Geometric approaches to stretched Kostka quasi-polynomials

Sam Jeralds

University of Sydney

Combinatorics, Geometry, and Representation Theory
ICTS

Kostka numbers

Fix $n \in \mathbb{Z}_{\geq 1}$ and two partitions $\lambda, \mu \vdash n$, where

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$$
$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0)$$
$$\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \mu_i = n$$

The **Kostka numbers** $K_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ associated to λ, μ are ubiquitous in algebraic combinatorics and representation theory.

One definition: $K_{\lambda,\mu} = \#SSYT$ of shape λ, content μ

Example: $\lambda = (2,1)$, $\mu = (1,1,1)$, $K_{\lambda,\mu} = 2$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Example: $\lambda = (4,2)$, $\mu = (2,2,2)$, $K_{\lambda,\mu} = 3$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Exercise: $\lambda = (6,3)$, $\mu = (3,3,3)$, $K_{\lambda,\mu} = 4$.

Good News: Determining if $K_{\lambda,\mu}$ vanishes is a textbook exercise: $K_{\lambda,\mu} \neq 0$ if and only if $\mu \trianglelefteq \lambda$ in **dominance order**:

$$\sum_{i=1}^{\ell} \mu_i \leq \sum_{i=1}^{\ell} \lambda_i \ \forall 1 \leq \ell \leq k.$$

Bad News: Computing $K_{\lambda,\mu}$ is "not formulaic," in that there is unlikely to be a "nice" formula for $K_{\lambda,\mu}$ in terms of the parts of the partitions, binomials, factorials,...

Goal: Determine any useful information about $K_{\lambda,\mu}$ as a function of λ, μ in a direct way.

Stretched Kostka polynomials

Proposition (Kirillov–Reshetikhin '86): For two partitions $\lambda, \mu \vdash n$, the function $K_{\lambda,\mu}(\cdot) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $K_{\lambda,\mu}(N) := K_{N\lambda,N\mu}$ is a polynomial in N.

We call this the **stretched Kostka polynomial** or **stretched Kostka coefficient**.

Example: For $\lambda = (2,1)$, $\mu = (1,1,1)$, $K_{\lambda,\mu}(N) = N + 1$.

King, Tollu, and Toumazet ('04, '06) gave many conjectural properties of the stretched Kostka polynomials, including:

Conjecture: $K_{\lambda,\mu}(N)$ has positive coefficients.
–This one is still open, to my knowledge

Conjecture: (now **Theorem:** (McAllister '08)): If $\mu \lhd \lambda$, $\lambda, \mu \in \mathbb{Z}^r_{\geq 0}$, and writing

$$\lambda = \left(\kappa_1^{v_1}, \kappa_2^{v_2}, \ldots, \kappa_s^{v_s}\right),$$

then the stretched Kostka polynomial $K_{\lambda,\mu}(N)$ has degree

$$\binom{r-1}{2} - \sum_{p=1}^{s} \binom{v_p}{2},$$

where we interpret $\binom{1}{2} = 0$.

Example: For $\lambda = (2,1,0) = (2^1, 1^1, 0^1)$ and $\mu = (1^3)$, we have $r = 3$, $v_1 = v_2 = v_3 = 1$, and so

$$\deg K_{\lambda,\mu}(N) = \binom{3-1}{2} - \binom{1}{2} - \binom{1}{2} - \binom{1}{2} = 1$$

as expected.

Example: For $\lambda = (4^1, 2^2, 0^3)$, $\mu = (3, 1^5)$, we have $r = 6$, $v_1 = 1$, $v_2 = 2$, $v_3 = 3$, and

$$\deg K_{\lambda,\mu}(N) = \binom{6-1}{2} - \binom{1}{2} - \binom{2}{2} - \binom{3}{2} = 6.$$

Some comments:

1. As stated, the theorem only works for $\mu \lhd \lambda$, which we call a **primitive pair**. But, any Kostka number $K_{\lambda,\mu}$ can be factored into a product of Kostka numbers K_{λ_i,μ_i} for primitive pairs $\mu_i \lhd \lambda_i$.

2. McAllister's proof is via **Ehrhart theory**, realizing the stretched Kostka polynomial $K_{\lambda,\mu}(N)$ as the Ehrhart function of certain Gelfand–Tsetlin polytopes $GT_{\lambda,\mu}$, and $\deg K_{\lambda,\mu}(N)$ as the volume of $GT_{\lambda,\mu}$.

Weight space dimensions

Representation-theoretic interpretation: Kostka numbers $K_{\lambda,\mu}$ encode the dimensions of weight spaces in finite-dimensional, irreducible highest weight (polynomial) representations of $GL_k(\mathbb{C})$. This is the interpretation we generalize.

The usual suspects: G semisimple, simply-connected alg. group / $\mathbb{C}$, $T \subset B \subset G$ maximal torus and Borel subgroups, Φ^+ positive roots, $X^*(T)^+$ dominant characters of T.

To $\lambda, \mu \in X^*(T)^+$ we associate V_λ the irreducible highest-weight λ representation of G, and the weight space $V_\lambda(\mu)$ of V_λ of weight μ.

Textbook nonvanishing:
$\dim V_\lambda(\mu) \neq 0 \iff \lambda - \mu \in Q^+ := \mathbb{Z}_{\geq 0}\Phi^+$, or $\mu \preceq \lambda$ in the dominance order on the weight lattice.

Question: For $N \in \mathbb{Z}_{\geq 0}$, how does the function

$$K^G_{\lambda,\mu}(N) := \dim V_{N\lambda}(N\mu)$$

behave?

This is the **stretched Kostka quasi-polynomial** associated to G, λ, and μ.

Example: If $G = SL_n(\mathbb{C})$, then $K^G_{\lambda,\mu}(N)$ can be recognized as a stretched Kostka polynomial.

For G arbitrary, $K^G_{\lambda,\mu}(N)$ is in general **quasi-polynomial:** or **polynomial on residue classes:** there exists an integer $d \geq 1$ and polynomials $p_0, p_1, \dots, p_{d-1}$ such that $K^G_{\lambda,\mu}(N) = p_i(N)$ for $N \equiv i \mod d$. An integer d which satisfies this is called a **period** of the quasi-polynomial.

Example: For G of type B_2, $\lambda = \varpi_2$, $\mu = 0$,

$$K^G_{\lambda,\mu}(N) = \lfloor N/2 \rfloor.$$

Two natural questions:

1. What is $\deg K^G_{\lambda,\mu}(N)$? That is, the maximal degree of its constituents.
2. What is the minimal period d of $K^G_{\lambda,\mu}(N)$?

Generalizing McAllister's work, S. Gao and Y. Gao ('24) proved the following partial answer to the first question.
Theorem: Let G be of **classical type**. Then

$$\deg K^G_{\lambda,\mu}(N) = |\Phi_1^+| - |\Phi_2^+| - \mathrm{rank}(\Phi_1),$$

where if $\lambda = \sum d_i \varpi_i$ and $\lambda - \mu = \sum c_i \alpha_i$, the root systems Φ_1 and Φ_2 are given by

$$\Phi_1 = \mathrm{span}\{\alpha_i : c_i \neq 0\}, \quad \Phi_2 = \mathrm{span}\{\alpha_i : c_i \neq 0,\ d_i = 0\}.$$

Some comments:

1. Their proof again uses Ehrhart theory, now applied to certain Berenstein–Zelevinsky polytopes.

2. Their proof uses a similar key reduction to "primitive" cases: it suffices to prove the theorem for G simple and those (λ, μ) with $\lambda - \mu = \sum c_i \alpha_i$ and $c_i > 0$ for all i. In this case,

$$\deg K^G_{\lambda,\mu}(N) = |\Phi^+| - |\Phi^+_o| - \mathrm{rank}(\Phi)$$

for $\Phi_o = \mathrm{span}\{\alpha_i : d_i = 0\}$.

3. Bounds on $\deg K^G_{\lambda,\mu}(N)$ of this form have been in the literature for a while (Dehy '98). The real content of McAllister's/Gao-Gao's results are the exact formulas for the degree determined by simple root-theoretic data.

The briefest sketch of Geometric Invariant Theory

Fix X a projective variety, G a reductive group acting on X, $\mathcal{L}$ a G-linearized line bundle on X.

Very roughly, **Geometric Invariant Theory** gives a translation between more complicated geometric data *(invariant sections)* on an easier geometric object (X) and easier geometric data *(honest sections)* on a harder geometric object *(GIT quotients)*.

GIT quotients

The (projective) **GIT quotient** of X by G with respect to the line bundle $\mathcal{L}$ is given by

$$Y = X//_{\mathcal{L}}G := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(X^{ss}, \mathcal{L}^{\otimes n})^G\right)$$

where X^{ss} is the open subset of *semistable points* of X determined by $\mathcal{L}$.
This is a projective variety of dimension at most $\dim(X) - \dim(G)$, which is obtained when X has *stable* points.

Minor inconvenience: It need not be the case that there is a line bundle $\widehat{\mathcal{L}}$ on $X//_{\mathcal{L}}G$ such that

$$H^0(X//_{\mathcal{L}}G, \widehat{\mathcal{L}}) \cong H^0(X, \mathcal{L})^G$$

as desired. But, two possible corrections:

- If $q : X^{ss} \to X//_{\mathcal{L}}G$ is the quotient map, then the isomorphism holds if one takes $H^0(X//_{\mathcal{L}}G, q_*^G(\mathcal{L}))$, sections of the invariant pushforward (a rank one reflexive sheaf).

- Some power of $\mathcal{L}$ **descends** to $X//_{\mathcal{L}}G$: there is some line bundle $\widehat{\mathcal{L}}$ on $X//_{\mathcal{L}}G$ and $d \geq 1$ such that $q^*(\widehat{\mathcal{L}}) = \mathcal{L}^{\otimes d}$ on X^{ss}. Then
$$H^0(X//_{\mathcal{L}}G, \widehat{\mathcal{L}}) \cong H^0(X, \mathcal{L}^{\otimes d})^G.$$

GIT and $K^G_{\lambda,\mu}(N)$

A twist on the Borel–Weil theorem gives

$$K^G_{\lambda,\mu}(N) = \dim V_{N\lambda}(N\mu) = \dim H^0(G/P_\lambda, \mathcal{L}^{\otimes N})^T$$

where

$P_\lambda \supset B$ the standard parabolic determined by λ,

G/P_λ the associated partial flag variety,

$\mathcal{L} := L_\lambda \otimes \mathbb{C}_\mu$ a T-linearized line bundle on G/P_λ

Fixing $d \geq 1$ so that $\mathcal{L}^{\otimes d}$ descends to $\widehat{\mathcal{L}}$ on the GIT quotient, and $\mathcal{F}_k := q^G_*(\mathcal{L}^{\otimes k})$ for $0 \leq k < d$, **asymptotic Riemann–Roch** applied to

$$H^0(G/P_\lambda//_{\mathcal{L}}T, \mathcal{F}_k \otimes \widehat{\mathcal{L}}^{\otimes n})$$

provides a quasi-polynomial structure of period d to $K^G_{\lambda,\mu}(N)$.

Theorem (Besson-J.-Kiers '25)

For G simple and $\lambda, \mu \in X^*(T)^+$ a primitive pair, the GIT quotient $(G/P_\lambda)//_{\mathcal{L}}T$ has dimension

$$\dim(G/P_\lambda) - \dim(T) = |\Phi^+| - |\Phi_P^+| - rank(G).$$

Corollary

For G simple and $\lambda, \mu \in X^*(T)^+$ a primitive pair, the stretched Kostka quasi-polynomial $K_{\lambda,\mu}^G(N)$ is precisely of degree

$$|\Phi^+| - |\Phi_P^+| - rank(G).$$

Some open questions

- Do the $K^G_{\lambda,\mu}(N)$ have positive coefficients?

  Conjecturally, yes (extending the conjecture of King–Tollu–Toumazet for type A)

  Combinatorial incarnation: **Ehrhart positivity** of the appropriate polytopes

  Geometric incarnation: Positivity in computations in the **Hirzebruch–Riemann–Roch** formula on the GIT quotient

- What are the minimal periods of $K^G_{\lambda,\mu}(N)$?

  Polynomality is "combinatorially surprising" in type A!

  Geometrically: **minimal** $d \geq 1$ such that $\mathcal{L}^{\otimes d}$ descends to the GIT quotient?

What about Schubert varieties?

Given $\lambda \in X^*(T)^+$, $\mu \in X^*(T)$, and $w \in W$ an element of the Weyl group, the **Demazure module** V_λ^w is a B-submodule of V_λ.

Question: How do the weight multiplicities $V_{N\lambda}^w(N\mu)$ grow as functions of N?

These again are quasi-polynomial in general. (Dehy '98) gives natural upper bounds on the degree, using Ehrhart theory applied to polytopes coming from Littelmann's path model. These is recoverable via GIT methods applied to Schubert varieties.

Hurdle: To get *exact* values of the degree, one needs a notion of "primitive" pairs (λ, μ) along with the appropriate **reduction rule** to reduce dimension computations to the primitive case. Such a reduction rule is the topic of forthcoming work with M. Besson and J. Kiers.

Thank You