that $PU_k^\dagger U_l P = \delta_{kl} P$.
- We design a recovery $R \sim \{PU_k^\dagger\}$ Not optimal.

Validation of the Recovery

- The aim is to recovery or map these orthogonal subspaces which we have obtained from the channel with Kraus $\{E_i\}$ to the code space.

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- $E_i|\psi\rangle$ & $A_i|\psi\rangle$ are linearly dependent. $\implies$ $|\langle\psi|E_i^\dagger A_i|\psi\rangle| = \sqrt{\langle\psi|E_i^\dagger E_i|\psi\rangle}\sqrt{\langle\psi|A_i^\dagger A_i|\psi\rangle}$
- This bound is reflected in the fidelity as well.

Recovery for approximate quantum error correction (QEC)

Worst-case fidelity $F^2_{\min} = \min_{|\psi\in\rangle\mathcal{C}} \langle\psi|\mathcal{R}\circ\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle$

- Through a numerical search (semi-definite programming (SDP)) we can obtain an optimal recovery [3]. But it hard to execute the SDP.
- There exists an analytical way to construct a near-optimal [4] and universal recovery [5] - - this recovery is known as the Petz map.

$$\mathcal{R}_{P,\mathcal{E}} \sim \{PE_i^\dagger \mathcal{E}(P)^{-1/2}\}$$

- Note that for a perfect code $\mathcal{R}_{P,\mathcal{E}} \sim \{\frac{1}{\sqrt{\lambda_{ii}}} PE_i^\dagger\}$ [4].
- The Petz map is an optimal recovery [6] under the measure of entanglement fidelity ($F^2_{Ent} = \frac{1}{d^2}\sum_{k,l} |\mathrm{Tr}(R_k E_l)|^2$),

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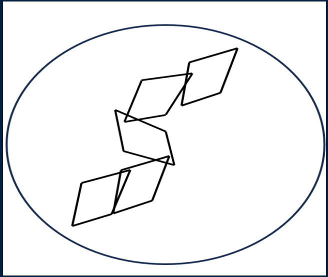
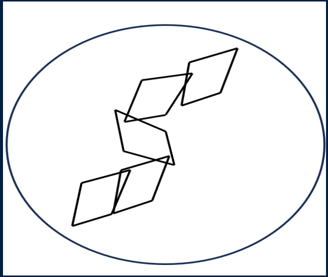
Route to optimal recovery



(a) After the noise

Route to optimal recovery



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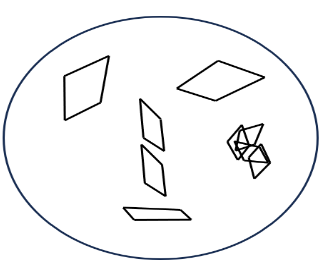

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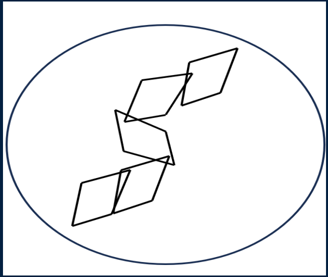

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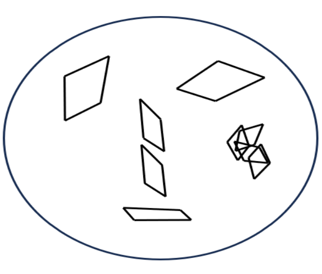

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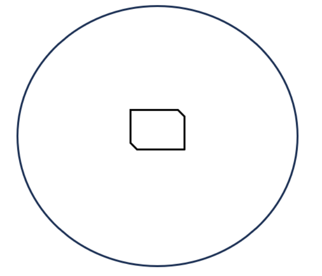

(c) After Petz Recovery

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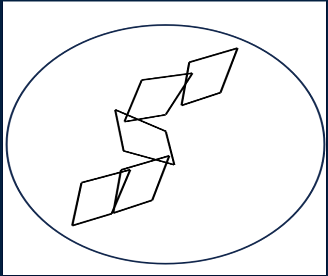

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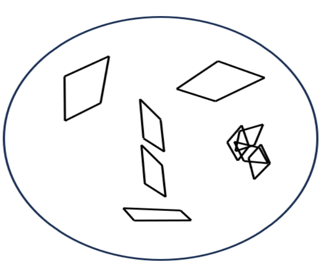

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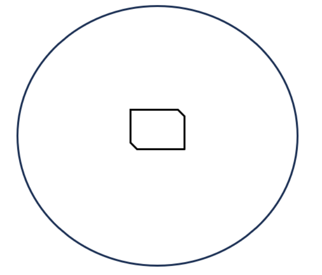

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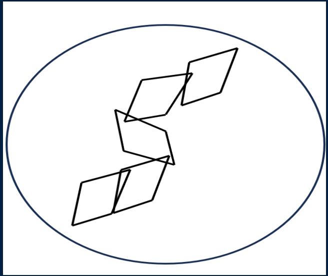

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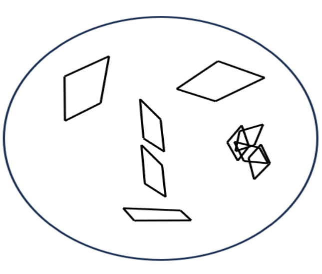

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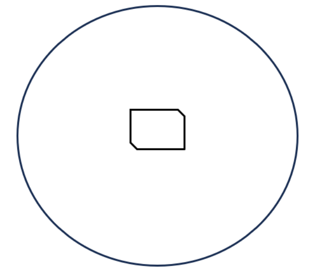

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- Petz map (adapted to the channel $\mathcal{E}$): $\mathcal{R}_{P,\mathcal{E}} \sim \{PE_i^\dagger \mathcal{E}(P)^{-1/2}\}$, $\mathcal{E} \sim \{E_i\} \leftarrow$ The newly constructed kraus.

Examples

- Noise model: Amplitude-damping noise

$$D_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad D_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

  $\gamma \rightarrow$ damping strength / probability of losing a photon/ probability that the qubit decay from $|1\rangle \rightarrow |0\rangle$.
- Consider the [4,1]-Leung code

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

$$|1_L\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)$$

- This code satisfies $\langle m_L|E_k^\dagger E_l|n_L\rangle = \lambda_{kl}^{mn}$.
- Because of our orthogonalisation algorithm, we accommodate either of the $\{E_{0011}, E_{1100}\}$ in the correctable set of errors.

Performance of the recovery operations

	[4,1]-Leung code	
Recovery	Worst case fidelity	Entanglement fidelity
Leung (noise process)	$1 - 2.75\gamma^2$	$1 - 3\gamma^2$
Petz (original noise process)	$1 - 1.75\gamma^2$	$1 - 1.75\gamma^2$
$\mathcal{R}_{P,\mathcal{E}}$ (adapted to $\mathcal{E}$)	$1 - 1.15\gamma^2$	$1 - 1.25\gamma^2$

- Entanglement fidelity from the optimal recovery obtained from SDP[7]:

$$1 - 1.25\gamma^2 + \mathcal{O}(\gamma^3).$$

[7] Fletcher *et al.* Phys. Rev. A 75, 012338

Comparison with the SDP optimised recovery

R_0	$\|0_L\rangle(\alpha\langle 0000\| + \beta\langle 1111\|) + \frac{1}{\sqrt{2}}\|1_L\rangle(\langle 0011\| + \langle 1100\|)$
R_1	$\|0_L\rangle(\beta\langle 0000\| - \alpha\langle 1111\|) + \frac{1}{\sqrt{2}}\|1_L\rangle(\langle 0011\| - \langle 1100\|)$
R_2	$\|0_L\rangle\|\langle 1110\| + \|1_L\rangle\|\langle 0010\|$
R_3	$\|0_L\rangle\|\langle 1101\| + \|1_L\rangle\|\langle 0001\|$
R_4	$\|0_L\rangle\|\langle 1011\| + \|1_L\rangle\|\langle 1000\|$
R_5	$\|0_L\rangle\|\langle 0111\| + \|1_L\rangle\|\langle 1000\|$
R_6	$\|0_L\rangle\|\langle 0110\|$
R_7	$\|0_L\rangle\|\langle 1001\|$
R_8	$\|0_L\rangle\|\langle 1010\|$
R_9	$\|0_L\rangle\|\langle 1010\|$

Figure: Optimal recovery from SDP.

R_0	$\|0_L\rangle(\alpha\langle 0000\| + \beta\langle 1111\|) + \frac{1}{\sqrt{2}}\|1_L\rangle(\langle 0011\| + \langle 1100\|)$
R_1	$\|1_L\rangle(\beta\langle 0000\| - \alpha\langle 1111\|) + \frac{1}{\sqrt{2}}\|0_L\rangle(\langle 0011\| - \langle 1100\|)$
R_2	$\|0_L\rangle\|\langle 1110\| + \|1_L\rangle\|\langle 0010\|$
R_3	$\|0_L\rangle\|\langle 1101\| + \|1_L\rangle\|\langle 0001\|$
R_4	$\|0_L\rangle\|\langle 1011\| + \|1_L\rangle\|\langle 1000\|$
R_5	$\|0_L\rangle\|\langle 0111\| + \|1_L\rangle\|\langle 1000\|$
R_6	$\|0_L\rangle\|\langle 0110\|$
R_7	$\|0_L\rangle\|\langle 1001\|$
R_8	$\|0_L\rangle\|\langle 1010\|$
R_9	$\|0_L\rangle\|\langle 1010\|$

Figure: Petz recovery of the modified channel.

Summary and outlook

- We have proposed a framework to perform approximate quantum error correction despite having overlapping syndrome-subspaces.
- We show that the recovery through a Petz map can be made optimal for the four qubit code.
- Does the canonical Petz map serve as an optimal recovery (for any arbitrary codes and noise)?
- The Petz map can be implemented on a circuit [8].
- Can we implement the Petz recovery for the modified channel ($\mathcal{E}$) on the circuit with fewer resources?

[8] D. Biswas , G. vaidya, P. Mandayam , Phys. Rev. Res. 6, 043034 (2024)

Thank You