PRECISION TESTS OF BULK ENTANGLEMENT ENTROPY

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INTRODUCTION

- The Ryu-Takayanagi formula provides an expression for the entanglement entropy across sub-regions, an intrinsically quantum object in terms of a classical quantity:

the area of the minimal surface.

• Consider the Ryu-Takayanagi formula in the context of AdS/CFT:

The formula can in principle be tested using computations in the dual CFT.

$$S_{\mathrm{EE}}^{\mathrm{CFT}}(A) = \frac{Area(\gamma_A)}{4G_N}$$

A is the subregion of interest in the boundary CFT and $S_{\mathrm{EE}}^{\mathrm{CFT}}(A)$ is corresponding entanglement entropy.

γ_A is the Ryu-Takayanagi surface whose boundary is A.

G_N is the Newton's constant in the bulk AdS.

• This formula admits a well known and widely used modification at order G_N^0.

$$S_{\mathrm{EE}}^{\mathrm{CFT}}(A) = \frac{\mathrm{Area}(\gamma_A)}{4G_N} + S_{\mathrm{bulk}}^{\mathrm{EE}}(\Sigma_A).$$

Σ_A is the region which extends between γ_A and A. $S_{\mathrm{bulk}}^{\mathrm{EE}}$ is the entanglement of all fields present in the bulk effective field theory.

Faulkner, Lewkowycz and Maldacena (2013)

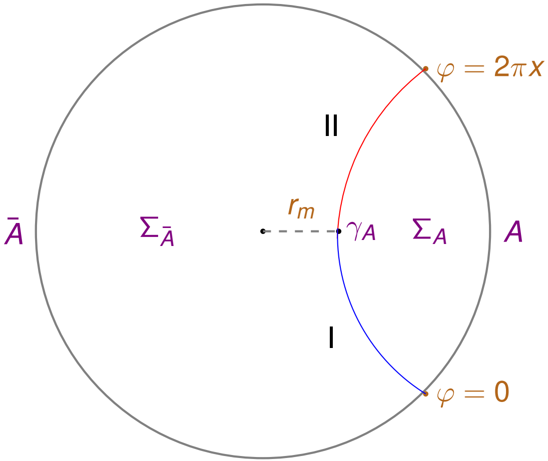


Figure: The $t = 0$ slice of AdS_3. We consider the entanglement between system A and $\bar{A}$ in the boundary. The minimal surface γ_A is the geodesic in the bulk connecting the end points of A. $S^{\mathrm{EE}}_{\mathrm{bulk}}$ is obtained by tracing over bulk fields in $\Sigma_{\bar{A}}$.

- The FLM proposal and its generalizations have played an fundamental role in our recent understanding of quantum gravity and the information paradox.
- The generalizations involve extending the formula to any sub-region in a theory of quantum gravity and the notion of quantum extremal surface.

- The FLM formula is within the framework of AdS/CFT.

We can in principle test this by evaluating entanglement entropy of various states in the CFT and comparing them with the result from the FLM formula.

- Apart from the verification of the FLM formula what can we learn from performing tests.

• We would need to develop techniques to study excited states in the CFT.

Using the replica trick, this involves knowing analytically the $2n$-point function of operators creating the excitations in the CFT.

• We would need to develop techniques to evaluate $S^{\mathrm{EE}}_{\mathrm{bulk}}(\Sigma_A)$.

This involves performing a trace in the bulk geometry including excitations.

• In the CFT we would not naturally obtain a split of the entanglement separately into $\frac{\mathrm{Area}(\gamma_A)}{4G_N}$ and $S^{\mathrm{EE}}_{\mathrm{bulk}}(\Sigma_A)$.

In the bulk we can examine how the geometric part and the quantum part combines together to reproduce the CFT answer.

• Considering a linear combination of the excitations:

We would be able to verify that the precise non-linear behaviour of the $2n$-point in the CFT is reproduced by an apparently a very different structure in the bulk.

• Considering linear combination of states which are related by symmetries, say the conformal symmetries, we can construct situation in which the isometries of the bulk are broken.

Test the FLM formula in less symmetric situations.

• The one instance the FLM formula has been tested is in the context of AdS_3/CFT_2.

Belin, Iqbal, Lokhande (2018)

The entanglement entropy of the excitation of the CFT vacuum by a primary operator, was shown to agree with that given by the FLM formula in the leading and sub-leading short distance expansion.

The operator dimensions of the primary $h << c$ and it was assumed to be a generalised free field.

In the bulk the excitation is dual to a single particle excitation of a minimally coupled scalar field.

- We will generalize this observation and also extend the methods both in the CFT and gravity for other single particle excitations as well as linear combinations of single particle excitations.

In general the excitations break the isometries in the bulk.

• In all cases the FLM formula precisely reproduces the CFT result.

The agreement involves interesting cancellations of contributions from the 2 terms in the FLM formula $\frac{\mathrm{Area}(\gamma_A)}{4G_N}$ and $S^{\mathrm{EE}}_{\mathrm{bulk}}(\Sigma_A)$.

• In the bulk, the agreement requires evaluation of the back reacted geometry to compute corrections to $\frac{\mathrm{Area}(\gamma_A)}{4G_N}$

• To evaluate $S^{\mathrm{EE}}_{\mathrm{bulk}}(\Sigma_A)$ we use the map from global AdS to the Rindler BTZ as in

Belin, Iqbal, Lokhande (2018)

For general single particle excitations, we would need to generalise the computation of Bogoliubov coefficients which relate the excitations in these spaces.

ENTANGLEMENT:

EXCITED STATES IN CFT

Alcaraz, Berganza, Sierra (2011, 2011), N. Lashkari (2014), G. Sáorsi, T. Ugajin (2016),

B. G. Chowdhury, J. R. David (2022)

Consider the state

$$\mathcal{O}|0\rangle = |\mathcal{O}\rangle$$

where $\mathcal{O}$ is an operator in CFT_2 on the cylinder It can be a primary, its descendants or composites.

We consider the reduced density matrix

$$\rho_{\mathcal{O}} = \mathrm{Tr}_{[0,2\pi x]}\Big((|\mathcal{O}\rangle\langle\mathcal{O}|\Big).$$

The entangling interval is of length $2\pi x$.

In the path integral, the density matrix is represented as

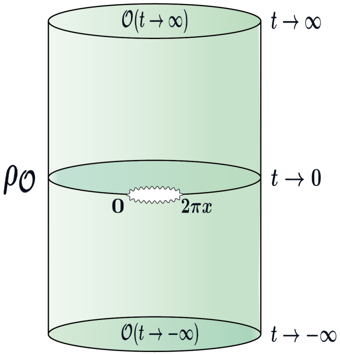


Figure: This figure shows the cut cylinder which represents the path integral for the density matrix $\rho_{\mathcal{O}}$.

• The expression for the entanglement entropy, which contains the subtraction of the entanglement entropy of the vacuum is

$$S_n(\rho_{\mathcal{O}}) = \frac{1}{1-n} \log \left(\frac{\mathrm{Tr}\rho_{\mathcal{O}}^n}{\mathrm{Tr}\rho_{(0)}^n} \right), \qquad S(\rho) = \lim_{n \to 1} S_n(\rho_{\mathcal{O}}).$$

Here $\rho_{(0)}$ refers to the density matrix without any operator insertions.

• From the path integral and conformal transformations the ratio of the traces of the density matrices can be written as the following $2n$-point function on the plane

$$\frac{\mathrm{Tr}\rho_{\mathcal{O}}^n}{\mathrm{Tr}\rho_{(0)}^n} = \frac{1}{\left(\langle\mathcal{O}|\mathcal{O}\rangle\right)^n}\Big\langle \prod_{k=0}^{n-1} w\circ\mathcal{O}(w_k)\prod_{k'=0}^{n-1}\hat{w}\circ\mathcal{O}^*(\hat{w}_k)\Big\rangle.$$

Here $w\circ\mathcal{O}(z)$ refers to the action of the conformal transformation $w(z)$ on the operator $\mathcal{O}$

If $\mathcal{O}$ is a primary of weight h, then $w\circ\mathcal{O}(z) = (\frac{\partial w}{\partial z})^h O(w(z))$

• The map $w(z)$ takes the complex plane to a wedge of angle $2\pi/n$. It glues the n cylinders together.

$$w(z) = \left(\frac{z-u}{z-v}\right)^{\frac{1}{n}},$$

and

$$u = e^{2\pi i x}, \qquad v = 1.$$

$\hat{w} \circ \mathcal{O}(\hat{z})$ refers to the conformal transformation

$$\hat{w}(\hat{z}) = \left(\frac{\frac{1}{\hat{z}} - u}{\frac{1}{\hat{z}} - v}\right)^{\frac{1}{n}}.$$

Under this map, the operators placed at $t \to -\infty$ on each replica cylinder are mapped to

$$w_k = e^{\frac{2\pi i(k+x)}{n}} = \lim_{z \to 0_k} \left(\frac{z-u}{z-v} \right)^{\frac{1}{n}},$$

k labels the wedges.

Similarly, the operators placed at $t \to +\infty$ on each replica cylinder are mapped to

$$\hat{w}_k = \lim_{\hat{z} \to \hat{0}_k} \left(\frac{\frac{1}{\hat{z}} - u}{\frac{1}{\hat{z}} - v} \right)^{\frac{1}{n}} = e^{\frac{2\pi i k}{n}}.$$

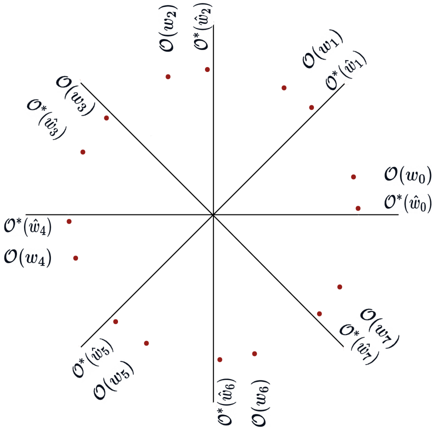


Figure: The figure shows the uniformized plane for the $n = 8$ replica surface. Each cylinder is mapped to a wedge on the uniformed plane. There are $2n$ operators with a pair of operators on each wedge. The operators are located on the unit circe separated by an arc length of $2\pi x$.

- The formulation is general.

If the operators are primary, it is easy to perform the conformal transformations. If they are descendants or composites , it is harder to evaluate.

The $2n$-point function is difficult to evaluate,

only in the case of the free boson, free fermion, such correlators are known.

- However one can set up a systematic expansion in short distance x of the $2n$ point function.

- The leading term is given by:

factorizing the $2n$ point function into n, 2-point functions on each wedge

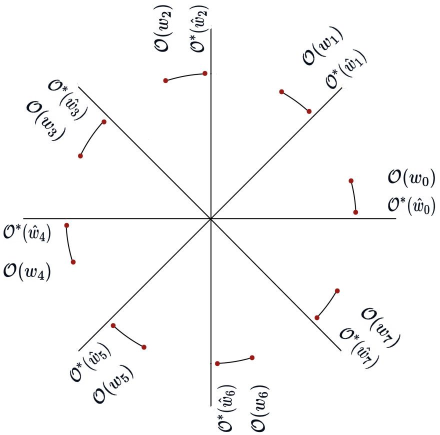


Figure: A $n = 8$ uniformized plane showing the contraction structure of the $2n$-point function for the leading term in the single interval entanglement entropy. The $2n$-point function is factored into n 2-point function with pairs of operators on the same wedge.

- The sub-leading contribution is obtained by factorizing the correlator as demonstrated in the figure

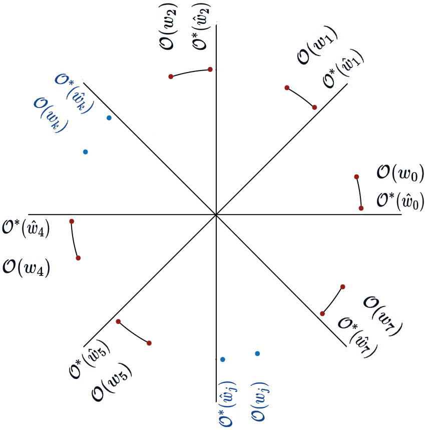


Figure: The figure shows a $n = 8$ uniformized plane with the factorisation of the $2n$-point function into $(n - 2)$, 2-point functions which are contracted on the same wedge and a 4-point functions involving operators on a pair of wedges which are in blue. The contributions from all such pairs are summed over to obtain the sub-leading contribution to the single interval entanglement entropy.

- In equations, the $2n$ point function is expanded as

$$\mathcal{C}_{2n} = \mathcal{C}_{2n}^{(0)} + \mathcal{C}_{2n}^{(1)} + \cdots ,$$

$$\mathcal{C}_{2n}^{(0)} = \prod_{k=0}^{n-1} \langle w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle ,$$

$$\mathcal{C}_{2n}^{(1)} = \sum_{i,j=0, i\neq j}^{n-1} \left(\prod_{k=0, k\neq i,j}^{n-1} \langle w \circ \mathcal{O}(w_k) \hat{w} \circ \mathcal{O}^*(\hat{w}_k) \rangle \right) \Big\langle w \circ \mathcal{O}(w_i) w \circ \mathcal{O}^*(\hat{w}_i) w \circ \mathcal{O}(w_j) w \circ \mathcal{O}^*(\hat{w}_j) \Big\rangle_c .$$

where the subscript 'c' refers to the connected correlator.

The first sub-leading correction arises from the 4-point function of the operators on the replica geometry in which a pair of operators are placed on one of the wedges and another pair in another wedge.

• Once the $2n$-point function is evaluated we can substitute it in the expression for the entanglement entropy and obtain

$$S(\rho_{\mathcal{O}}) = \lim_{n\to 1} \frac{1}{1-n} \log\left(\frac{\mathcal{C}_{2n}}{(\langle\mathcal{O}|\mathcal{O}\rangle)^n}\right).$$

• The four point function is evaluated using the OPE expansion.
For eg in case of the primaries in the jth, k th wedge.

$$\langle w \circ \mathcal{O}(w_j)\hat{w} \circ \mathcal{O}^*(\hat{w}_j) w \circ \mathcal{O}(w_k)\hat{w} \circ \mathcal{O}^*(\hat{w}_k)\rangle_c = (B_j \hat{B}_j B_k \hat{B}_k)^h \times$$
$$\frac{1}{(w_j - \hat{w}_j)^{2h}(w_k - \hat{w}_k)^{2h}} \left(\sum_{q=1}^{\infty} \chi_{\mathrm{vac,q}} w^2 {}_2F_1(q, q, 2q, w) + \sum_p C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C^{\mathcal{O}_p}_{\mathcal{O}\mathcal{O}} w^{h_p} \mathcal{F}(c, h, h_p, w) \right).$$

The cross ratio w is defined as

$$w = \frac{(w_j - \hat{w}_j)(w_k - \hat{w}_k)}{w_j - w_k)(\hat{w}_j - \hat{w}_k)} = \left(\frac{\sin \frac{\pi x}{n}}{\sin \frac{\pi}{n}(j-k)} \right)^2.$$

• The first term in the round bracket is the expansion of the Virasoro block corresponding to the stress tensor exchange in terms of the global $SL(2,C)$ blocks represented by the hypergeometric function ${}_2F_1(q,q,2q,w)$.

The first two coefficients which is sufficient for our purpose are

$$\chi_1 = 0, \qquad \chi_2 = \frac{2h^2}{c},$$

where c is the central charge of the CFT.

• The second term in the round bracket represent the contribution of the Virasoro blocks of the primaries of dimension h_p of the theory.

$C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C^{\mathcal{O}_P}_{\mathcal{O}\mathcal{O}}$ is the product of the corresponding structure constants.

The Virasoro block admits the following expansion

$$\mathcal{F}(c, h, h_p; w) = 1 + O(w).$$

The B 's are derivatives of the conformal transformations to the uniformised plane.

Considered operators of dimensions $(h, 0)$ for simplicity.

• The CFT dependence is encoded in the structure constants.

• To apply this result to holographic CFT's there are 2 cases we can consider

Generalised free fields $h << c$:

The exchanged operator is the composite $:\mathcal{O}^2:$ with $h_p = 2h$.

The product of structure constants

$$C_{\mathcal{O}\mathcal{O}\mathcal{O}_p} C^{\mathcal{O}_p}_{\mathcal{O}\mathcal{O}} = 2, \qquad\qquad h_p = 2h,$$

Heavy states:

The second case we can consider is $h \sim O(c)$,

The contribution from the stress tensor exchange is the leading contribution.

- We have obtained results for both cases, but we will focus on perturbative excitations or generalised free field.

CFT RESULTS

- Consider states

$$L_{-1}^{l}|h,0\rangle = \partial_l \mathcal{O}^{(h,0)}|0,0\rangle$$

in an arbitrary CFT.

$$\begin{aligned}\hat{S}(\rho_{\mathcal{O}[-l]}) &= 2(h+l)(1-\pi x\cot\pi x) - \frac{8h^2}{15c}[D_{\mathcal{O}[l]}(h,2)]^2(\sin\pi x)^4 \\ &\quad -C_{\mathcal{O}\mathcal{O}\mathcal{O}_p}C^{\mathcal{O}_p}_{\mathcal{O}\mathcal{O}}[D_{\mathcal{O}[-l]}(h,h_p)]^2\frac{\Gamma(\frac{3}{2})\Gamma(h_p+1)}{2\Gamma(h_p+\frac{3}{2})}(\sin\pi x)^{2h_p}+\cdots,\end{aligned}$$

$$D_{\partial^l\mathcal{O}}(h,h_p) = \frac{l!\,\Gamma(2h)}{\Gamma(2h-h_p)\Gamma(2h+l)}\sum_{k=0}^{l}\frac{\Gamma(2h-h_p+k)}{k!}\left[\frac{\Gamma(h_p+l-k)}{(l-k)!\Gamma(h_p)}\right]^2.$$

We have verified this by direct computation in the free boson theory for the operator e^{ikX}.

Consider the linear combination in a generalised free field theory

$$|\hat{\Psi}\rangle = \sum_{l=0}^{\infty} c_l L_{-1}^{l} |h, h\rangle .$$

$$S(\rho_{|\hat{\Psi}\rangle}) = \sum_{l,l'=0}^{\infty} \frac{c_l c_{l'}^* \hat{g}_{ll'}(x)}{\langle\hat{\Psi}|\hat{\Psi}\rangle} + 2h(1 - \pi x \cot \pi x) - \frac{\Gamma(\frac{3}{2})\Gamma(4h+1)}{\Gamma(4h+\frac{3}{2})} \frac{(\pi x)^{8h}}{\langle\hat{\Psi}|\hat{\Psi}\rangle^2} \times \left(\sum_{l,l'=0}^{\infty} c_l c_{l'}^* D_{ll'}(h, 2h)\right)^2 + \cdots .$$

There coefficients $g_{ll'}(x)$ and

$$D_{ll'}(h, 2h) = \frac{\Gamma(2h+l)\Gamma(2h+l')}{\left(\Gamma(2h)\right)^2} .$$

$$
\begin{aligned}
\hat{g}_{ll'}(x) &= \partial_z^l \partial_{\hat{z}}^{l'} \hat{G}(z,\hat{z})\Big|_{(z,\hat{z})=(0,0)} \\
\hat{G}(z,\hat{z}) &= -\frac{h}{(1-z\hat{z})^{2h}} \left\{ 2 + \log\left(\frac{z-u}{z-v}\right) + \log\left(\frac{1-u\hat{z}}{1-v\hat{z}}\right) \right. \\
&\qquad \left. + \frac{2}{(u-v)(1-z\hat{z})} \left[(z-u)(1-v\hat{z}) \log\left(\frac{z-u}{z-v}\right) + (v-z)(1-u\hat{z}) \log\left(\frac{1-u\hat{z}}{1-v\hat{z}}\right) \right] \right\}.
\end{aligned}
$$

• A similar expression can be written for the entanglement for semi-classical states, $h >> c$.

We have cross checked the results for these coefficients by comparing our results to the entanglement entropy certain coherent states, Bañados states evaluated by:

Caputa, Ge (2022)

• The results for the linear combination, which will be the focus of our discussion in the bulk.

$$|\Phi\rangle = c_0|h\, h\rangle + c_1 L_{-1}|h\, h\rangle.$$

Let us write the entanglement entropy as

$$S(\rho_{|\Phi\rangle}) = S^{(0)}(\rho_{|\Phi\rangle}) + S^{(1)}(\rho_{|\Phi\rangle}),$$

$$S^{(0)}(\rho_{|\Phi\rangle}) = \frac{1}{|c_0|^2 + 2h|c_1|^2} \times \Big(\Big[4h|c_0|^2 + 4h(2h+1)|c_1|^2\Big](1 - \pi x \cot \pi x)$$

$$+h\Big[(c_0 c_1^* + c_1 c_0^*)\cot \pi x + i(c_0 c_1^* - c_1 c_0^*)\Big](2\pi x - \sin 2\pi x)\Big).$$

$$S^{(1)}(\rho_{|\Phi\rangle}) = -\frac{\Gamma(\frac{3}{2})\Gamma(4h+1)}{\Gamma(4h+\frac{3}{2})}(\sin \pi x)^{8h} \times \left[\frac{|c_0 + 2hc_1|^2}{|c_0|^2 + 2h|c_1|^2}\right]^2$$

- Our goal is to reproduce this result using the FLM formula.

SINGLE PARTICLE STATES IN BULK

• A primary and its global descendants are dual to single particle excitations of a minimally coupled scalar in AdS_3.

The metric of global AdS_3 is given by

$$ds^2 = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2 d\varphi^2 \qquad \varphi \sim \varphi + 2\pi.$$

• The equations of motion of the scalar in this background is given by

$$(\nabla^2 - M^2)\phi(x) = 0\,.$$

$$M^2 = 2h(2h-2)$$

• We expand the solutions in terms of modes

$$\phi(t,r,\varphi)=\sum_{m,n}\left(a_{m,n}e^{-i\Omega_{m,n}t}e^{im\varphi}f_{m,n}(r)+a^{\dagger}_{m,n}e^{i\Omega_{m,n}t}e^{-im\phi}f^{*}_{m,n}(r)\right)\ .$$

m runs over the set of integers due to the periodic boundary conditions on φ.

n labels the radial wave function, n is also quantized, $n=0,1,\cdots$.

$$\Omega_{m,n}=2h+n+|m|$$

• The canonical commutation relations of ϕ and $\dot{\phi}$ result in

$$[a_{m,n}, a^{\dagger}_{m,n}] = \delta_{n,n'}\delta_{m,m'}$$

Therefore single particle states on the global AdS_3 vacuum are given by

$$|\psi_{m,n}\rangle = a^{\dagger}_{m,n}|0\rangle.$$

• The wave functions and quantum numbers of a few low lying modes are

m	n	$f_{m,n}(r)$	$L_0 + \bar{L}_0$	$L_0 - \bar{L}_0$
0	0	$\frac{1}{\sqrt{2\pi}\left(r^2+1\right)^h}$	$2h$	0
0	1	$\frac{1}{\sqrt{2\pi}} \frac{2hr^2-1}{\left(r^2+1\right)^{h+1}}$	$2h+2$	0
0	2	$\frac{1}{\sqrt{2\pi}} \frac{h(2h+1)r^4-2(2h+1)r^2+1}{\left(r^2+1\right)^{h+2}}$	$2h+4$	0
1	0	$\frac{\sqrt{h}r}{\sqrt{\pi}\left(r^2+1\right)^{h+\frac{1}{2}}}$	$2h+1$	1
2	0	$\frac{\sqrt{h(2h+1)}r^2}{\sqrt{2\pi}\left(r^2+1\right)^{h+1}}$	$2h+2$	2

Table: This table lists the explicit wave functions of the single particle states for low values of m and n. The last 2 columns lists out the quantum numbers of L_0, $\bar{L}_0$ of the corresponding dual state in the CFT.

The states in the bulk are normalized using the Klein-Gordan inner product.

• From examining the quantum numbers of the state and comparing normalizations in the CFT and the bulk:

we obtain the following identification

$$
\begin{aligned}
|\Phi\rangle &= c_0|h,h\rangle + c_1 L_{-1}|h,h\rangle \\
&\leftrightarrow c_0|\psi_{0,0}\rangle + \sqrt{2h}c_1|\psi_{1,0}\rangle \equiv |\hat{\phi}\rangle
\end{aligned}
$$

BACK REACTED GEOMETRY

• Once we excite global AdS_3 vacuum by this state, the energy density induced by the excited state back reacts when G_N is non-vanishing and deforms the geometry.

• At the leading oder in G_N we can obtain the back reacted geometry by solving the Einsteins equations with the expectation value of stress tensor as source.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu} = 8\pi G_N \langle \hat{\phi} | T_{\mu\nu} | \hat{\phi} \rangle$$

- The stress tensor is given by

$$T_{\mu\nu} =: \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left((\nabla \phi)^2 + m^2 \phi^2 \right) :$$

We can substitute the mode expansions of the scalar field ϕ and evaluate the expectation value on any state in particular the state $|\hat{\phi}\rangle$

• Note that we have considered the normal ordered stress tensor.

⇒ This ensures we are insensitive to the UV divergence.

⇒ However this divergence does not dependent on the state, it is also same for the AdS_3 vacuum.

⇒ Since we are interested in the difference in entanglement entropy between vacuum and the excited state,

⇒ it is sufficient to work with normal ordered stress tensor.

- For definiteness here is the expectation value *tt*- component

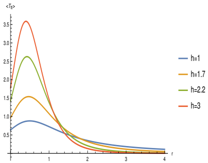
$$\frac{\langle\hat{\phi}|T_{tt}|\hat{\phi}\rangle}{\langle\hat{\phi}|\hat{\phi}\rangle} = \frac{1}{|c_0|^2+2h|c_1|^2}\frac{1}{\pi}\Big\{2h(2h-1)\left(r^2+1\right)^{1-2h}|c_0|^2+4h^2\left(r^2+1\right)^{-2h}\left(4h^2r^2-2hr^2+1\right)|c_1|^2$$
$$+4h^2(2h-1)r\left(r^2+1\right)^{\frac{1}{2}-2h}\cos(t+\varphi)\left(c_1c_0^*+c_0c_1^*\right)$$
$$-4ih^2(2h-1)r\left(r^2+1\right)^{\frac{1}{2}-2h}\sin(t+\varphi)\left(c_1c_0^*-c_0c_1^*\right)\Big\},$$

Note the time and angular dependence.

All stress-tensor components need to be evaluated.

We have performed a cross check that the stress tensor satisfies the conservation law.

Let us examine the energy profile at various angles for $t=0, c_0=c_1=1$.



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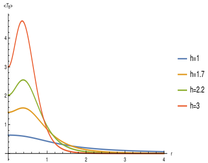
r
h=1
h=1.7
h=2.2
h=3

(a)



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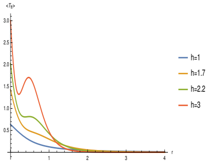

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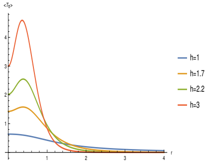

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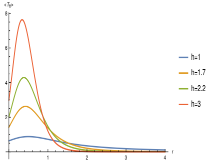

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h=1.7
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h=3

(e)

• We solve the Einstein equation at the linear order in G_N with the stress tensor as the source with the metric anstaz

$$
\begin{aligned}
ds^2 &= \left[1 + r^2 + J_1(t, r, \varphi)\right] dt^2 + 2J_2(t, r, \varphi) dr dt + 2J_3(r) dt d\varphi \\
&\quad + \frac{dr^2}{1 + r^2 + J_4(t, r, \varphi)} + r^2 d\varphi^2.
\end{aligned}
$$

The functions J_i can be found, we present the component J_4.

$$J_4(t,r,\varphi)=\frac{G_N}{(|c_0|^2+2h|c_1|^2)}\Big\{2|c_0|^2 d_{0,0}(r)+2(2h)|c_1|^2 d_{1,0}(r)$$
$$+2\sqrt{2h}[R_4(r)(c_1c_0^*+c_0c_1^*)\cos(t+\varphi)-i\tilde{R}_4(r)(c_1c_0^*-c_0c_1^*)\sin(\varphi+t)]\Big\}.$$

$$d_{0,0}(r)=-8h+\frac{8h}{(r^2+1)^{2h-1}},$$

$$d_{1,0}(r) = 4\left(4h^2r^2+2h+1\right)\left(r^2+1\right)^{-2h}-4(2h+1).$$

$$R_4(r) = \frac{-4\sqrt{2h}\sqrt{r^2+1}}{r}+\frac{4\sqrt{2h}\left(2hr^2+1\right)}{r\left(r^2+1\right)^{2h-\frac{1}{2}}},$$
$$\tilde{R}_4(r) = \frac{4\sqrt{2h}\sqrt{r^2+1}}{r}+\frac{4\sqrt{2h}\left(2hr^2+1\right)}{r\left(r^2+1\right)^{2h-\frac{1}{2}}},$$

• The metric satisfies the following property:

From the metric we can read out boundary stress tensor using the Fefferman-Graham coordinates.

The boundary stress tensor read out using these co-ordinates agrees with that of the expectation value of the CFT stress tensor evaluated in the state $|\Phi\rangle$.

$$\frac{\langle\Phi|T_{tt}(t,\varphi)|\Phi\rangle}{\langle\Phi|\Phi\rangle} = -\frac{c}{12}+\frac{2h|c_0|^2+2h(2h+1)|c_1|^2}{|c_0|^2+2h|c_1|^2} + \frac{2h}{|c_0|^2+2h|c_1|^2}\Big[(c_0^*c_1+c_0c_1^*)\cos(t+\varphi)+i(c_0^*c_1-c_1^*c_0)\sin(t+\varphi)\Big].$$

PERTURBED MINIMAL AREA

• We have a time dependent metric which breaks rotational symmetry.

To evaluate the perturbed minimal area at order G_N , we need not correct the Ryu-Takayanagi surface.

The shift in the area is given by evaluating the length of the RT geodesic using the perturbed metric.

$$\delta A = -\frac{1}{2}\int_{\gamma_A} dr J_4(t=0,r,\varphi)\frac{(r^2-r_{\mathrm{min}}^2)^{\frac{1}{2}}}{r(1+r^2)^{\frac{3}{2}}},$$

The integral is along the RT-geodesic.

J_4 also depends on the angle we need the explicit equation of the RT-surface.

r_m is the minimum radial distance at the turning point of the RT surface. It is related to the interval length as

$$\pi x = \arctan \frac{1}{r_m} = \mathrm{arccot}\, r_m \equiv \frac{\theta}{2}.$$

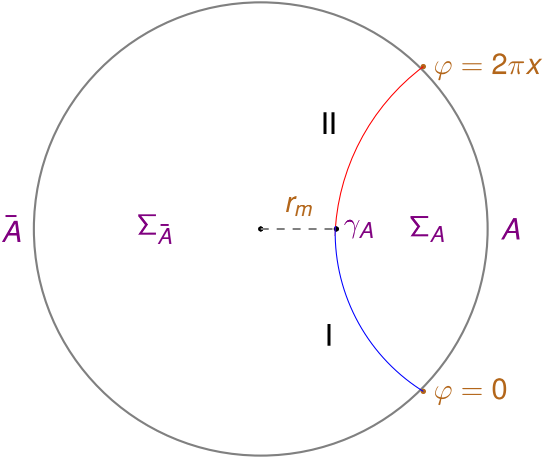

Figure: The $t = 0$ slice of AdS_3. We consider the entanglement between system A and $\bar{A}$ in the boundary. The minimal surface γ_A is the geodesic in the bulk connecting the end points of A. γ_A consists of: Branch I, where $\varphi'(r) < 0$ and Branch II with $\varphi'(r) > 0$. When perturbation breaks φ isometry one need to evaluate the minimal area for the above branches separately.

• The equations of the 2 branches of the RT surface are given by

$$\cos\varphi = \frac{r_m^2\sqrt{1+r^2} + \sqrt{r^2 - r_m^2}}{r(1+r_m^2)}, \qquad \text{Branch I},$$
$$\cos\varphi = \frac{r_m^2\sqrt{1+r^2} - \sqrt{r^2 - r_m^2}}{r(1+r_m^2)}, \qquad \text{Branch II}.$$

Performing the integral for the perturbed minimal area, we obtain

$$
\begin{aligned}
\frac{\delta A}{4G_N} = & \frac{1}{|c_0|^2 + 2h|c_1|^2} \times \Bigg(\Big[4h|c_0|^2 + 4h(2h+1)|c_1|^2 \Big] (1 - \pi x \cot \pi x) \\
& + h\Big[(c_0 c_1^* + c_1 c_0^*) \cot \pi x + i(c_0 c_1^* - c_1 c_0^*) \Big] (2\pi x - \sin 2\pi x) \Bigg) \\
& - \frac{\Gamma(\frac{3}{2})\Gamma(2h+1)}{\Gamma(2h+\frac{3}{2})} \frac{(\pi x)^{4h}}{(|c_0|^2 + 2h|c_1|^2)} \times \Big(|c_0|^2 + 2h(c_0 c_1^* + c_0^* c_1) + (2h)^2 |c_1|^2 \Big) \\
& + \cdots
\end{aligned}
$$

The leading term precisely agrees with the leading term from the CFT.

• The 2nd term has a similar structure to that in the CFT but dependence is $(\pi x)^{4h}$ not $(\pi x)^{8h}$.

Such a term is not there in the CFT result.

It should be cancelled from the contribution in the bulk entanglement entropy $S_{\mathrm{bulk}}(\Sigma_A)$?

BULK ENTANGLEMENT ENTROPY

• If $|\psi\rangle$ is the single particle state of interest, then the reduced density matrix in the bulk is given by

$$\rho_{\text{bulk}} = \text{Tr}_{\bar{\Sigma}_A}\Big(|\psi\rangle\langle\psi|\Big).$$

We need to trace over $\bar{\Sigma}_A$, the region to the left of γ_A.

Once this is performed we can evaluate the corresponding Von-Neumann entropy.

This calculation is non-trivial to do directly.

• It is convenient to go to a co-ordinate system in which the RT surface becomes a Rindler horizon.

Casini, Huerta, Myers 2011

Then the partial trace over $\bar{\Sigma}_A$ would reduce the density matrix to a thermal density matrix.

In the presence of excitations, it would reduce to evaluating thermal n-point functions.

• The map arises by parametrising the AdS_3 hyperboloid in 2 different ways as follows

$$
\begin{aligned}
X_0 &= \sqrt{r^2+1}\,\sin t, \\
&= \sqrt{\rho^2-1}\,\sinh\tau, \\
X_1 &= \sqrt{r^2+1}\,\cos t, \\
&= \rho\cosh\eta\cosh x + \sqrt{\rho^2-1}\,\sinh\eta\cosh\tau, \\
X_2 &= r\sin(\varphi-\frac{\theta}{2}), \\
&= \rho\sinh x, \\
X_3 &= r\cos(\varphi-\frac{\theta}{2}), \\
&= \cosh\eta\sqrt{\rho^2-1}\,\cosh\tau + \rho\sinh\eta\cosh x,
\end{aligned}
$$

θ is the opening angle of the RT geodesic and related to the interval πx , the AdS_3 hyperboloid is defined by

$$-X_0^2 - X_1^2 + X_2^2 + X_3^3 = -1$$

The metric in the (ρ, τ, x) co-ordinates reduces to

$$ds^2 = -(\rho^2 - 1)d\tau^2 + \frac{d\rho^2}{\rho^2 - 1} + \rho^2 dx^2, \qquad x \in \mathbb{R}.$$

The Rindler BTZ metric, with inverse temperature $\beta = 2\pi$.

We can also obtain the inverse transformation: global AdS_3 co-orodinates (r, t, φ) to the Rindler BTZ coordinates (ρ, τ, x) which is given by

$$
\begin{aligned}
r &= \sqrt{\rho^2 \sinh^2 x + \left(\sqrt{\rho^2 - 1} \cosh\eta \cosh\tau + \rho \cosh x \sinh\eta\right)^2}\,, \\
t &= \arctan\left(\frac{\sinh\tau\sqrt{\rho^2-1}}{\rho\cosh x\cosh\eta + \sqrt{\rho^2-1}\cosh\tau\sinh\eta}\right)\,, \\
\varphi - \frac{\theta}{2} &= \arctan\left(\frac{\rho\sinh x}{\sqrt{\rho^2-1}\cosh\tau\cosh\eta + \rho\cosh x\sinh\eta}\right)\,.
\end{aligned}
$$

- The usefulness of the Rindler co-ordinates is that:

the horizon at $t = 0, \rho = 1$ is the image of the Ryu-Takayanagi geodesic in global AdS_3

once we make the identification

$$\cosh \eta = \frac{1}{\sin \frac{\theta}{2}} = \sqrt{r_m^2 + 1}.$$

• The Rindler BTZ coordinates does not the cover global AdS.

This can be seen from the equation for φ, we have chosen one branch for the arctan.

Choosing the branch sifted by π covers another part of the global AdS.

Therefore we have 2 patches, the left and the right Rindler wedges.

• A scalar field in global *AdS* will have support in both branches and therefore will need to be expanded as

$$\phi(\rho,\tau,x)=\sum_{L,R}\int_{\omega>0}\frac{d\omega dk}{4\pi^2}\left(e^{-i\omega\tau}b_{\omega,k,l}(\rho,x)g_{\omega,kl}(\rho,x)+e^{i\omega\tau}b^{\dagger}_{\omega,k,l}(\rho,x)g^{*}_{\omega,k,l}(\rho,x)\right)$$

with the commutation relations

$$[b_{\omega,k,l},b^{\dagger}_{\omega',k',l'}]=4\pi^2\delta(\omega-\omega')\delta(k-k')\delta_{ll'}.$$

The wave functions $g_{\omega,kl}(\rho,x)$ are known and are hypergeometric functions.

- We have 2 Rindler wedges, global AdS_3 vacuum can be written as a thermofield double state in the Rindler left and right vaccua

$$|0\rangle = \sum_n e^{-\frac{2\pi E_n}{2}} |n^*\rangle_L |n\rangle_R.$$

Inverse temperature is 2π.

• From the fact that the field ϕ can be expanded in modes in either the Rindler coordinates or in terms of the global AdS_3 coordinates, we must have the relation

$$a_{m,n} = \sum_{l,\omega,k} (\alpha_{m,n;\omega,k,l} b_{\omega,k,l} + \beta_{m,n;\alpha,k,l} b^{\dagger}_{\omega,k,l}),$$

where $\alpha_{m,n,\omega,k,l}$, $\alpha_{m,n,\omega,k,l}$

are the Bogoliubov coefficients relating the particle creation and annihilation operators in the two coordinates.

• The thermofield double satisfies

$$b_{\omega,k,L}|0\rangle = e^{-\pi\omega} b^{\dagger}_{\omega,-k,R}|0\rangle, \qquad b^{\dagger}_{\omega,k,L}|0\rangle = e^{\pi\omega} b_{\omega,-k,R}|0\rangle,$$

Action of the operators on the left wedge can be converted to the action on the right wedge.

• The global AdS_3 vacuum is annihilated by the operators $a_{m,n}$, we see that Bogoliubov coefficients must satisfy the relations

$$\alpha_{\omega,k,L} = -e^{\pi\omega}\beta^*_{\omega,-k,R}, \qquad \beta^*_{\omega,k,L} = -e^{-\pi\omega}\alpha_{\omega,-k,R}.$$

• These relations allow us to write the single particle state in terms of an operator acting on the right sector alone

$$a^{\dagger}_{m,n}|0\rangle = |\psi_{m,n}\rangle = \sum_{\omega,k}\left[(1-e^{-2\pi\omega})\alpha^{*}_{m,n;\omega,k,R}b^{\dagger}_{\omega,k,R} + (1-e^{2\pi\omega})\beta_{m,n;\omega,k,R}b_{\omega,k,R}\right]|0\rangle.$$

Since excitations can be written in terms of the right sector alone, the trace over $\bar{\Sigma}_A$ is easily done.

The trace over the left sector results in a thermal vacuum over which we have single particle excitation.

For eg. on the thermofield double

$$\rho_0 = \mathrm{Tr}_{\bar{\Sigma}_A}\Big(|0\rangle\langle 0|\Big) = e^{-2\pi H_R}.$$

here $\hat{H}_R$ is the Hamiltonian of the single particle excitations in the right wedge given by

$$H_R = \sum_{\omega,k} w b^{\dagger}_{\omega,k,R} b_{\omega,k,R}.$$

ρ_0 is a thermal state with inverse temperature 2π.

• Consider a linear combination of single particle excitations

$$
\begin{aligned}
|\psi\rangle &= \sum_{m,n} B_{m,n} a^\dagger_{mn}|0\rangle, \\
&= \sum_{m,n,\omega,k} B_{m,n}\left[(1-e^{-2\pi\omega})\alpha^*_{m,n;\omega,k,R} b^\dagger_{\omega,k,R} + (1-e^{2\pi\omega})\beta_{m,n;\omega,k,R} b_{\omega,k,R}\right]|0\rangle.
\end{aligned}
$$

Here the coefficients B_{mn} are such that the sate is normalised to unity

$$\sum_{mn} |B_{mn}|^2 = 1.$$

We can ignore the subscript R,

writing the density matrix of the state $|\psi\rangle$,

$$\rho = \sum_{m,n,\omega,k} B_{m,n} \left[(1-e^{-2\pi\omega})\alpha^*_{m,n;\omega,k} b^\dagger_{\omega,k} + (1-e^{2\pi\omega})\beta_{m,n;\omega,k} b_{\omega,k}\right] |0\rangle\langle 0|$$
$$\times \sum_{m',n',\omega',k'} B^*_{m',n'} \left[(1-e^{-2\pi\omega'})\alpha_{m',n';\omega,k'} b_{\omega',k'} + (1-e^{2\pi\omega'})\beta^*_{m',n';\omega',k'} b^\dagger_{\omega',k'}\right].$$

Tracing out the left sector we obtain

$$\rho_{\rm bulk} \equiv {\rm Tr}_{\mathcal{H}_\mathcal{L}} \rho.$$

For the excited state we obtain the following reduced density matrix

$$\rho_{\text{bulk}} = \sum_{m,n,\omega,k} B_{m,n}\left[(1-e^{-2\pi\omega})\alpha^{*}_{m,n;\omega,k}b^{\dagger}_{\omega,k} + (1-e^{2\pi\omega})\beta_{m,n;\omega,k}b_{\omega,k}\right]\rho_0$$
$$\times \sum_{m',n',\omega,k} B^{*}_{m',n'}\left[(1-e^{-2\pi\omega'})\alpha_{m',n';\omega',k'}b_{\omega',k'} + (1-e^{2\pi\omega'})\beta^{*}_{m',n';\omega',k'}b^{\dagger}_{\omega',k}\right].$$

• We need evaluate the difference in single interval entanglement between the single particle excitations and the ground state in the CFT.

This difference in the contribution to the bulk entanglement can be written as

$$S_{\mathrm{bulk}}(\Sigma_A) = \lim_{n\to 1} S_{n\,:\mathrm{bulk}}(\Sigma_A), \qquad S_{n\,:\mathrm{bulk}}(\Sigma_A) = \frac{1}{1-n}\log\frac{\mathrm{Tr}(\rho_{\mathrm{bulk}})^{\mathrm{n}}}{\mathrm{Tr}\rho_0^{\mathrm{n}}}.$$

• In principle it is possible to evaluate the trace $\mathrm{Tr}(\rho_{\mathrm{bulk}})^n$ using the creation and annihilation operator algebra and the two point function

$$\mathrm{Tr}(\rho_0 b^\dagger_{\omega,k} b_{\omega' k'}) = \frac{4\pi^2 \delta(\omega - \omega')\delta(k - k')}{e^{2\pi\omega} - 1}.$$

For this we would need the Bogoliubov coefficients and also perform integrals over ω and k and then analytically continue in the replica index n to obtain the entanglement entropy.

• However there is a simplification when one is interested in the short distance expansion.

Belin, Iqbal, Lokhande (2018).

The Bogoliubov coefficients for single particle can be analytically evaluated,

eg for the state $a^{\dagger}_{0,0}|0\rangle$

$$\alpha_{0,0;\omega,k} = \frac{1}{(\cosh\eta)^{2h}}\left[\frac{e^{\eta}-i}{e^{\eta}+i}\right]^{i\omega} F(\omega,k),$$

$$\beta_{0,0;\omega,k} = -\frac{1}{(\cosh\eta)^{2h}}\left[\frac{e^{\eta}+i}{e^{\eta}-i}\right]^{i\omega} F(\omega,k).$$

$$F(\omega,k) = \frac{2^{2h}\sqrt{\omega}}{\Gamma(2h)\sqrt{4\pi}}\left|\Gamma(iw)\Gamma\left(h+i\frac{\omega-k}{2}\right)\Gamma\left(h-i\frac{\omega-k}{2}\right)\right|.$$

- In the short interval limit

$$\pi x = \arctan \frac{1}{r_m} = \mathrm{arccot}\, r_m \equiv \frac{\theta}{2}, \quad \cosh \eta = \frac{1}{\sin \frac{\theta}{2}} = \sqrt{r_m^2 + 1}.$$

The Bogoliubov coefficients are suppressed as $(\pi x)^{2h}$

This is true for the Bogoliubov coefficients for other excited states as well.

We can set up a perturbation theory in the short distance expansion to evaluate $S_{\mathrm{bulk}}(\Sigma_A)$

• The leading contributions to the $S_{\rm bulk}(\Sigma_A)$ is quadratic in the Bogoliubov coefficients and is given by

$$S^{(1)}_{\rm bulk}(\Sigma_A) = 2\pi \sum_{\omega_1, k_1} \omega_1 \left(|B \cdot \alpha_1^*|^2 + |B \cdot \beta_1|^2 \right).$$

$$B \cdot \alpha_i^* = \sum_{m,n} B_{m,n} \alpha^*_{m,n;\omega_i,k_i}, \qquad B \cdot \beta_i = \sum_{m,n} B_{m,n} \beta_{m,n;\omega_i,k_i}.$$

This contribution is proportional to $(\pi x)^{4h}$.

- To evaluate this for the excited state $\hat{\phi}$

$$|\hat{\phi}\rangle = c_0|\psi_{0,0}\rangle + \sqrt{2h}c_1|\psi_{1,0}\rangle,$$

we choose

$$B_{00} = \frac{c_0}{\sqrt{|c_0|^2 + 2h|c_1|^2}}, \qquad B_{10} = \frac{\sqrt{2h}c_1}{\sqrt{|c_0|^2 + 2h|c_1|^2}}$$

Substituting in $S^{(1)}_{\rm bulk}(\Sigma_A)$, along with the expressions for Bogoliubov coefficients and the B's and performing the integrals over ω, k

$$S^{(1)}_{\rm bulk}(\Sigma_A)|_{|\hat{\phi}\rangle} \quad = \quad + \left[\frac{\left|c_0 + 2hc_1\right|^2}{|c_0|^2 + 2h|c_1|^2}\right] \frac{\Gamma(2h+1)\Gamma(\frac{3}{2})}{\Gamma(2h+\frac{3}{2})}(\pi x)^{4h} + \cdots$$

This precisely cancels the ‘unwanted’ term from the correction in the minimal area.

• We proceed with the next order correction to $S_{\text{bulk}}(\Sigma_A)$, which is given by

$$S^{(2)}(\Sigma_A) = \frac{1}{2}\sum_{\substack{\omega_1,\omega_2\\ k_1,k_2}} \Bigg[2\pi(\omega_1-\omega_2)\frac{(1-e^{-2\pi\omega_1})(1-e^{2\pi\omega_2})}{1-e^{2\pi(\omega_2-\omega_1)}}\Big|(B\cdot\alpha_1^*)(B^*\cdot\alpha_2)+(B^*\cdot\beta_1^*)(B\cdot\beta_2)\Big|^2$$

$$2\pi(\omega_1+\omega_2)\frac{(1-e^{2\pi\omega_1})(1-e^{2\pi\omega_2})}{1-e^{2\pi(\omega_1+\omega_2)}}2\Big\{|B\cdot\alpha_1^*|^2|B^*\cdot\beta_2|^2+(B\cdot\alpha_1^*)(B^*\cdot\beta_2^*)(B^*\cdot\alpha_2)(B\cdot\beta_1)\Big\}\Bigg].$$

This correction is quartic in the Bogoliubov coefficients and

involves a double integral over ω, k

• Substituting again the Bogoliubov coefficients and the *B*'s the second order contribution to the bulk entanglement is given by

$$S^{(2)}(\Sigma_A)|_{|\hat{\phi}\rangle} = -\left[\frac{\left|c_0 + 2hc_1\right|^2}{|c_0|^2 + 2h|c_1|^2}\right]^2 \frac{\Gamma(\frac{3}{2})\Gamma(4h+1)}{\Gamma(4h+\frac{3}{2})}(\pi x)^8.$$

This precisely agrees with that evaluated from the CFT.

- Therefore the LHS and RHS of

$$S_{\mathrm{EE}}^{\mathrm{CFT}}(A) = \frac{\mathrm{Area}(\gamma_A)}{4G_N} + S_{\mathrm{bulk}}^{\mathrm{EE}}(\Sigma_A).$$

agrees when evaluated in the short distance expansion on the state

$$\begin{aligned} |\Phi\rangle &= c_0|h\,h\rangle + c_1 L_{-1}|h\,h\rangle \\ &\leftrightarrow |\hat{\phi}\rangle = c_0 a^\dagger_{0,0}|0\rangle + c_1\sqrt{2h}a^\dagger_{1,0}|0\rangle \end{aligned}$$

To summarize:

We have performed such checks for the following 6 low lying states

$$|\hat{\psi}^{(1,0)}\rangle = L_{-1}|h\, h\rangle, \qquad |\hat{\psi}^{(2,0)}\rangle = L_{-1}^{2}|h\, h\rangle,$$

$$|\hat{\psi}^{(1,1)}\rangle = L_{-1}\bar{L}_{-1}|h\, h\rangle, \qquad |\hat{\psi}^{(2,2)}\rangle = L_{-2}\bar{L}_{-2}|h\, h\rangle,$$

$$|\Phi\rangle = c_0|h\, h\rangle + c_1 L_{-1}|h\, h\rangle, \qquad |\Upsilon\rangle = c_0|h\, h\rangle + c_1 L_{-1}\bar{L}_{-1}|h\, h\rangle.$$

In each case there is an extra 'spurious' contribution from the perturbed minimal area term which is precisely cancelled by the leading contribution from $S_{\rm bulk}(\Sigma)$

In each case the CFT answer precisely agrees with that given by the FLM formula.

CONCLUSIONS/DISCUSSIONS

- The details involving the verification of the FLM formula are interesting and several features of holography are tested.

$\Rightarrow$ CFT evaluation of entanglement entropy of excited states involving descendants,

$\Rightarrow$ Use of generalised free fields for perturbative states,

$\Rightarrow$ Evaluation of Bogoliubov coefficients from global AdS to the Rindler BTZ.

$\Rightarrow$ The nonlinearity of the entanglement entropy in the superposition of excited states.

$\Rightarrow$ FLM formula as well as the RT formula in less symmetric situations.

- Recently similar tests have been done for scalar excitations and $U(1)$ current excitations in $d > 2$

S. Colin-Ellerin, G. Lin (2024)

• An interesting generalization to this work is to study the single interval entanglement entropy of the descendants of the vacuum $L_{-(l+2)}|0\rangle, \; l = 0, 1, 2, \cdots$

$$
\begin{aligned}
S(\rho_{L_{-(l+2)}|0\rangle}) \;\; &= \;\; 2(l+2)(1-\pi x \cot \pi) - \frac{8(l+2)^2}{15c} \sin^4 \pi x \\
&\quad - \left[\frac{(l+3)(l+2)(l+1)}{3!}\right]^2 \frac{128}{315} \sin^8 \pi x + \cdots \; .
\end{aligned}
$$

This has been evaluated in the CFT.

B. G. Chowdhury, J. R. David (2022)

Note that the coefficients are universal numbers.

- These states are dual to excitations of the graviton in the bulk. In 3d, these would be topological.

Reproducing this from the bulk would help us understand more about the entanglement properties of graviton as well as the FLM formula and its generalisations better.