Ising spin and gauge on hyperbolic lattices

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$\mathbb{Z}_2$ gauge

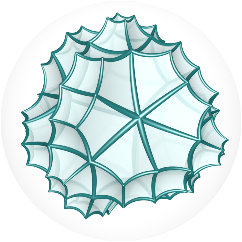
$\mathbb{Z}_2$ gauge on a hyperbolic lattice

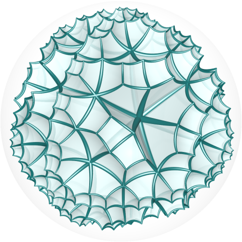
- Ising spins are placed on the **links** of a $\{4, 3, 5\}$ lattice.

$$Z = \sum_{\{\tau\}} \exp(\beta \sum_{a} \prod_{\langle ij \rangle \in \Box_a} \tau_{ij})$$

- $\beta = 1/g^2$
- $\tau = \pm 1$
- $\langle ij \rangle$ is a sum over links
- $\{\tau\}$ sum is over all possible spin configurations

The $\{4, 3, 5\}$ lattice





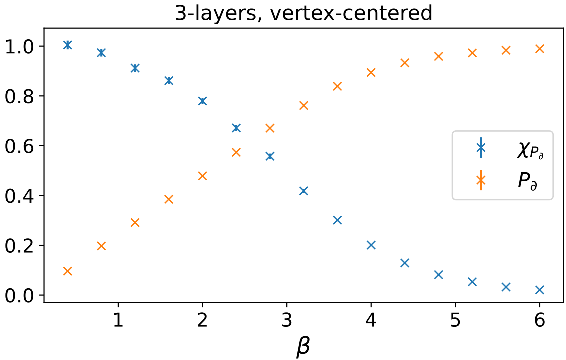
Simulation details

Staple action:

$$S = -\frac{\beta}{4} \sum_{\ell} \tau_\ell \underbrace{V_\ell}_{staples}$$

- Sequential heat-bath Monte Carlo
- Bin for autocorrelations
- Single-elimination jackknife
- Correlated fits for reliable p-value

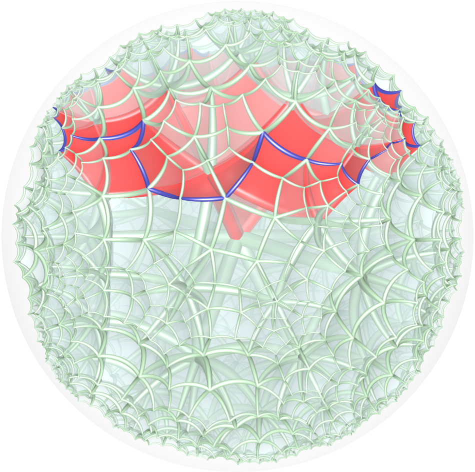
Boundary observables



$$P_\partial = \frac{1}{N_{p,\partial}} \sum_{a \in \partial} p_a$$

$$\chi_{P_\partial} = N_{p,\partial}(\langle P_\partial^2 \rangle - \langle P_\partial \rangle^2)$$

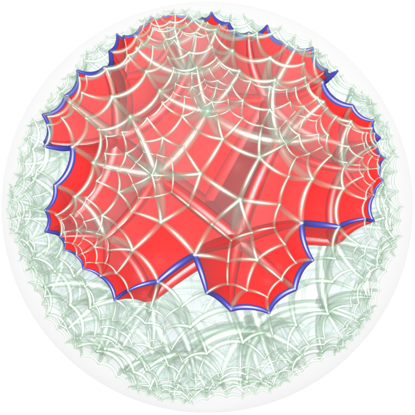
Wilson loop



For a **closed** curve, γ, the **Wilson loop**:

$$W(\gamma) = \prod_{\ell \in \gamma} \tau_\ell$$

Boundary Wilson loop



Strong-coupling expansion:

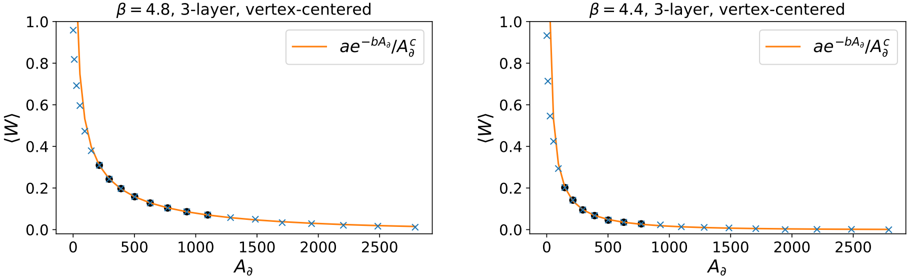
$$
\begin{aligned}
W(A_b) &\propto \tanh^{A_b} \beta \\
&= e^{-A_b \log(\coth \beta)} \\
&= e^{-2A_b \Delta}
\end{aligned}
$$

Use Eq. (1)

$$
\begin{aligned}
W(A_\partial) &\propto e^{-2\alpha \log(|K| A_\partial)\Delta/|K|} \\
&\sim A_\partial^{-2\alpha\Delta/|K|}
\end{aligned}
$$

$$A_b = \alpha \log(A_\partial |K|)/|K| \qquad (1)$$

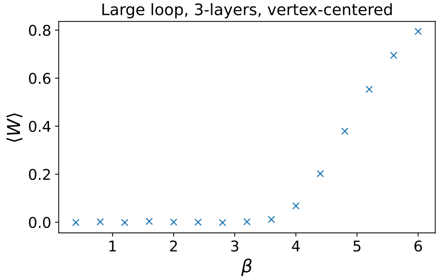
Boundary Wilson loop

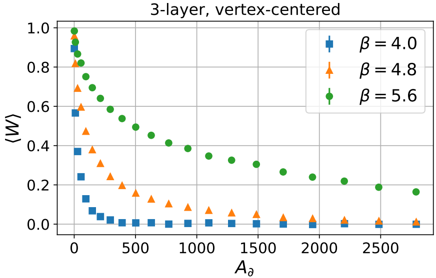


$$A_b = \underbrace{\alpha_0 \log(|K|A_\partial)/|K|}_{\textit{hyperbolic}} + \underbrace{\alpha_1 A_\partial}_{\textit{flat}}$$

$$\implies e^{-2A_b\Delta} \propto \frac{e^{-2\Delta\alpha_1 A_\partial}}{A_\partial^{2\alpha_0\Delta/|K|}}$$

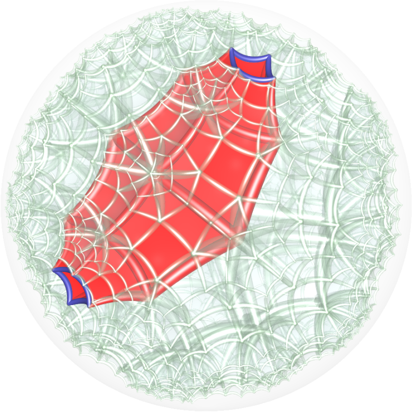
Boundary Wilson loop





Approx. 12×12 loop.

Boundary Plaquette correlator



$$R = \alpha L \log(r/L) \qquad (2)$$

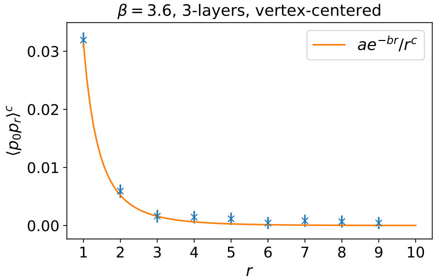
Strong coupling:

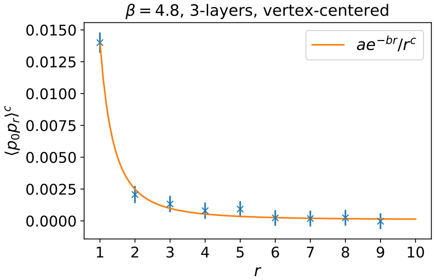
$$\begin{aligned}\langle p_0 p_R \rangle &\propto \tanh^{4R} \beta \\ &= e^{-4\log(\coth\beta)R} \\ &= e^{-8\Delta R}\end{aligned}$$

Using Eq. (2)

$$\begin{aligned}\langle p_0 p_r \rangle &\propto e^{-8\Delta R} \\ &= e^{-8\Delta\alpha L \log(r/L)} \\ &= \left(\frac{r}{L}\right)^{-8\Delta\alpha L}\end{aligned}$$

Boundary Plaquette correlator

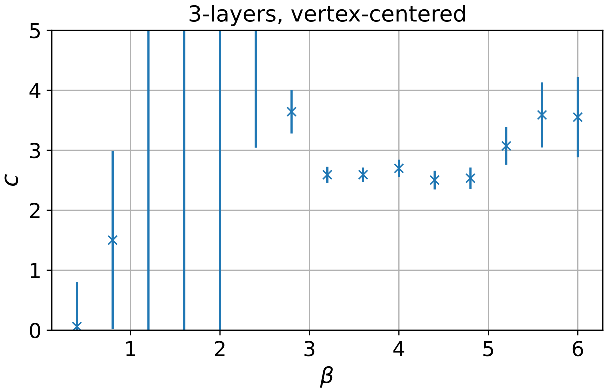




$$R = \underbrace{\alpha_0 L \log(r/L)}_{\textit{hyperbolic}} + \underbrace{\alpha_1 r}_{\textit{flat}}$$

$$\implies e^{-8R\Delta} \propto \frac{e^{-8\Delta\alpha_1 r}}{r^{8\alpha_0 \Delta L}}$$

Boundary power-law



Gauge remarks

- Boundary Wilson loop has zero string tension plus discretization effects.
- Boundary plaquette-plaquette correlator has massless glueball plus discretization effects.
- Very little volume dependence

$\mathbb{Z}_2$ spin

Ising spins on hyperbolic lattices

- Ising spins are placed on the **vertices** of a $\{3,7\}$ lattice.

$$Z = \sum_{\{s\}} \exp(\beta \sum_{\langle ij \rangle} s_i s_j)$$

- $\beta = 1/T$
- $s = \pm 1$
- $\langle ij \rangle$ is a sum over nearest-neighbor pairs
- $\{s\}$ sum is over all possible spin configurations

Boundary physics

Boundary theory

- Ising model: $Z = \sum_{\{s\}} e^{\beta \sum_{\langle ij \rangle} s_i s_j}$ on **whole lattice**
- Some physics more transparent in different variables
  $\rightarrow$ **High-temperature expansion**
- A theory of currents, or fluxes
- Expand $e^{\beta s_i s_j} = \cosh \beta (1 + s_i s_j \tanh \beta)$
- This allows you to **do** the $\sum_{\{s\}}$ exactly

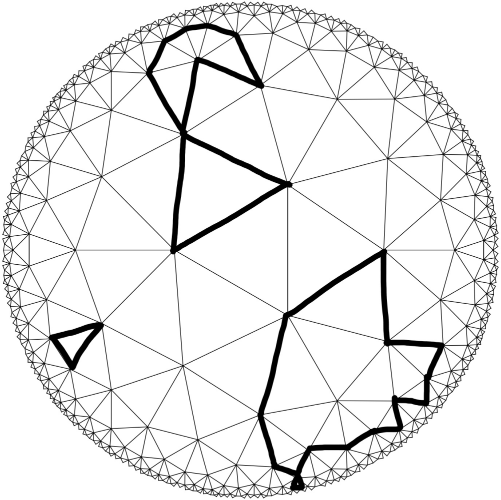
High-temperature expansion

High-temperature expansion

$$Z \propto \sum_r n(r) \tanh^r \beta$$

$n(r)$ is the number of graphs with r bonds.

$\rightarrow$ **Closed graphs**

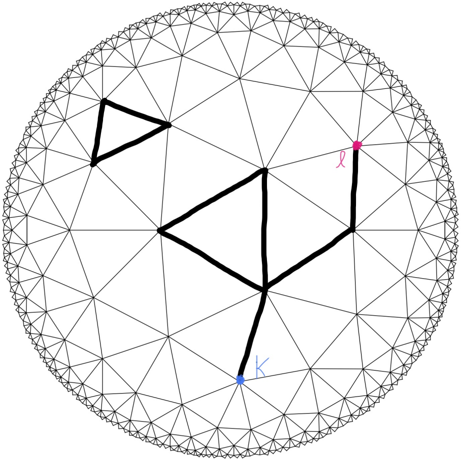


Spin-spin correlator

$$\langle s_k s_\ell \rangle \propto$$

$$\sum_{\{s\}} s_k s_\ell \left(\prod_{\langle ij \rangle} (1 + s_i s_j \tanh \beta) \right)$$

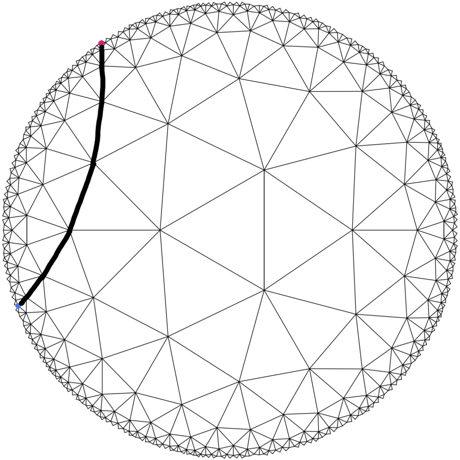
- **Connected** paths between k and ℓ.



Spin-spin correlator

Leading order behavior **through the bulk** for two **boundary** points.

$$\langle s_k s_\ell \rangle(R) \propto \tanh^R \beta$$
$$= \exp(R \log \tanh \beta)$$
$$= \exp(-\log(\coth \beta) R)$$

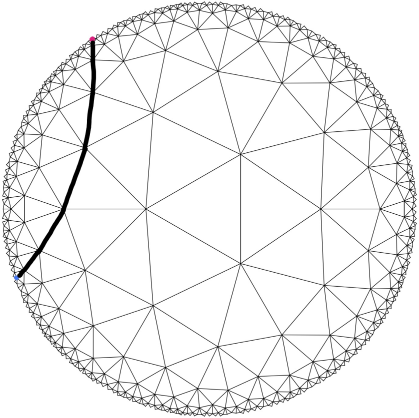


Spin-spin correlator

$$\langle s_k s_\ell \rangle(R) \propto \exp(-\log(\coth\beta)R)$$

The bulk geodesics are related to the boundary "geodesics"

$$R = \alpha L \log(r/L)$$



Spin-spin correlator

So,

$$\exp(-\log(\coth\beta)R) \to \exp(-\log(\coth\beta)\alpha L\log(r/L))$$
$$= (r/L)^{-\alpha L\log(\coth\beta)}$$

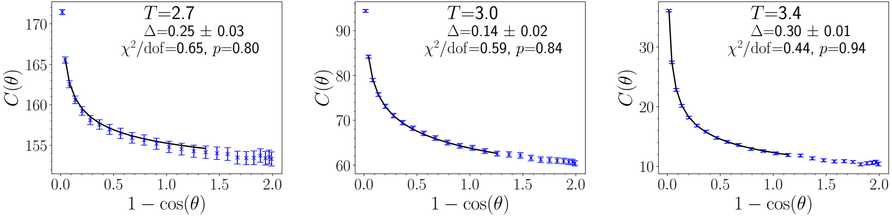
$$\Delta \equiv \frac{1}{2}\alpha L \log\coth\beta$$

$$\to (r/L)^{-2\Delta} \tag{3}$$

Spin-spin correlator

Fit function

$$C(\theta) = a(T) + b(T)(1 - \cos\theta)^{-\Delta(T)},$$



Result

For a range of βs, the two-point correlation function is power-law.

Boundary observables

- Absolute magnetization per site at boundary $\langle M_\partial \rangle / N_\partial$

$$\langle |M_\partial| \rangle / N_{0,\partial} = m_\partial = \frac{1}{\mathcal{N}} \sum_{C} \Big(\frac{1}{N_{0,\partial}} \Big| \sum_{i \in \partial} s_i \Big| \Big)$$

- Energy per site at boundary

$$\langle E_\partial \rangle / N_{0,\partial} = \frac{1}{\mathcal{N}} \sum_{C} \Big(\frac{1}{N_{0,\partial}} \sum_{\langle ij \rangle \in \partial} s_i s_j \Big)$$

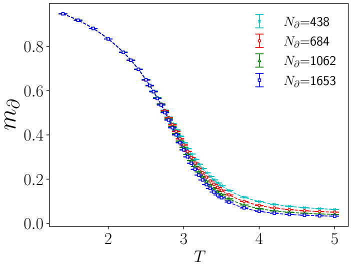
- Susceptibility of magnetization

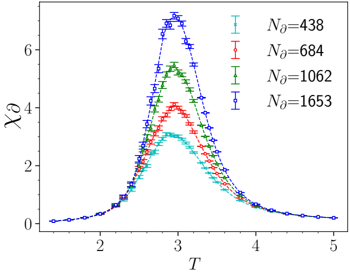
$$\chi(M_\partial) = (\langle M_\partial^2 \rangle - \langle |M_\partial| \rangle^2)/(N_{0,\partial} T)$$

- Susceptibility of energy

$$\chi(E_\partial) = (\langle E_\partial^2 \rangle - \langle E_\partial \rangle^2)/(N_{0,\partial} T^2)$$

Boundary observables



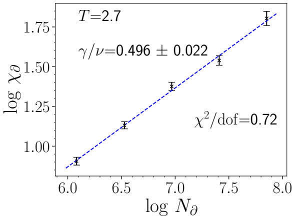


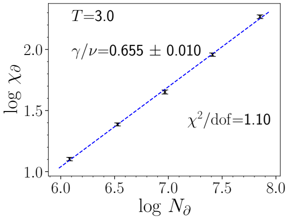
- m_∂ doesn't appear to change spontaneously.
- Broad peak in χ_∂, does not narrow as the volume is increased
- Symmetric around transition point

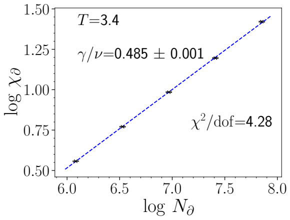
Scaling exponent in boundary susceptibility

$$\chi \sim \int C \mathrm{d}V \sim \int_0^{N_\partial} \frac{1}{r^{2\Delta}} \mathrm{d}r \sim N_\partial^{1-2\Delta}$$

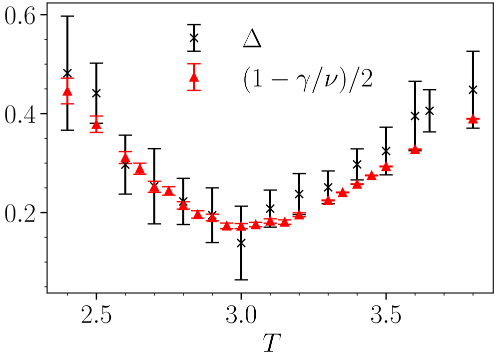
$$\frac{\gamma_\partial}{\nu_\partial} = 1 - 2\Delta$$







Comparison of Δs



The dual lattice

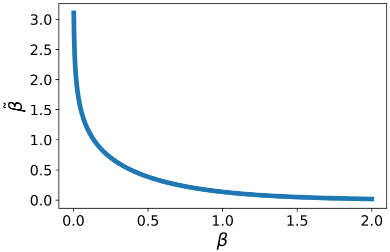
Duality

- Previous numerical results motivated by high-temperature calculations
- Can we understand the low-temperature conformal behavior analytically?
- Yes. Enter duality.

Duality

Ising model duality

- High-low temperature duality
- Exact, $\tilde{\beta} = \frac{1}{2} \log \coth \beta$
- Another Ising model!



The Ising model on $\{7,3\}$ lattice

$$Z = \sum_{\{\tilde{s}\}} e^{\tilde{\beta} \sum_{\langle ij \rangle} \tilde{s}_i \tilde{s}_j}$$

$\tilde{s} = \pm 1$

$$\Longleftrightarrow$$

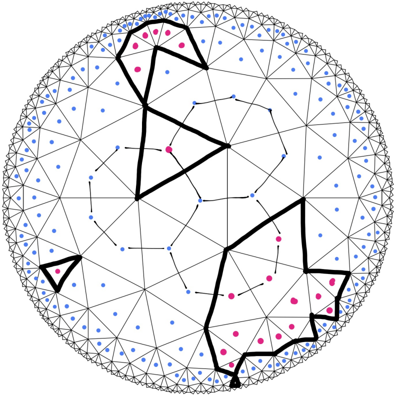
The Ising model on $\{3,7\}$ lattice

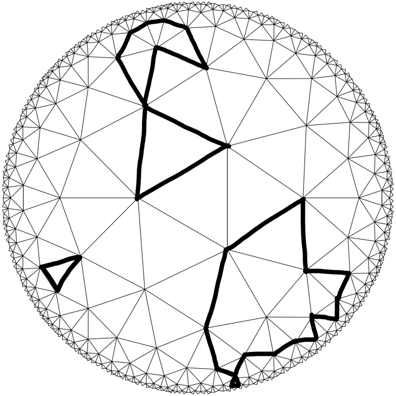
$$Z = \sum_{\{s\}} e^{\beta \sum_{\langle ij \rangle} s_i s_j}$$

$s = \pm 1$

Duality

Ising model duality



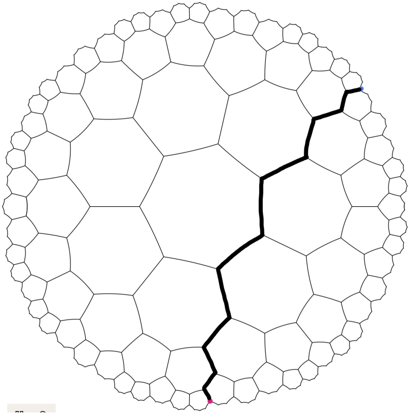


Duality

- Use this duality
- Dual high temperature

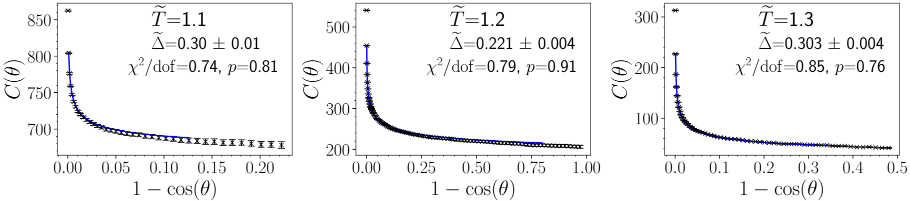
  $\Updownarrow$

  direct low temperature
- Study dual high-temperature correlations



Correlators of the dual spin variable

$$C(\theta) = a(\tilde{T}) + b(\tilde{T})(1 - \cos\theta)^{-\tilde{\Delta}(\tilde{T})},$$



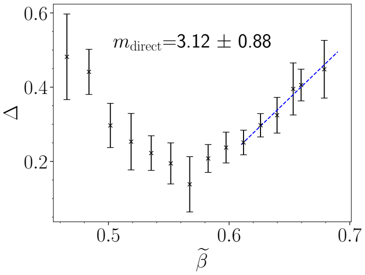
Result

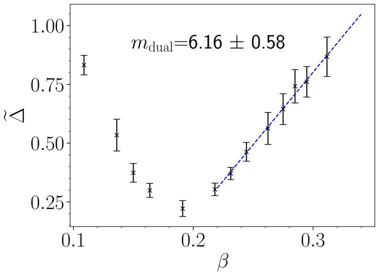
Dual-lattice two-point correlations also power-law for a range of $\tilde{\beta}$.

Fits to the Δs

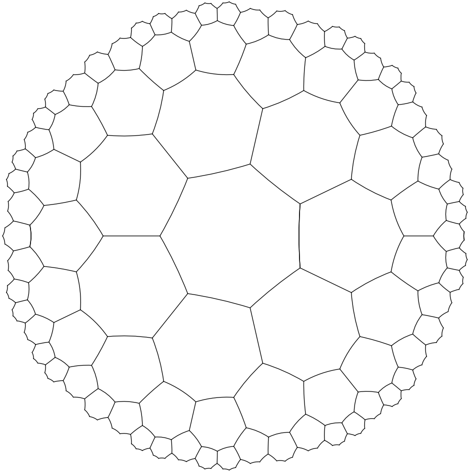
- $\Delta \sim \frac{1}{2}\alpha L \log\coth\beta$
- $\tilde{\beta} = \frac{1}{2}\log\coth\beta$

$$\implies \Delta \sim \alpha L\tilde{\beta} \quad \text{and} \quad \tilde{\Delta} \sim \alpha\tilde{L}\beta$$





Dual lattice



The radii of curvature are known exactly

$$1/L = 2\cosh^{-1}\left(\frac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{q}}\right)^2$$

- $\{3,7\}$ disk: $L \approx 0.63$
- $\{7,3\}$ disk: $\widetilde{L} \approx 1.24$
- $\frac{\widetilde{L}}{L} = 1.97$
- $m_{dual}/m_{direct} = 6.16(58)/3.12(88) = 1.97(59)$,

Spin conclusions

- We find **power-law** correlations in the **boundary** theory
- These power-law correlations support a conformal field theory on the boundary
- The conformal nature can be traced back to the **geometry** of the lattice

  $\rightarrow R = \alpha L \log(r/L)$
- There does not appear to be a typical phase transition in the boundary theory.

Thank you!