Intermittency, cascades and thin sets in $3D$ Navier-Stokes turbulence

J. D. Gibbon : Imperial College London

TPIMP-ICTS 2020

The 3D incompressible Navier-Stokes equations

Consider the 3D Navier-Stokes equations in the domain $[0, L]^3_{per}$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x}) \qquad \mathrm{div}\,\boldsymbol{u} = 0 \qquad (\mathrm{div}\,\boldsymbol{f} = 0)$$

1. I became interested in intermittency after reading Batchelor & Townsend (1949) and the multi-fractal approach described in Uriel's book (1995).
2. A generation of computations of the $3D$ incompressible NSEs on a periodic cube $[0, L]^3_{per}$ show the same typical *intermittent behaviour* in the vorticity and energy dissipation fields :
   - Intense regions begin to flatten into quasi-$2D$ pancakes ;
   - These pancakes then roll up into tubular/filamentary structures which ultimately become even finer & begin to fragment.
3. *I will address how this visual manifestation of intermittency could be connected to NS weak solns and invariant scaling.*

Plot courtesy of J. R. Picardo and S. S. Ray at ICTS

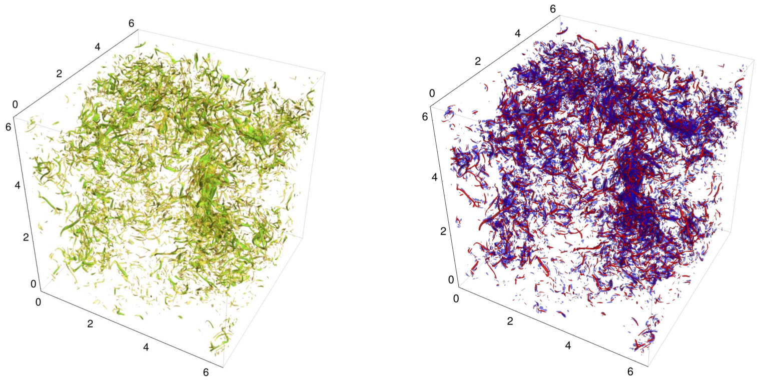


Figure: **Left :** snapshot of the energy dissipation field $\varepsilon = 2\nu S_{i,j}S_{j,i}$ of a forced 512^3 NS flow at $Re_\lambda = 196$ which is colour-coded such that yellow is 4 times the mean and blue denotes 6 times the mean. **Right :** the field $Q = \frac{1}{2}\left(|\boldsymbol{\omega}|^2 - |S|^2\right)$ with colours corresponding to $-2Q_{rms}$ (blue) and $5Q_{rms}$ (red). Plot courtesy of J. R. Picardo and S. S. Ray.

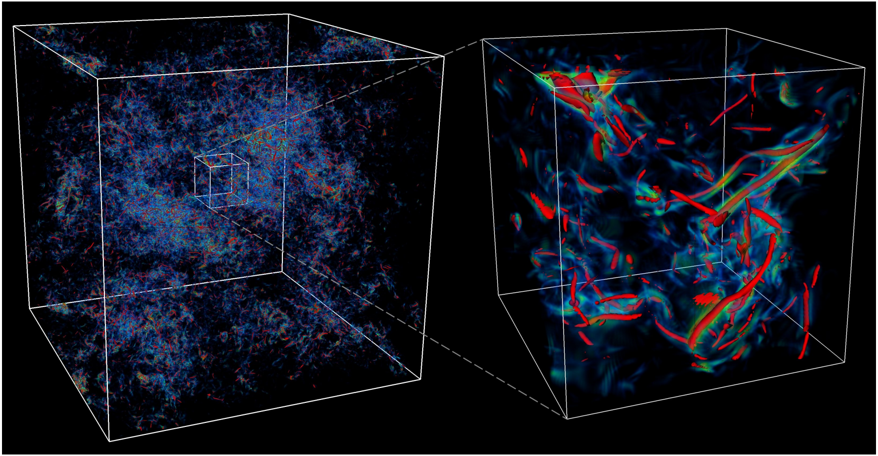


Figure: The enstrophy $|\boldsymbol{\omega}_i|^2$ (red) & the energy dissipation $\varepsilon = 2\nu S_{i,j}S_{j,i}$ (blue-green) at $Re_\lambda = 1000$.

Just for the record :

1. Orszag & Patterson 1972;
2. Kerr 1985;
3. Eswaran & Pope 1988 ;
4. Jimenez *et al* 1993 ;
5. Moin & Mahesh (Ann Rev Fluid Mech 20, 1998) ;
6. Kurien & Taylor 2005;
7. Ishihara, Gotoh, Kaneda (Ann Rev Fluid Mech 2009) ;
8. A 4096^3 computation by Donzis, Yeung & Sreenivasan 2012 at TAMU using 10^5 processors.
9. Hunt, Ishihara, Worth & Kaneda (2013, 2017). Latest is an 8000^3 computation – Ishihara, Elsinga & Hunt (PrRS 2020).

Estimates of the energy dissipation rate

We begin with the forced $3D$ NSEs on a periodic domain $\mathcal{V} = [0,\ L]^3$:

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla P = \nu\Delta\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x})\,, \qquad \mathrm{div}\,\boldsymbol{u} = 0 = \mathrm{div}\,\boldsymbol{f}\,.$$

Formally, we find the energy equation by dotting $\boldsymbol{u}$ across the NSEs :

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}|\boldsymbol{u}|^2\,dV = -\nu\int_{\mathcal{V}}|\nabla\boldsymbol{u}|^2\,dV + \int_{\mathcal{V}}\boldsymbol{u}\cdot\boldsymbol{f}\,dV$$

How do we deal with $\int_{\mathcal{V}}\boldsymbol{u}\cdot\boldsymbol{f}\,dV$? Use the Cauchy-Schwarz inequality to write

$$\left|\int_{\mathcal{V}}\boldsymbol{u}\cdot\boldsymbol{f}\,dV\right| \leq \|\boldsymbol{u}\|_2\|\boldsymbol{f}\|_2\,, \qquad \text{where} \qquad \|\boldsymbol{f}\|_2 \equiv \left(\int_{\mathcal{V}}|\boldsymbol{f}|^2\,dV\right)^{1/2}$$

Thus, upon time averaging $\langle\cdot\rangle_T = T^{-1}\int_0^T\cdot\,dt$, we have

$$\nu\left\langle\int_{\mathcal{V}}|\nabla\boldsymbol{u}|^2\,dV\right\rangle_T \leq \langle\|\boldsymbol{u}\|_2\|\boldsymbol{f}\|_2\rangle_T + O\left(T^{-1}\right)\,.$$

Some definitions for $3D$ NSEs

$$\underbrace{Re = \frac{UL}{\nu}}_{Reynolds\ No}; \qquad \underbrace{Gr = \frac{L^3 f_{rms}}{\nu^2}}_{Grashof\ No}, \qquad \underbrace{U^2 = L^{-3}\left\langle \|\boldsymbol{u}\|_2^2 \right\rangle_T}_{Average\ vel.\ field}.$$

Thus the energy dissipation rate $\mathcal{E}$ per unit volume is

$$\mathcal{E} = \nu \mathbf{L^{-3}} \left\langle \int_{\mathcal{V}} |\nabla \boldsymbol{u}|^2 dV \right\rangle_T \leq L^{-4}\nu^3 Gr Re + O\left(T^{-1}\right).$$

Doering & Foias (2002) have shown that for NS-solns $Gr \leq c\, Re^2$.

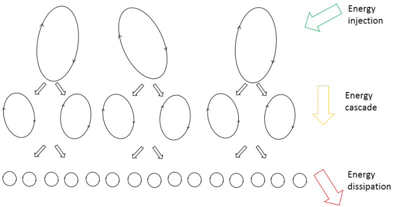
$$\mathcal{E} \leq \mathbf{c\, L}^{-4}\nu^{\mathbf{3}}\left[\mathbf{Re^3} + \mathbf{O}\left(\mathbf{Re^2}\right)\right] \qquad\qquad Gr \to \infty,$$

from which we find an estimate for the inverse Kolmogorov length

$$L\lambda_k^{-1} = \left(\frac{\mathcal{E}}{\nu^3}\right)^{1/4} \leq c\, Re^{3/4}.$$

The u.b. coincides with K41. **Question :** in the NS-sense, are there scales smaller than this?

Turbulent cascades & length-scales smaller than λ_k?



Numerical simulations of the $3D$ Navier-Stokes equations show that finer and finer vortical structures appear as resolution increases involving inverse scales much smaller than λ_k.

1. Richardson (1922) & K41 believed that viscosity ultimately halts this process (Frisch 1995) – **Peter Lynch's book on Richardson 2006.**
2. Mandelbrot (1974) suggested that the cascade process is fractal : i.e., the inverse length scales associated with the cascades of vorticity diverge to infinity.

Question : For NSEs, is Richardson's parody of Swift's poem true :

Big whirls have little whirls that feed on their velocity, and
little whirls have lesser whirls and so on to viscosity.

In other words, is there a finite limit or do vortices spin down to mol-scales?

Cascades & higher derivatives

A cascade is a sequential process that involves vorticity & strain being driven down to smaller length scales. This process should show up in estimates of both spatially & temporally averaged gradients of $\boldsymbol{u}(\boldsymbol{x}, t)$.

Define a doubly-labeled set of norms in dimensionless form

$$F_{n,m} = \nu^{-1} L^{1/\alpha_{n,m}} \|\nabla^n \boldsymbol{u}\|_{2m}\,, \qquad \alpha_{n,m} = \frac{2m}{2m(n+1)-3}\,,$$

for $1 \leq n < \infty$ and $1 \leq m \leq \infty$, where

$$\|\nabla^n \boldsymbol{u}\|_{2m} = \left(\int_V |\nabla^n \boldsymbol{u}|^{2m} dV\right)^{1/2m}.$$

1. Derivatives are sensitive to ever finer length scales in the flow.
2. Higher values of m pick out the larger spikes, with the $m = \infty$ case representing the maximum norm.

The NSEs have the scale invariance (on the whole domain) :

$$\boldsymbol{u}(\boldsymbol{x},\, t) \to \lambda^{-1}\boldsymbol{u}\left(\boldsymbol{x}/\lambda,\, t/\lambda^2\right) \qquad \Rightarrow \qquad F_{n,m} \to F_{n,m}\,.$$

Thus the $F_{n,m}$ are invariant at every length and time scale in the flow.

Theorem (JDG 2018)

With the definition $\alpha_{n,m} = \frac{2m}{2m(n+1)-3}$, *and for*

- $n \geq 1$ *with* $1 \leq m \leq \infty$
- $n = 0$ *with* $3 < m \leq \infty$,

Leray's weak solutions of the $3D$ *NSEs satisfy*

$$\left\langle F_{n,m}^{\alpha_{n,m}}\right\rangle_T \leq c_{n,m} Re^3 + O\left(T^{-1}\right)\,.$$

JDG, J. Nonlin. Sci., **29**(1), 215–228, 2019.

Historical Table of weak solution results

Note that $\alpha_{1,1} = 2$ gives the standard 'Leray's energy inequality' result

$$\left\langle F_{1,1}^2 \right\rangle_T = \nu^{-2} L \left\langle \|\nabla \boldsymbol{u}\|_2^2 \right\rangle_T \leq c\, Re^3 \,.$$

n, m	$\alpha_{\mathbf{n},\mathbf{m}} = \frac{\mathbf{2m}}{\mathbf{2m(n+1)-3}}$	**Known for weak**
$n = 0,\ m = \infty$	$\alpha_{0,\infty} = 1$	$\langle \|\boldsymbol{u}\|_\infty \rangle_T \leq c\, L^{-1} \nu Re^3$ Tartar78
$n = 0,\ m > 3$	$\alpha_{0,m} = \frac{2m}{2m-3}$	$\left\langle \|\boldsymbol{u}\|_{2m}^{\alpha_{0,m}} \right\rangle_T \leq L^{-1} \nu^{\alpha_{0,m}} Re^3$
$n = 1,\ m = 1$	$\alpha_{1,1} = 2$	$\left\langle F_{1,1}^2 \right\rangle_T < c\, \nu^2 L^{-1} Re^3$ Leray34
$n = 1,\ m \geq 1$	$\alpha_{1,m} = \frac{2m}{4m-3}$	$\langle D_m \rangle_T \leq c\, Re^3$ JDG2011
$n \geq 1,\ m = 1$	$\alpha_{n,1} = \frac{2}{2n-1}$	$\left\langle H_{n,1}^{\frac{1}{2n-1}} \right\rangle_T \leq \nu^{\alpha_{n,1}} L^{-1} Re^3$ FGT81

Table: Estimates for a range of n and m.

Concerning the $D_m = F_{1,m}$:

- Donzis, Gupta, JDG, Kerr, Pandit & Vincenzi, JFM (2013); Nonl'y (2014).
- For $3D$ Euler : Kerr JFM, **729**, R2, (2013).
- JDG, J. Nonlin. Sci., **29**(1), 215–228, 2019.

Theorem

For any $n \geq 1$ & $1 \leq m \leq \infty$; (ii) for $n = 0$ & $3/2 \leq m \leq \infty$, sufficient conditions for ***strong solutions*** *of the 3D NSEs to exist are*

$$\left\langle \mathbf{F}_{\mathbf{n},\mathbf{m}}^{\mathbf{2}\alpha_{\mathbf{n},\mathbf{m}}} \right\rangle_{\mathbf{T}} < \infty\,.$$

The Prodi-Serrin condition for regularity?? If $\boldsymbol{u} \in L^p[(0,\ T),\ L^q]$ then $\exists$ a unique smooth soln **if p and q obey**

$$\frac{2}{p} + \frac{3}{q} = 1\,, \qquad\qquad q > 3\,.$$

Choose $n = 0$ with $p = 2\alpha_{0,m}$ and $q = 2m$ then

$$\frac{2}{2\alpha_{0,m}} + \frac{3}{2m} = 1\,.$$

Definition of a sequence of length scales $\lambda_{n,m}(t)$

Define a set of t-dependent length-scales $\{\lambda_{n,m}(t)\}$ s.t.

$$\lambda_{n,m}^{-2m(n+1)+3}\nu^{2m} = \left(\frac{L}{\lambda_{n,m}}\right)^{-3} H_{n,m} \qquad \text{where} \qquad H_{n,m} = \int_{\mathcal{V}} |\nabla^n \boldsymbol{u}|^{2m} dV$$

$$\left(L\lambda_{n,m}^{-1}\right)^{n+1} = F_{n,m}\,, \qquad \alpha_{n,m} = \frac{2m}{2m(n+1)-3}\,.$$

Lemma

For weak solutions

$$\left\langle L\lambda_{n,m}^{-1}\right\rangle_T \leq c_{n,m} Re^{\frac{3}{(n+1)\alpha_{n,m}}} + O\left(T^{-1}\right)\,.$$

when : i) $n \geq 1$ *and* $1 \leq m \leq \infty$ *; ii)* $n = 0$ *and* $3 < m \leq \infty$ *.*

The upper bound has a finite limit : Richardson and Kolmogorov were correct!

$$\lim_{\mathbf{n,m}\to\infty} \frac{\mathbf{3}}{(\mathbf{n}+\mathbf{1})\alpha_{\mathbf{n,m}}} \to \mathbf{3}$$

$\lambda_k \sim 1$mm while mean free paths are $\sim 5 \times 10^{-5}$mm = 50nm. Thus, $\exists$ a bandwidth of realistic *Re* for these estimates to lie within the validity of the NSE.

1. In 1978 Fournier and Frisch introduced the idea of turbulence in D dimensions where D is no longer an integer but restricted to $D \geq 2$. They achieved this by analytically continuing the Taylor expansion in time of the energy spectrum $E_k(t)$, assuming Gaussian initial conditions.

2. The idea of a non-integer dimension has taken root in the many papers on the beta, bi-fractal and multi-fractal models – see Frisch 1995.

3. Can the Navier-Stokes estimates be found on a domain of non-integer dimension?
    - In a fully rigorous sense, the answer is in the negative.
    - For instance, there are no proofs of the Divergence Theorem or the Sobolev inequalities on fractal domains.

Estimates made so far are true for weak solutions in a $D = 3$ domain. How can we generalize this to a D-dim domain for $D = 1, 2, 3$?

$$F_{n,m,D} = \nu^{-1} L^{1/\alpha_{n,m,D}} \|\nabla^n \boldsymbol{u}\|_{2m}\,, \qquad \alpha_{n,m,D} = \frac{2m}{2m(n+1) - D}\,.$$

The $F_{n,m,D}$ possess the same invariance properties as $F_{n,m}$.

Theorem

For $D = 2, 3$, and for $n \geq 1$ and $1 \leq m \leq \infty$, we have the estimate

$$\left\langle F_{n,m,D}^{(4-D)\alpha_{n,m,D}} \right\rangle_T \leq c_{n,m,D}\, Re^3\,.$$

For $D = 1$ the same result holds for Burgers' equation.

JDG : *Turbulent cascades & thin sets in 3D NS-turbulence* EPL 2020.

Scaling of the exponent in integer D dimensions

1. The above Theorem **shows how the exponent of $F_{n,m,D}$ scales with D.**
2. The surprising but crucial factor of $4 - D$ in the exponent multiplying $\alpha_{n,m,D}$ deserves some remarks :
3. When $D = 3$, the factor of $4 - D$ is simply unity ;
4. When $n = m = 1$ this factor cancels to make $(4 - D)\alpha_{1,1,D} = 2$ for every value of D, as it should. It also furnishes us with the correct bound on the averaged energy dissipation rate $\mathcal{E}$.
5. When $D = 2$ we achieve the

$$[(4 - D)\alpha_{n,m,D}]_{D=2} = 2\alpha_{n,m,2}$$

The factor of 2 in the upper bound gives us full regularity. **Thus the case $D = 2$ is critical for regularity, as is well-known.**

Examine the exponent of $F_{n,m,D}$: one finds that

$$(4-D)\alpha_{n,m,D} = \frac{2m(4-D)}{2m(n+1)-D} \quad \text{increases} \qquad \text{as} \qquad D \searrow 0\,.$$

- An increasing exponent of $F_{n,m}$ implies more, not less, regularity. *This is the direction of increasing dissipation.*
- *This suggests that a flow may adjust itself to find the smoothest, most dissipative set on which to operate.*
- This also runs counter to a commonly held theory of viscous turbulence in which singularities have been long-standing candidates as the underlying cause of turbulent dynamics.