Macroscopic description of Integrable systems in confining traps

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(A work in progress)

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- Integrable systems have infinitely many conserved quantities.
- Realistic systems are not integrable.

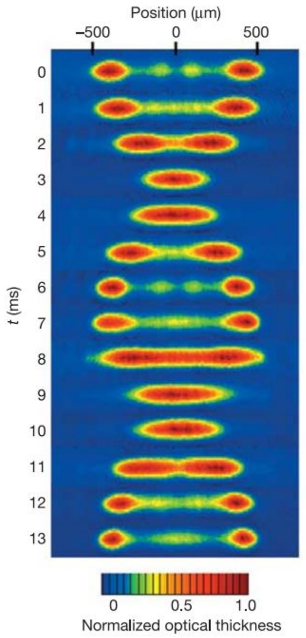
Question: What is the effect of Integrability breaking?

The effect of integrability breaking on many of the cornerstone ideas of statistical physics have been studied.

- Transport:
  Pinned Toda chain [Dhar et. al., J Stat Phys (2019)].

- Chaos and Ergodicity:
  Fermi-Pasta-Ulam-Tsingou equipartition problem [Fermi et. al., document LA-1940 (1955)].

- Thermalization:
  1) Quantum: Newton's Cradle [Kinoshita et. al., Nat. Letts. (2006)].
  2) Classical: Hard rods in Harmonic trap [Cao et. al., PRL (2019)].



Q. Thermalization of isolated Integrable systems when confined to external trap.

1-D Models studied (short ranged):

1) Hard Rods: Eq. 1

2) Hyperbolic Calogero Model: Eq. 2

3) Toda Model: Eq. 3 (Crossing allowed)

$$H^{(\mathrm{rod})} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} \frac{x_i^\delta}{\delta} + \sum_{i=1}^{N-1} \Theta(x_{i+1} - x_i). \tag{1}$$

$$\Theta(r) = \begin{cases} \infty & \text{for } r \leq a, \\ 0 & \text{for } r > a. \end{cases}$$

$$H^{(\mathrm{hcm})} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} \frac{x_i^\delta}{\delta} + \frac{J}{2} \sum_{i \neq j}^{N} \frac{1}{\sinh^2(x_j - x_i)}. \tag{2}$$

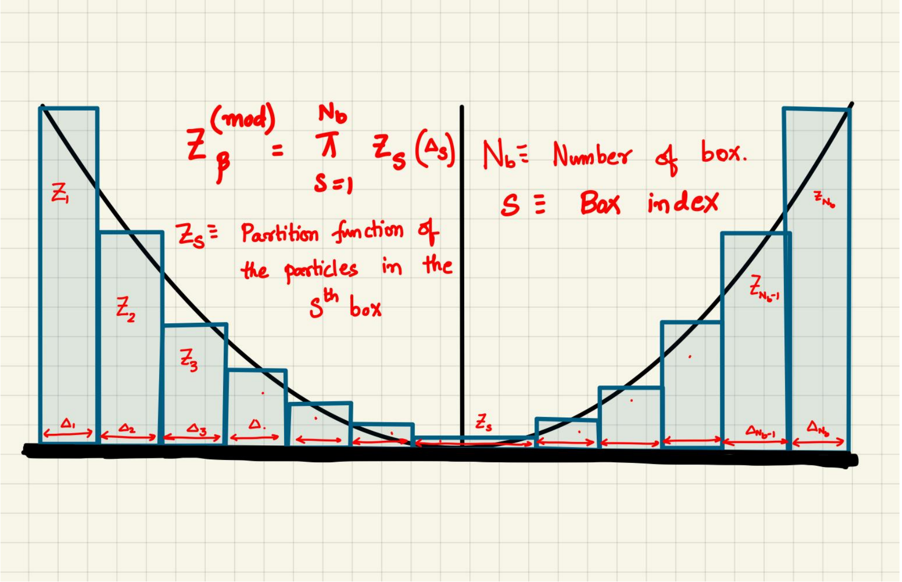
$$H^{(\mathrm{tod})} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N} \frac{x_i^\delta}{\delta} + J \sum_{i=1}^{N-1} \exp\left(-\frac{x_{i+1} - x_i}{a}\right). \tag{3}$$

SubQ.

1) Free energy and density at equilibrium.
2) Do they thermalize: Virial theorem and density comparison.

Free Energy and average density at Equilibrium

Difficult to solve the partition function exactly in the presence of external trap.



Assumption:

1) Trap breaks all the conservation laws except the Energy conservation. (Will be verified post hoc when we study thermalization.)

An approximate scheme:

Assumption:

2) The partition function can be written as a product of partition function of smaller subsystems.
3) Each subsystem experiences a constant potential.

Average thermal density is obtained by minimising the Free energy with the normalization constraint.

Hard Rods

Free energy,

$$\mathcal{F}_{\rm hrd}\left[\rho_N(x), T\right] = \int dx\, N\rho_N(x)\left(V_{\rm ext}(x) - T\log\left(\frac{1 - aN\rho_N(x)}{N\rho_N(x)}\right)\right). \quad (4)$$

Chemical potential,

$$\mu_N = V_{\rm ext}(x) - T\left[\log\left(\frac{1 - aN\rho_N(x)}{N\rho_N(x)}\right) - \frac{1}{1 - aN\rho_N(x)}\right]. \quad (5)$$

Scaling form of density,

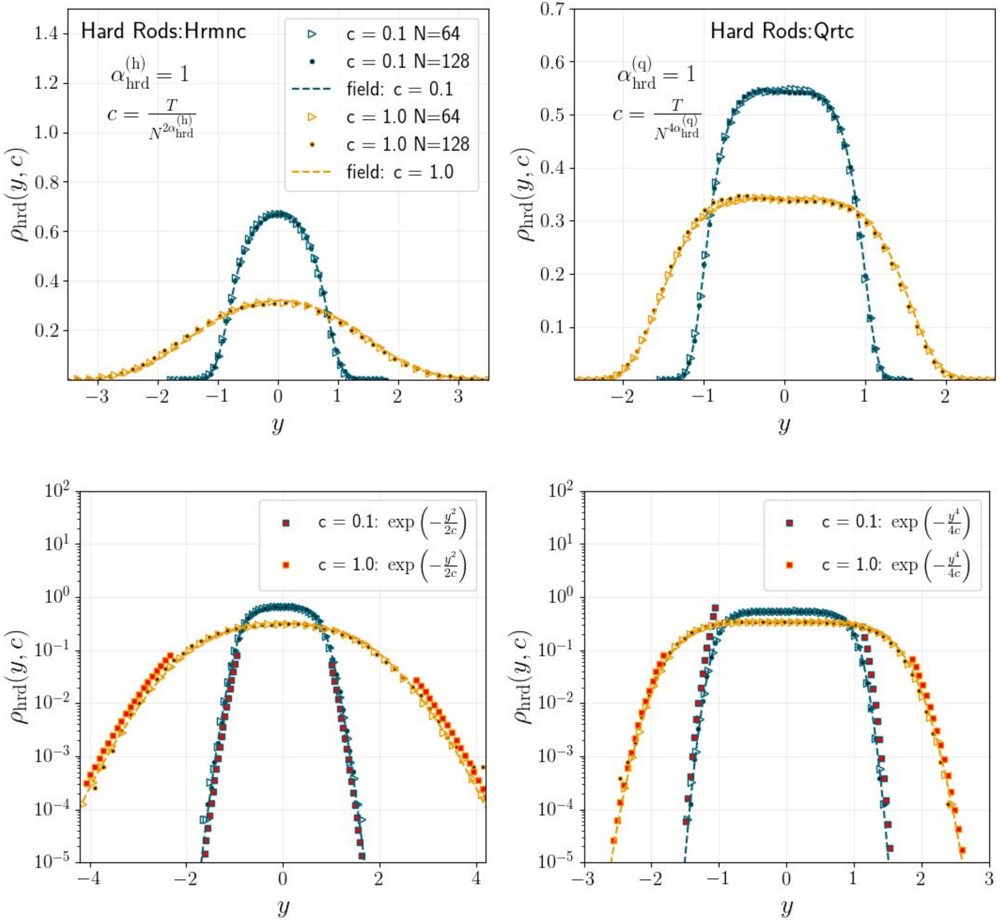
$$\rho_N(x, T) = N^{-\alpha^{(\delta)}_{\rm hrd}}\rho_{\rm hrd}\left(\frac{x}{N^{\alpha^{(\delta)}_{\rm hrd}}}, \frac{T}{N^{\gamma^{(\delta)}_{\rm hrd}}}\right). \quad (6)$$

Scaled chemical potential,

$$\mu_{\rm hrd} = \frac{y^\delta}{\delta} - c\left[\log\left(\frac{1 - a\rho_{\rm hrd}(y, c)}{\rho_{\rm hrd}(y, c)}\right) - \frac{1}{1 - a\rho_{\rm hrd}(y, c)}\right], \quad (7)$$

where,

$$c = \frac{T}{N^{\gamma^{(\delta)}_{\rm hrd}}} \quad \text{with} \quad \gamma^{(\delta)}_{\rm hrd} = \delta\alpha^{(\delta)}_{\rm hrd} \quad \text{and} \quad \alpha^{(\delta)}_{\rm hrd} = 1. \quad (8)$$



Hyperbolic Calogero Model

Free energy,

$$\mathcal{F}_{\rm hcm}\left[\rho_N(x), T\right] = \int dx \left(N\rho_N(x)V_{\rm ext}(x) + J\zeta(2)N^3\rho_N(x)^3 + TN\rho_N(x)\log N\rho_N(x)\right). \tag{9}$$

Chemical Potential,

$$\mu_N = V_{\rm ext}(x) + 3N^2\zeta(2)\rho_N(x)^2 + T\left(1+\log N\rho_N(x)\right). \tag{10}$$

Scaling form of density,

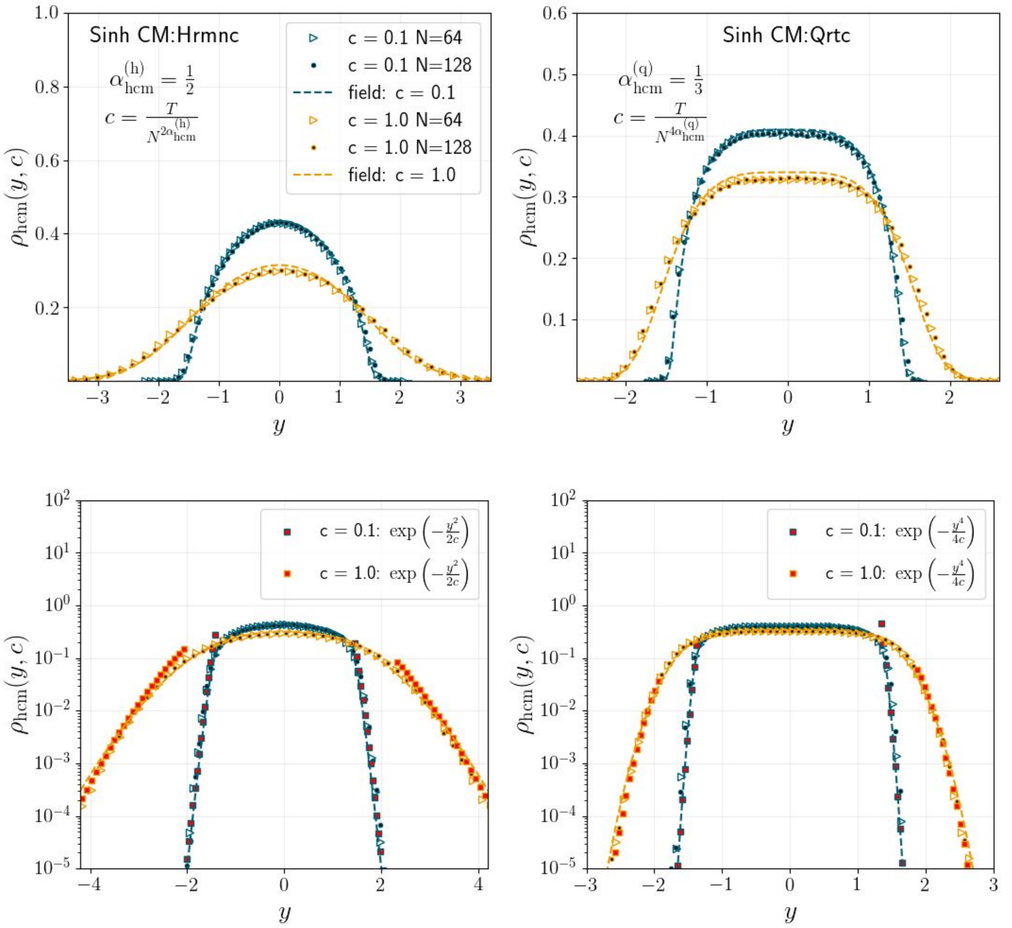
$$\rho_N(x,T) = N^{-\alpha^{(\delta)}_{\rm hcm}}\rho_{\rm hcm}\left(\frac{x}{N^{\alpha^{(\delta)}_{\rm hcm}}}, \frac{T}{N^{\gamma^{(\delta)}_{\rm hcm}}}\right). \tag{11}$$

Scaled chemical potential,

$$\mu_{\rm hcm} = \frac{y^\delta}{\delta} + 3\zeta(2)\rho_{\rm hcm}(y,c)^2 + c\log(\rho_{\rm hcm}(y,c)), \tag{12}$$

where,

$$c = \frac{T}{N^{\gamma^{(\delta)}_{\rm hcm}}} \quad \text{with} \quad \gamma^{(\delta)}_{\rm hcm} = \delta\alpha^{(\delta)}_{\rm hcm} \quad \text{and} \quad \alpha^{(\delta)}_{\rm hcm} = \frac{2}{2+\delta}. \tag{13}$$



Toda Model

Free energy,

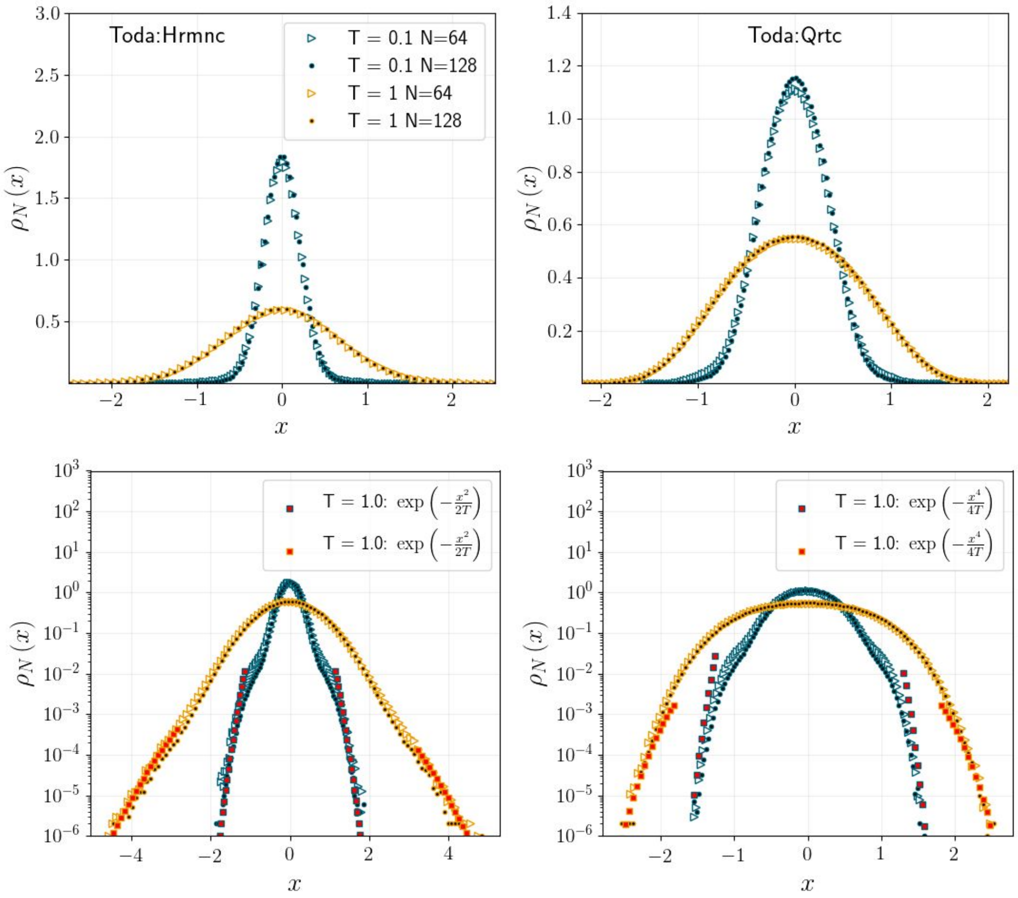
$$\mathcal{F}_{\rm tod}\left[\rho_N(x),\beta\right] = N\int dx\,\Bigg(\rho_N(x)V_{\rm ext}(x) + \frac{a\rho_N(x)P_N(x)\Gamma[a\beta P_N(x)]}{\Gamma'[a\beta P_N(x)]} - \beta^{-1}\rho_N(x)\log\Gamma[a\beta P_N(x)]\Bigg), \qquad (14)$$

Equation of state,

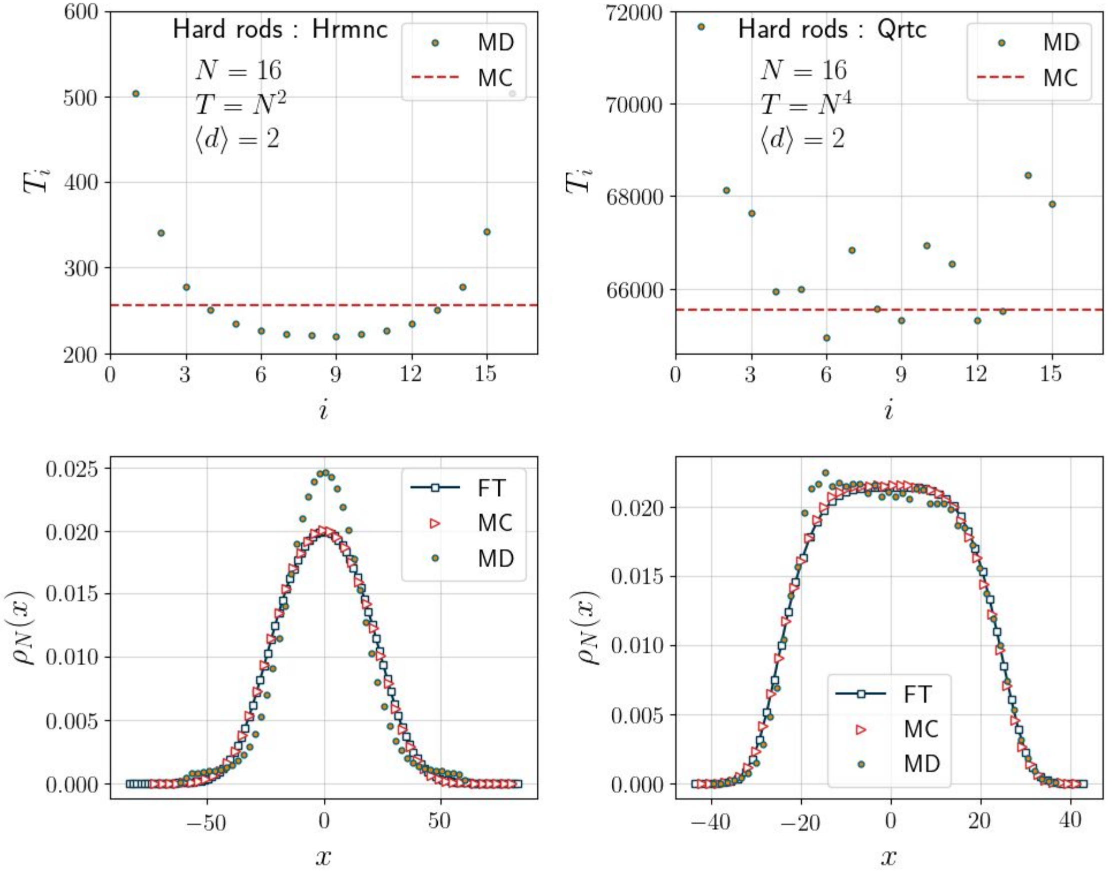
$$\frac{1}{aN\rho_N(x)} = \log(\beta J) - \frac{\Gamma[a\beta P_N(x)]}{\Gamma'[a\beta P_N(x)]}. \qquad (15)$$

Chemical potential,

$$\mu_{\rm tod} = V_{\rm ext}(x) + \beta^{-1}\left(a\beta P_N(x)\log(\beta J) - \log\Gamma[a\beta P_N(x)]\right). \qquad (16)$$

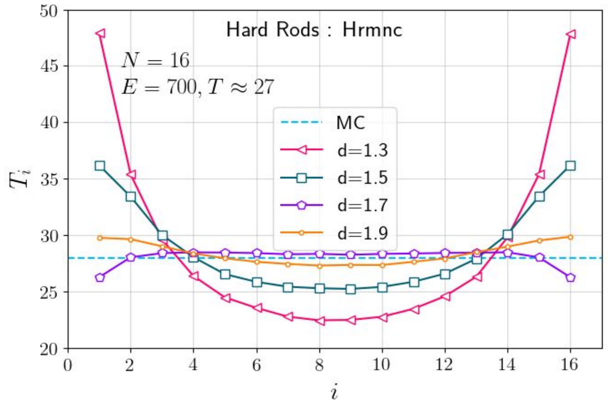


Dynamics: Hard Rods



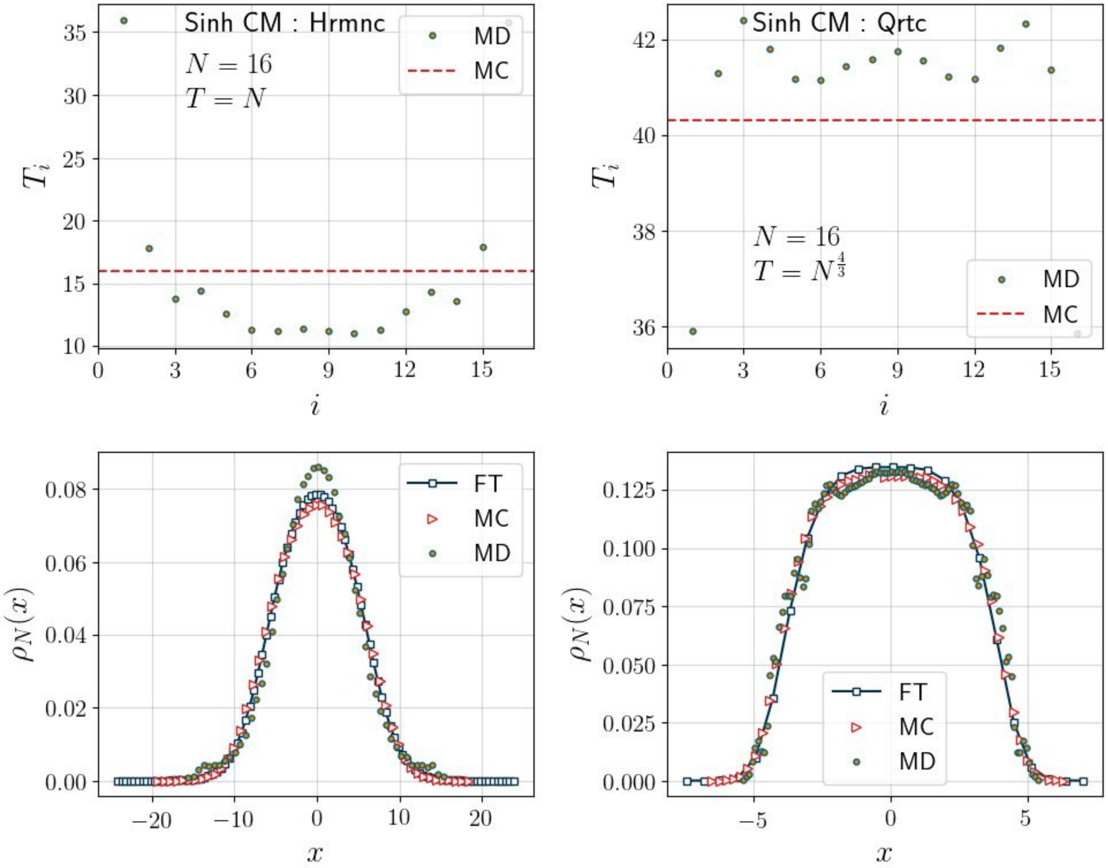
Microcanonical Ensemble:

- Harmonic confinement:
  - Does not thermalize.
- Initial Condition dependence



- Quartic confinement:
  - Thermalizes.

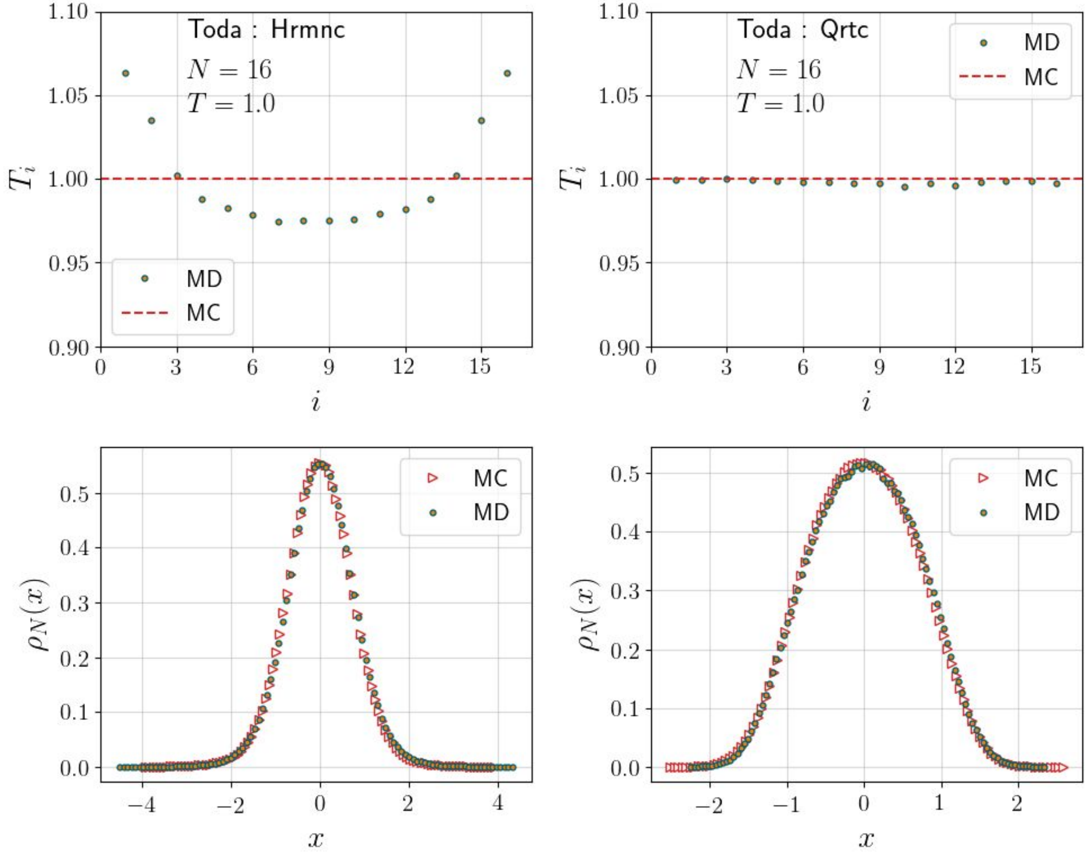
Dynamics: Hyperbolic Calogero Model



Microcanonical Ensemble:

- Harmonic confinement:
  - Does not thermalize.
- Quartic confinement:
  - Density almost Thermalizes.
  - Virial theorem fails at the edges.
  - Calogero Model is Integrable in quartic trap. [Polychronakos A., Phys. Lett. B. (1992)]

Dynamics: Toda Model



Microcanonical Ensemble:

- Harmonic confinement:
  - Density Thermalizes.
  - Virial theorem fails at the edges
- Quartic confinement:
  - Thermalizes

Conclusions and Continuations.

- Validity of approximate scheme for field theory.
- Scaling behaviour of the average densities.
- Particles on the tails of the support of densities are effectively non-interacting, here trap and Entropy becomes more important.

Density

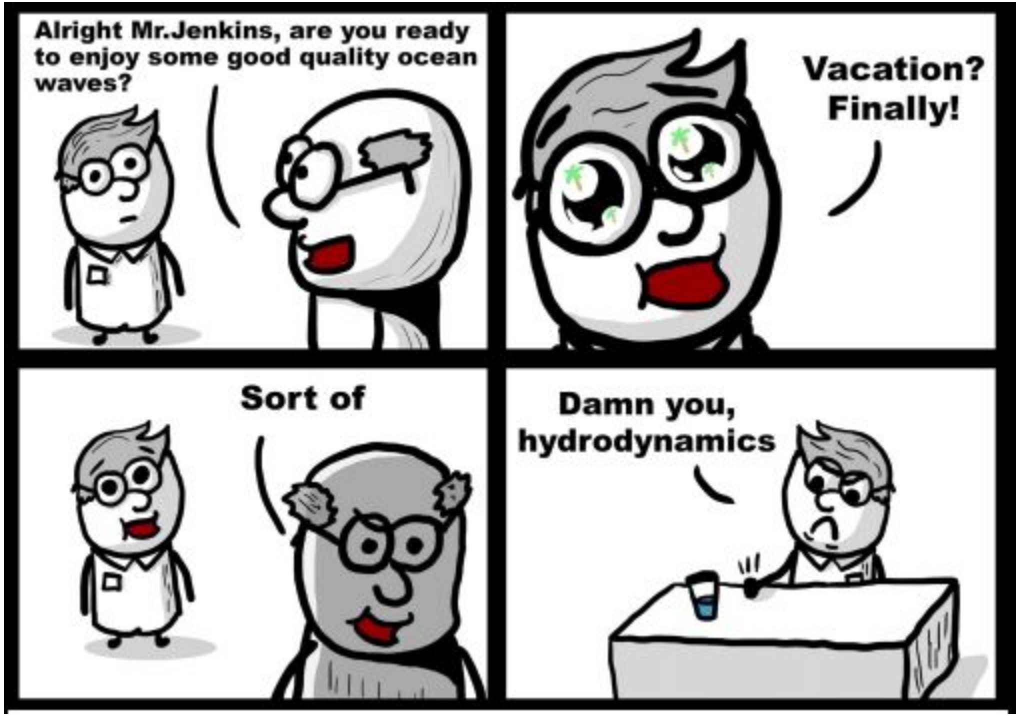
Model\Trap	Harmonic	Quartic
1) Hard Rods	No	Yes
2) Hyperbolic Calogero	No	Yes
3) Toda	Yes	Yes

Virial Theorem

Model\Trap	Harmonic	Quartic
1) Hard Rods	No	Yes
2) Hyperbolic Calogero	No	Yes
3) Toda	No	Yes

- Particles at the edges do not thermalize well.
- Initial condition dependence: hard rods and other models.

- Equation of state.
- Transient behaviour.
- Hydrodynamic approach.



Credit: Internet

Thank You.