Kavli Asian Winter School (KAWS): problems for Caron-Huot's lectures

Solutions will be discussed in the tutorial session [with corrected typo in problem 2].

1. (*chaos bound; short*) A transfer function satisfies $S(\omega) \approx 1 + iC\omega^\lambda$ over some range of $|\omega|$, where λ is real and $|C\omega^\lambda|$ is small. Show that $|S(\omega)| \leq 1$ in the upper-half plane implies: $-1\leq\lambda\leq 1$. And if $\lambda > \lambda_{\rm min}$ is sufficiently close to 1, then in addition: Re $C\geq 0$. (What $\lambda_{\rm min}$ ensures this?)

   In the context of out-of-time-order correlators for thermal systems [1], $\frac{\lambda}{2\pi T}$ is a Lyapunov exponent. Show that chaos is minimal (amplitude is elastic: ImC=0) when it is maximal (fastest growth: λ=1). I'am not sure what this riddle means.

2. (*resonant medium*) After a signal goes through a medium with resonant absorption at frequency ω_0, its Fourier transform is multiplied by the transfer function:

$$S(\omega) = \exp\left(\frac{ia}{\omega_0 - \omega - i\Gamma}\right).$$

   Here $a > 0$ measures the optical depth of the medium. Show that this medium is causal and unitary; and that the stationary-phase approximation predicts a time advance for frequencies near the resonance. (This model is similar to realistic media with faster-than-light group velocity [2].)

   Now send in a Gaussian pulse that is narrowly peaked around the frequency ω_0 (ie. $\sigma^{-1} \ll \Gamma$),

$$\hat{f}_{\rm in}(\omega) = e^{-(\omega-\omega_0)^2\sigma^2/2}\sqrt{2\pi}\sigma, \qquad f_{\rm in}(t) = e^{-i\omega_0 t - t^2/(2\sigma^2)}\,.$$

   Argue that for suitable choices of a and Γ, the Gaussian can come out mostly undistorted in shape and advanced in time by 5σ. Explain why this does not challenge causality? Numerically sketch or plot the Fourier transform. (Example parameters: $\sigma = 1$, $\Gamma = 15$, $a = 1125$, $\omega_0 = 10$.) [1]

3. (*null constraints*) This problem and the next one are based on [3]. Parametrize a crossing-symmetric amplitude for a real massless scalar, at low energies, by a sum of contact terms:

$$\mathcal{M}_{\rm low}(s,t) = -g_0 + g_2(s^2+t^2+u^2) + g_3(stu) + g_4(s^2+t^2+u^2)^2 + \ldots$$

   If the amplitude satisfies twice-subtracted dispersion relations and its imaginary part is only non-negligible for $m \geq M$, show that (in $d = 4$):

$$g_2 = \left\langle \frac{1}{m^4}\right\rangle, \quad g_3 = \left\langle \frac{3-2J(J+1)}{m^6}\right\rangle, \quad g_4 = \left\langle \frac{1}{2m^8}\right\rangle, \quad 0 = \left\langle \frac{J(J+1)(J(J+1)-8)}{m^8}\right\rangle \qquad (1)$$

   where $\langle \bullet \rangle \equiv 16\sum_{J \text{ even}}(2J{+}1)\int_{M^2}^\infty \frac{dm^2}{m^2}\text{Im } a_J(m^2)(\bullet)$, and phase shifts are normalized so that $|1+ia_J| \leq 1$. (The overall normalization is not always important, but work it out if you can!)

   The last "null constraint" encodes crossing symmetry. If we instead considered an amplitude $\Phi\bar{\Phi}\bar{\Phi}\Phi$ for a complex scalar, there would be two spectral densities (for $\Phi\Phi$ and $\Phi\bar{\Phi}$ channels) but also two distinct dispersion relations (fixed-t and fixed-u). How many linearly independent null constraints would you then expect at order $1/m^6$, and at order $1/m^8$? How does the counting change if we allow single-subtracted dispersion relations?

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[1] Mathematica's `NIntegrate[]` can compute the transform at discrete t's; you may require `WorkingPrecision->40`.

4. Find positive combinations of the sum rules (1) which show that $g_4 \le \frac{g_2}{2M^4}$ and $g_3 \le \frac{3g_2}{M^2}$ (easy).

   Use a computer to prove the optimal lower bound $g_3 \ge -10.61249\frac{g_2}{M^2}$ using only three of the sum rules (1). Some hints for linear programming are given below. If desired, check how the bound changes when you add the next two null constraints (n_5 and n_6 in (3.29) of [3]).

5. How narrow can a function be, if it is "positive in impact parameters and compact support in momentum space"? Consider a 1D version of this problem, where

$$f(b) = \frac{1}{\pi}\int_0^1 dp\hat{f}(p)\cos(pb) \ge 0 \;\forall\; b \in \mathbb{R},$$

   and we define "width" from $\langle b^2\rangle$:

$$\int_{-\infty}^{+\infty} f(b)db = 1, \qquad \langle b^2\rangle \equiv \int_{-\infty}^{+\infty} f(b)b^2 db = \text{mimized}.$$

   Discuss a possible ansatz for $\hat{f}(p)$: what constraints on its behavior as $p \to 0$ and $p \to 1$ ensure that $\langle b^2\rangle$ is finite? Show (numerically or analytically[hard]) that the narrowest function has $\langle b^2\rangle = \pi^2$.

On solving linear optimization programs

Given a matrix `mat` and vectors `obj` and `norm`, suppose we want the vector v which maximizes the "objective" `obj.v`, subject to the constraints `mat.v`$\ge$`0` and normalization `norm.v==1`. Mathematica's `LinearOptimization[-obj,{mat,0*mat[[;;,1]]},{{norm},{1}}]` will return just that.

Concretely, for question 4 above that uses the three sum rules $\{g_2, g_3, 0\}$, v is a three vector, `norm=`$\{0,1,0\}$ and `obj=`$\{-1,0,0\}$ (explain why!); `mat` is a $N\times 3$ matrix whose rows represent N discrete values of J and m. Since this method imposes positivity at only discrete values, it is important to plot the outcome and verify positivity for all $m \ge 1$, refining your sampling if necessary!

For larger problems, or problems involving semi-definite matrices, the powerful solver SDPB [4] is generally much more stable and efficient. SDPB can also deal directly with polynomials in a positive variable $x \ge 0$. It is worthwhile familiarizing yourself with SDPB if there is any chance you will encounter this type of semi-definite optimization problems. The easiest way to install it is often through the Docker or Singularity environments, depending on your system.

References

[1] J. Maldacena, S. H. Shenker and D. Stanford, "A bound on chaos," JHEP **08**, 106 (2016); arXiv:1503.01409 [hep-th].

[2] W. Withayachumnankul, B. M. Fischer, B. Ferguson, B. R. Davis and D. Abbott, "A Systemized View of Superluminal Wave Propagation," in https://ieeexplore.ieee.org/document/5535097, vol. 98, no. 10, pp. 1775-1786, Oct. 2010.

[3] S. Caron-Huot and V. Van Duong, "Extremal Effective Field Theories," JHEP **05**, 280 (2021); arXiv:2011.02957 [hep-th].

[4] D. Simmons-Duffin, "A Semidefinite Program Solver for the Conformal Bootstrap," JHEP **06**, 174 (2015) [arXiv:1502.02033 [hep-th]]. Repository: `https://github.com/davidsd/sdpb`