Quantum Hall effect: Topology and dynamics in arbitrary dimensions

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Women at the intersection of Mathematics and Theoretical Physics

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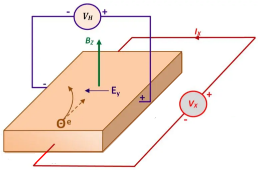

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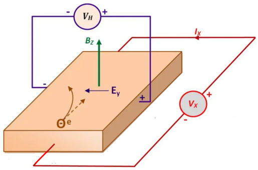

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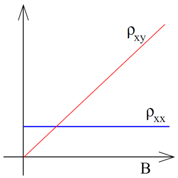

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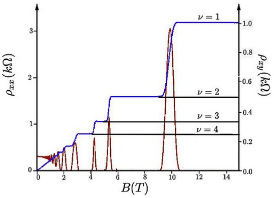

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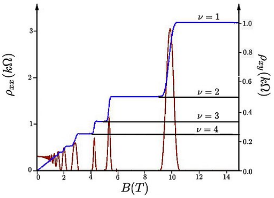

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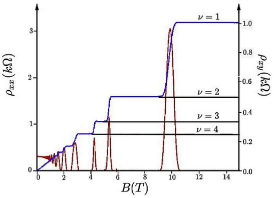

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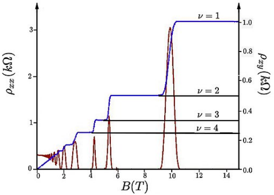

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- Framework for interesting ideas
  - conformal and topological field theories
  - non-commutative geometries, fuzzy spaces
  - bulk-edge dynamics, bosonization

Quantum mechanics of 2d charged particle moving in a strong magnetic field (Landau problem)

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\begin{aligned}
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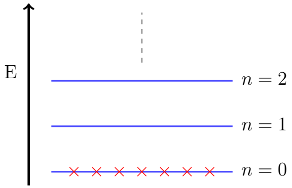

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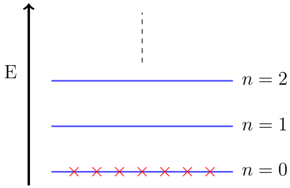

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Basic features of integer QHE

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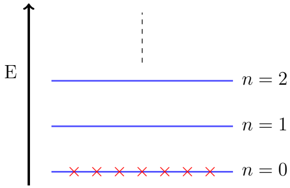

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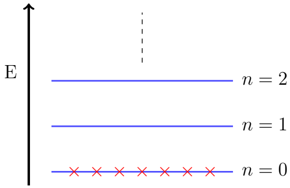

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Lowest Landau level (LLL) : $D_{\bar{z}}\Psi = (\partial_{\bar{z}} + z/2)\Psi = 0$

$$\psi_n \sim z^n e^{-|z|^2/2} \; , z = x + iy$$

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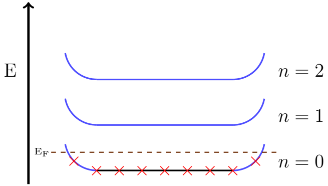

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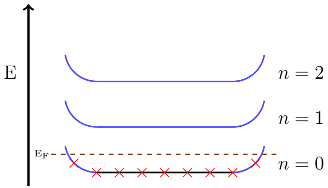

Low energy dynamics is confined on the edge.

Incompressible quantum Hall droplets with boundary fluctuations

Edge excitations $\Longleftrightarrow$ area preserving diffeomorphisms

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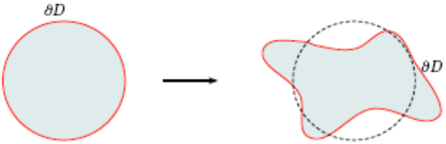

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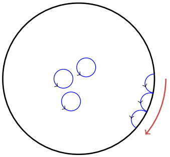

Effective actions

Interested in long wavelength description

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Anomaly cancellation between bulk and edge actions,

$$\delta S_{\text{bulk}} + \delta S_{\text{edge}} = 0$$

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$$S_{eff} = \frac{1}{4\pi} \int \left[[A + (s + \tfrac{1}{2})\omega] \, d[A + (s + \tfrac{1}{2})\omega] - \frac{1}{12}\omega d\omega \right] + \cdots$$

$$\omega = \text{spin connection} \qquad s = 0 \to LLL \,,\; s = 1 \to \text{1st LL}, \cdots$$

$$T^{ij} = \frac{2}{\sqrt{g}} \frac{\delta S_{eff}}{\delta g_{ij}} = \frac{\eta_H}{2} (\epsilon^{il} \dot{g}^{lj} + \epsilon^{jl} \dot{g}^{li})$$

η_H = Hall viscosity

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  QHE on $\mathbb{CP}^k$ (KARABALI AND NAIR, 2002...)

  - higher dimensionality
  - possibility of having both abelian and nonabelian magnetic fields

$\mathbb{CP}^k$: 2k dim space, locally parametrized by z_i, $i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2} = g_{i\,\bar{i}}\, dz^i d\bar{z}^{\bar{i}} \qquad \Omega = i\, g_{i\,\bar{i}}\, dz^i \wedge d\bar{z}^{\bar{i}}$$

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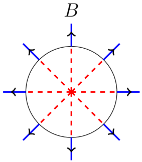
- Each Landau level forms an irreducible $SU(k+1)$ representation, whose degeneracy and energy is easy to calculate.

QHE on S^2 analyzed by Haldane



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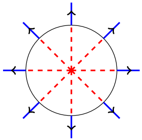

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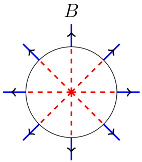

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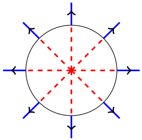

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- $\hat{R}_+, \hat{R}_- \to$ covariant derivatives $D_\pm = i\hat{R}_\pm / r$

$[\hat{R}_+, \hat{R}_-] = 2\hat{R}_3 \quad \Rightarrow \quad \hat{R}_3 \Psi = -\frac{n}{2}\Psi$

- A complete basis for wavefunctions on $SU(2)$ are given by Wigner $\mathcal{D}$-functions

$$\Psi^j_{m,m'} \sim \mathcal{D}^j_{m,m'}(g) = \langle\, j, m \;|\, \hat{g} \,|\; j, m' \,\rangle$$

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  - LLL : $\hat{R}_- \; |j, -\frac{n}{2}\rangle = 0 \;\Rightarrow\; |j, -\frac{n}{2}\rangle$ is the lowest weight state $\Rightarrow \dim(j) = n+1$
  - s-th LL : $|j, -\frac{n}{2} - s\rangle$ is the lowest weight state $\Rightarrow \dim(j) = n + 2s + 1$
- The spectrum decomposes into discrete Landau levels. Each LL forms an $SU(2)$ rep. whose degeneracy is easy to count.
- LLL wavefunctions

$$\Psi_m \sim \frac{z^m}{(1+\bar{z}z)^{n/2}} \qquad m = 0, \cdots, n$$

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1) \times (k+1)$-matrix $g \in SU(k+1)$ as a coordinate, where
  $g_{i,k+1} = z_i/\sqrt{1 + \bar{z} \cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1 + \bar{z} \cdot z}$

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- $\hat{R}_a,\ \hat{R}_{k^2+2k} \to$ gauge transformations $\quad (\ U(k)\ )$
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- How Ψ transforms under gauge transformations depends on choice of background fields

- Choose "uniform" $U(1)$ or $U(k)$ background magnetic fields.

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- Lowest Landau level: $\hat{R}_{-i}\Psi = 0$ Holomorphicity condition

($|\ \alpha\ \rangle$ is lowest weight state)

LLL wavefunctions for $U(1)$ magnetic field

For a $U(1)$ magnetic field the LLL wavefunctions can be written in terms of complex coordinates as

$$
\begin{aligned}
\Psi_{i_1 i_2 \cdots i_k} &= \sqrt{N}\left[\frac{n!}{i_1! i_2! \ldots i_k! (n-s)!}\right]^{\frac{1}{2}} \frac{z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{\frac{n}{2}}}\,, \\
s &= i_1 + i_2 + \cdots + i_k\,, \quad 0 \leq i_i \leq n\,, \quad 0 \leq s \leq n
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They form an $SU(k+1)$ representation of dimension

$$N = \mathrm{dim} J = \frac{(n+k)!}{n!\, k!}$$

- QHE on a compact space $M \Longrightarrow$ LLL defines an N-dim Hilbert space

  In the presence of confining potential $\Longrightarrow$ incompressible QH droplet

- K states are filled, $N - K$ unoccupied

  Occupancy matrix for ground state droplet : $\hat{\rho}_0$

$$\hat{\rho}_0 = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{bmatrix} \begin{matrix} \updownarrow K \\ \updownarrow N-K \end{matrix}$$

Matrix Formulation of LLL Dynamics

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- Under time evolution: $\hat{\rho}_0 \to \hat{\rho} = \hat{U}\, \hat{\rho}_0\, \hat{U}^\dagger$

  $\hat{U} = N \times N$ unitary matrix ; "collective" variable describing excitations within the LLL

The action for $\hat{U}$ is

$$S_0 = \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$

which leads to the evolution equation

$$i \frac{d\hat{\rho}}{dt} = [\hat{V}, \hat{\rho}]$$

S_0 : universal matrix action

No explicit dependence on properties of space on which QHE is defined, abelian or nonabelian nature of fermions, etc.

S_0 : action of a noncommutative field theory

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$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N)\text{ matrices}} \quad \Longrightarrow \quad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

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$(N \times N)$ matrices — symbols

- $A(\vec{x}, t) \equiv \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{A}_{ml}(t) \Psi_l^*(\vec{x})$

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S_0 = bosonic action describing the dynamics of LLL fermions

DAS, DHAR, MANDAL, WADIA; SAKITA :2d plane context

Edge effective action for $\nu = 1$

Large N, K limit with $N \gg K \gg 1$ (large n limit) $\Longrightarrow$ chiral boundary action

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EDGE EFFECTIVE ACTION FOR $\nu = 1$

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$$S_0 \sim \int_{\partial D} (\partial_t \phi + u\, \mathcal{L}\phi)\, \mathcal{L}\phi$$

$(2k-1)$ (space) dim chiral action defined on droplet boundary

$$\mathcal{L}\phi = (\Omega^{-1})^{ij} \hat{r}_j \partial_i \phi, \qquad \mathcal{L} = \begin{cases} \text{derivative along boundary of droplet} \\ \to \partial_\theta \text{ in 2 dim.} \end{cases}$$

B. Nonabelian background magnetic field $U(k)$

- Wavefunction is a nontrivial representation of $SU(k)$: $\psi_{m,\alpha} \;\; \alpha = 1, \cdots, N'$
- Symbol = $(N' \times N')$ matrix valued function $\longrightarrow$ action in terms of $G \in U(N')$

B. Nonabelian background magnetic field $U(k)$

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- Symbol = $(N' \times N')$ matrix valued function $\longrightarrow$ action in terms of $G \in U(N')$
- The effective edge action is a generalized gauged WZW action in $(2k-1,1)$ dimensions.

$$
\begin{aligned}
S_0 =& \frac{1}{4\pi}\int_{\partial D} \mathrm{tr}\left[\left(G^\dagger \dot{G} + u\, G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right] \\
&+ \frac{1}{4\pi}\int_D \mathrm{tr}\left[-d\left(i\bar{A}dGG^\dagger + i\bar{A}G^\dagger dG\right) + \frac{1}{3}\left(G^\dagger dG\right)^3\right] \wedge \left(\frac{\Omega}{2\pi}\right)^{k-1} \frac{1}{(k-1)!} \\
\equiv& S_{\mathrm{WZW}}(A^L = A^R = \bar{A})
\end{aligned}
$$

$\mathcal{L} = (\Omega^{-1})^{ij}\hat{r}_j D_i$ = covariant derivative along the boundary of droplet

- In the presence of gauge fluctuations one starts with a gauged matrix action.

$$\partial_t \to \hat{D}_t = \partial_t + i\hat{\mathcal{A}}$$

$$S = \int dt \, \mathrm{Tr}\left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} - \underbrace{\hat{\rho}_0 \, \hat{U}^\dagger \hat{\mathcal{A}} \hat{U}}_{\text{gauge interactions}} \right]$$

In terms of bosonic fields

$$S = N \int dt \, d\mu \, \mathrm{tr} \, \left[i\rho_0 * U^\dagger * \partial_t U \; - \; \rho_0 * U^\dagger * (V + \mathcal{A}) * U \right]$$

QUESTION: How is $\mathcal{A}$ related to the gauge fields coupled to the original fermions?

- S is invariant under

$$\begin{aligned} \delta U &= -i\lambda * U \\ \delta \mathcal{A}(\vec{x}, t) &= \partial_t \lambda(\vec{x}, t) - i\left(\lambda * (V + \mathcal{A}) - (V + \mathcal{A}) * \lambda\right) \end{aligned} \tag{1}$$

- Since S describes gauge interactions it has to be invariant under usual gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda + i[\bar{A}_\mu + A_\mu, \Lambda], \qquad \delta \bar{A}_\mu = 0 \tag{2}$$

Background ($\bar{A}_\mu$) Perturbation (A_μ)

The strategy is to choose

$$\begin{aligned} \mathcal{A} &= \text{function}(A_\mu,\ \bar{A}_\mu, V) \\ \lambda &= \text{function}(\Lambda, A_\mu, \bar{A}_\mu) \end{aligned}$$

such that the gauge transformation (2) induces $\delta\mathcal{A}$ in (1) (generalized Seiberg-Witten map)

- In the large N limit the result is $S = S_{\text{edge}} + S_{\text{bulk}}$

$S_{\text{edge}} \sim S_{\text{WZW}}\left(A^L = A + \bar{A}\,, A^R = \bar{A}\right)$ = Chirally gauged WZW action in $2k$ dim

$S_{\text{bulk}} \sim S_{CS}^{2k+1}(\tilde{A}) + \cdots$ = $(2k+1)$ dim CS action

$\tilde{A} = (A_0 + V, \bar{a}_i + \bar{A}_i + A_i)$ = background + fluctuations

- Gauge Invariance $\Longrightarrow$ Anomaly Cancellation

$$\delta S_{\text{edge}} \neq 0, \qquad \delta S_{\text{bulk}} \neq 0$$

$$\delta S_{\text{edge}} + \delta S_{\text{bulk}} = 0$$

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- So we can use $\frac{\delta S_{eff}}{\delta A_0} = J_0 =$ Dolbeault index density

  and integrate up to get S_{eff}.

BULK TOPOLOGICAL EFFECTIVE ACTION: EXAMPLES

- $\mathbb{CP}^1 = SU(2)/U(1)$; s-th LL

$$S_{3d} = \frac{1}{4\pi}\int\left\{\left(A+(s+\frac{1}{2})\omega\right)d\left(A+(s+\frac{1}{2})\omega\right)-\frac{1}{12}\omega\, d\omega\right\}$$

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- We have general results for arbitrary dimensions, higher Landau levels and nonabelian magnetic fields

We can calculate the electromagnetic response functions in all dimensions, $J^\mu = \frac{\delta S_{eff}}{\delta A_\mu}$.

- (2+1) dimensions

$$J^i = \frac{\epsilon^{ij}}{2\pi}\left(E_j + \frac{R_{j0}}{2}\right)$$

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$$J^i = \frac{\epsilon^{ijklrs}}{2^3(2\pi)^3}E_j\left[\left(F_{kl} + \frac{1}{2}\mathrm{Tr}R_{kl}\right)\left(F_{rs} + \frac{1}{2}\mathrm{Tr}R_{rs}\right) - \frac{1}{12}\mathrm{Tr}\left(R_{kl}R_{rs}\right)\right]$$

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One can calculate the energy-momentum tensor $T^{\mu\lambda}$

$$T^{\mu\lambda} = -\frac{2}{\sqrt{g}} \frac{\delta S_{eff}}{\delta g_{\mu\lambda}}$$

and from this the viscosity tensor η^{ijkl} defined as $T^{ij} = \eta^{ijkl}\, \dot{g}_{kl}$.

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- In two-dimensions

$$\begin{aligned} \sqrt{g}\, T^{ml} &= \frac{1}{2}\, \eta_H \left(g^{mi} \epsilon^{lk} + g^{li} \epsilon^{mk} \right) \dot{g}_{ki} \\ &+ \frac{1}{2}\, \eta_H^{(2)} \left(g^{mi} \epsilon^{lk} + g^{li} \epsilon^{mk} \right) \nabla_i \nabla_k \left(g^{rn} \dot{g}_{rn} \right) \end{aligned}$$

where the Hall viscosity η_H can be read off as ($\bar{s} = s + \frac{1}{2}$)

$$\begin{aligned} \eta_H &= \frac{1}{4\pi} \left[\bar{s}\, B + \left(\bar{s}^2 - \frac{1}{12} \right) \left(\frac{R}{2} - \nabla^2 \right) \right] \\ \eta_H^{(2)} &= \frac{1}{8\pi} \left(\bar{s}^2 - \frac{1}{12} \right) \end{aligned}$$

- In four-dimensions the expression for the viscosity tensor is quite involved. In the flat limit, where $\mathbb{CP}^2 \Rightarrow \mathbb{C} \times \mathbb{C}$

$$\eta_H = \left(\frac{(s+1)B}{4\pi} \right)^2$$

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  KARABALI AND NAIR, 2023

- We divide the system into two regions, D and its complementary D^C, and define the reduced density matrix

$$\rho_D = \mathrm{Tr}_{D^C} |GS\rangle \langle GS|$$

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- The entanglement entropy is defined as

$$S = -\mathrm{Tr}\, [\rho_D \log \rho_D]$$

- We choose D to be the spherically symmetric region of $\mathbb{CP}^k$ satisfying $z \cdot \bar{z} \leq R^2$. For $\mathbb{CP}^1 \sim S^2$, D is a polar cap around the north pole with latitude angle θ. $R = \tan\theta/2$ via stereographic projection.

- The entanglement entropy can also be written as

$$S = -\mathrm{Tr}\left[\rho_D \log \rho_D\right] = -\sum_{m=1}^{N} \Big[\lambda_m \log \lambda_m + (1-\lambda_m) \log(1-\lambda_m)\Big]$$

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- λ's are eigenvalues of the two-point correlator (PESCHEL)

$$C(r,r') = \sum_{m=1}^{N} \Psi_m^*(z)\,\Psi_m(z')\ , \qquad z, z' \in D$$

$$\int_D C(r,r')\Psi_l^*(z')d\mu(z') = \lambda_l\,\Psi_l^*(z)$$

where

$$\lambda_l = \int_D |\Psi_l|^2 d\mu$$

- For 2d gapped systems

$$S = c\,L - \gamma + \mathcal{O}(1/L)$$

L: perimeter of boundary

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- For integer QHE on $S^2 = \mathbb{CP}^1$ RODRIGUEZ AND SIERRA, 2009

For $\nu = 1$: $c = 0.204$

Some results on Kähler manifolds CHARLES AND ESTIENNE, 2019

A. QHE on $\mathbb{CP}^k$ with $U(1)$ magnetic field

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The LLL wavefunctions are essentially the coherent states of $\mathbb{CP}^k$.

$$\begin{aligned} \Psi_{i_1 i_2 \cdots i_k} &= \sqrt{N}\left[\frac{n!}{i_1! i_2! \ldots i_k! (n-s)!}\right]^{\frac{1}{2}} \frac{z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{\frac{n}{2}}} , \\ s &= i_1 + i_2 + \cdots + i_k , \quad 0 \le i_i \le n , \quad 0 \le s \le n \end{aligned}$$

They form an $SU(k+1)$ representation of dimension

$$N = \dim J = \frac{(n+k)!}{n!\, k!}$$

The volume element for $\mathbb{CP}^k$ is

$$d\mu = \frac{k!}{\pi^k} \frac{d^2 z_1 \cdots d^2 z_k}{(1+\bar{z}\cdot z)^{k+1}} \quad , \quad \int d\mu = 1$$

- The eigenvalues $\lambda = \int_D \Psi^* \Psi$ are given by

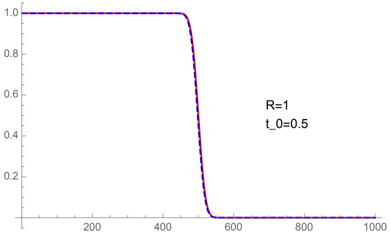
$$\lambda_{i_1 i_2 \cdots i_k} \equiv \lambda_s = \frac{(n+k)!}{(n-s)!(s+k-1)!} \int_0^{t_0} dt\, t^{s+k-1}\, (1-t)^{n-s}$$

where $t_0 = R^2/(1+R^2)$.

- The entanglement entropy is

$$\begin{aligned} S &= \sum_{s=0}^{n} \overbrace{\frac{(s+k-1)!}{s!(k-1)!}}^{degeneracy} H_s \\ H_s &= [-\lambda_s \log \lambda_s - (1-\lambda_s) \log(1-\lambda_s)] \end{aligned}$$

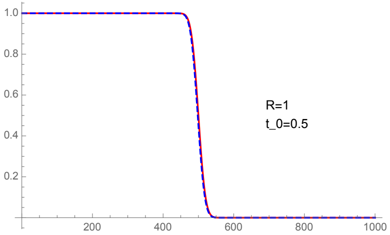
- For large n, this is amenable to an analytical semiclassical calculation for all $k \ll n$.



Graph of λ_s vs s

Transition ($\lambda = \frac{1}{2}$) at $s^* \sim n\, t_0$

$k = 1$, $k = 5$



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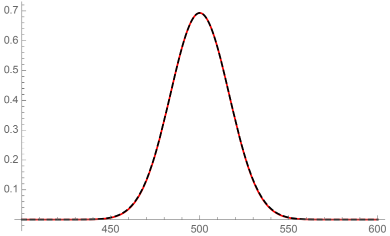

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—- exact

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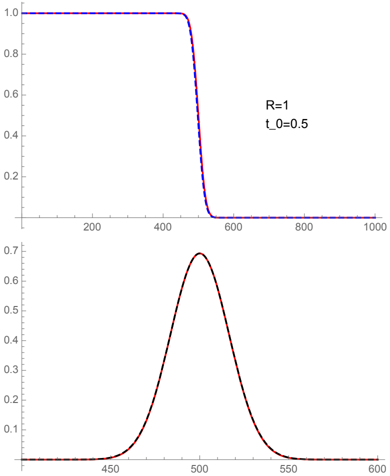

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Only wavefunctions localized around the boundary of the entangling surface contribute to entropy.

From semiclassical analysis

$$S \sim n^{k-\frac{1}{2}} \frac{\pi\,(\log 2)^{3/2}}{2\,k!} \underbrace{2k\frac{R^{2k-1}}{(1+R^2)^k}}_{\textit{geometric area}} \sim c_k \text{ Area}$$

In agreement with $k = 1$ result by RODRIGUEZ AND SIERRA

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- $V_{\text{phase space}} \to \frac{n^k}{k!} \int \Omega^k = \frac{n^k}{k!} \int d\mu$

$$A_{\text{phase space}} = \frac{n^{k-\frac{1}{2}}}{k!} A_{\text{geom}} = \frac{n^{k-\frac{1}{2}}}{k!}\, 2k \frac{R^{2k-1}}{(1+R^2)^k}$$

$$S \sim \frac{\pi}{2} (\log 2)^{3/2} A_{\text{phase space}}$$

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- The corresponding phase-space volume in this case is $V_{\text{phase space}} = \dim\tilde{J}\ \frac{n^k}{k!} \int d\mu$

$$S \sim \frac{\pi}{2} (\log 2)^{3/2} A_{\text{phase space}}$$

for any dimension and Abelian or non-Abelian background. (KARABALI, 2020)

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- Extend these ideas to fractional Hall effect (AGARWAL, KARABALI, NAIR, 2025)

THANK YOU!