GREEN'S FUNCTIONS FOR RANDOM RESISTOR NETWORKS

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RANDOM RESISTOR NETWORKS

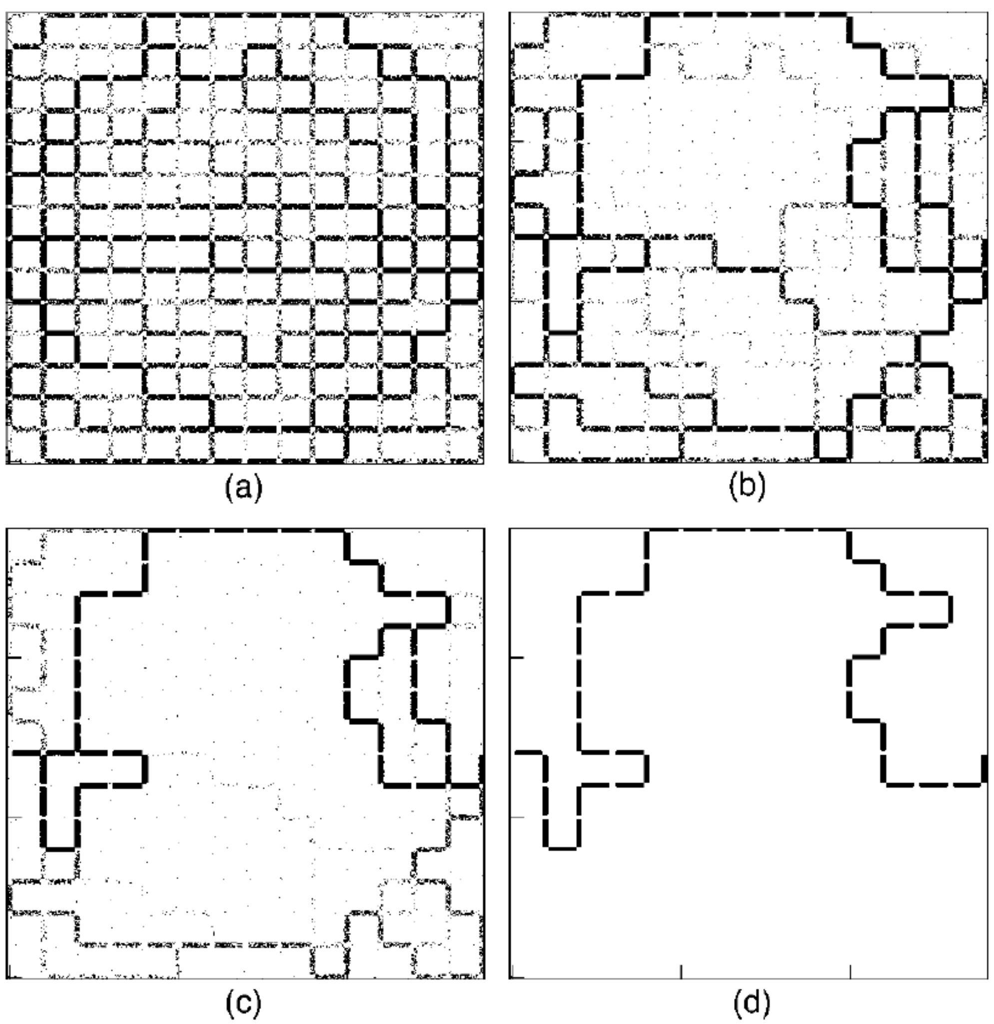
- *The conductivity of disordered random media is a problem of fundamental interest.*
- *Simple tractable models that are often used are electrical networks constituting random resistor elements.*
- *Typically there is a weak disorder regime where the the current distribution is delocalized throughout the lattice.*
- *There is also a strong disorder regime disorder regime where the current distribution collapses to a self-similar fractal optimal path.*
- *While critical exponents of the disorder regimes have been explored in depth, a scalable analytical toolbox to analyse such disordered networks has not yet been developed.*

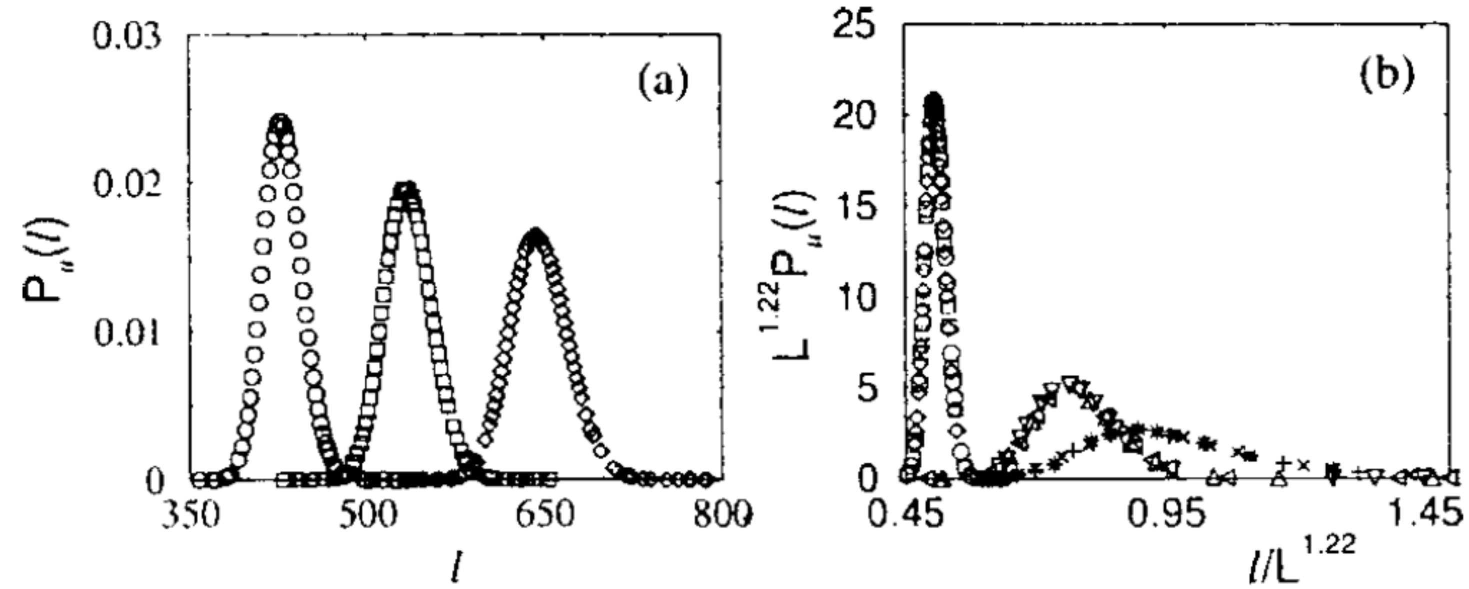
PREVIOUS WORK

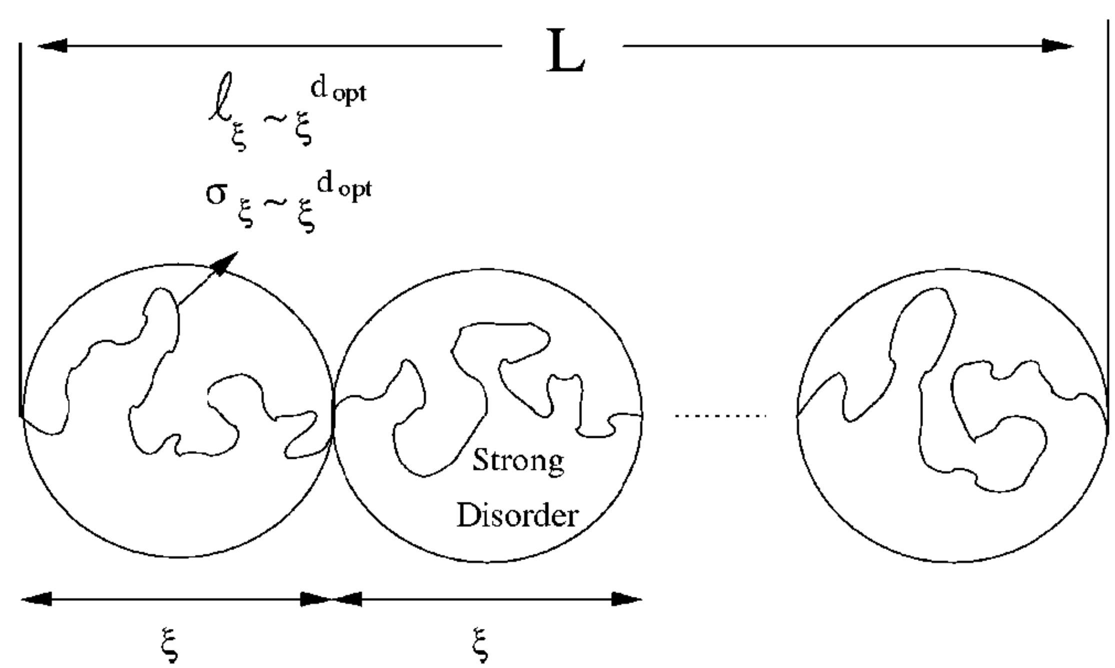
PHYSICAL REVIEW E **71**, 045101(R) (2005)

Current flow in random resistor networks: The role of percolation in weak and strong disorder

Zhenhua Wu,[1] Eduardo López,[1] Sergey V. Buldyrev,[1,2] Lidia A. Braunstein,[1,3] Shlomo Havlin,[4] and H. Eugene Stanley[1]







EXPONENTIAL DISORDER

- *Resistances are drawn from an exponential distribution*

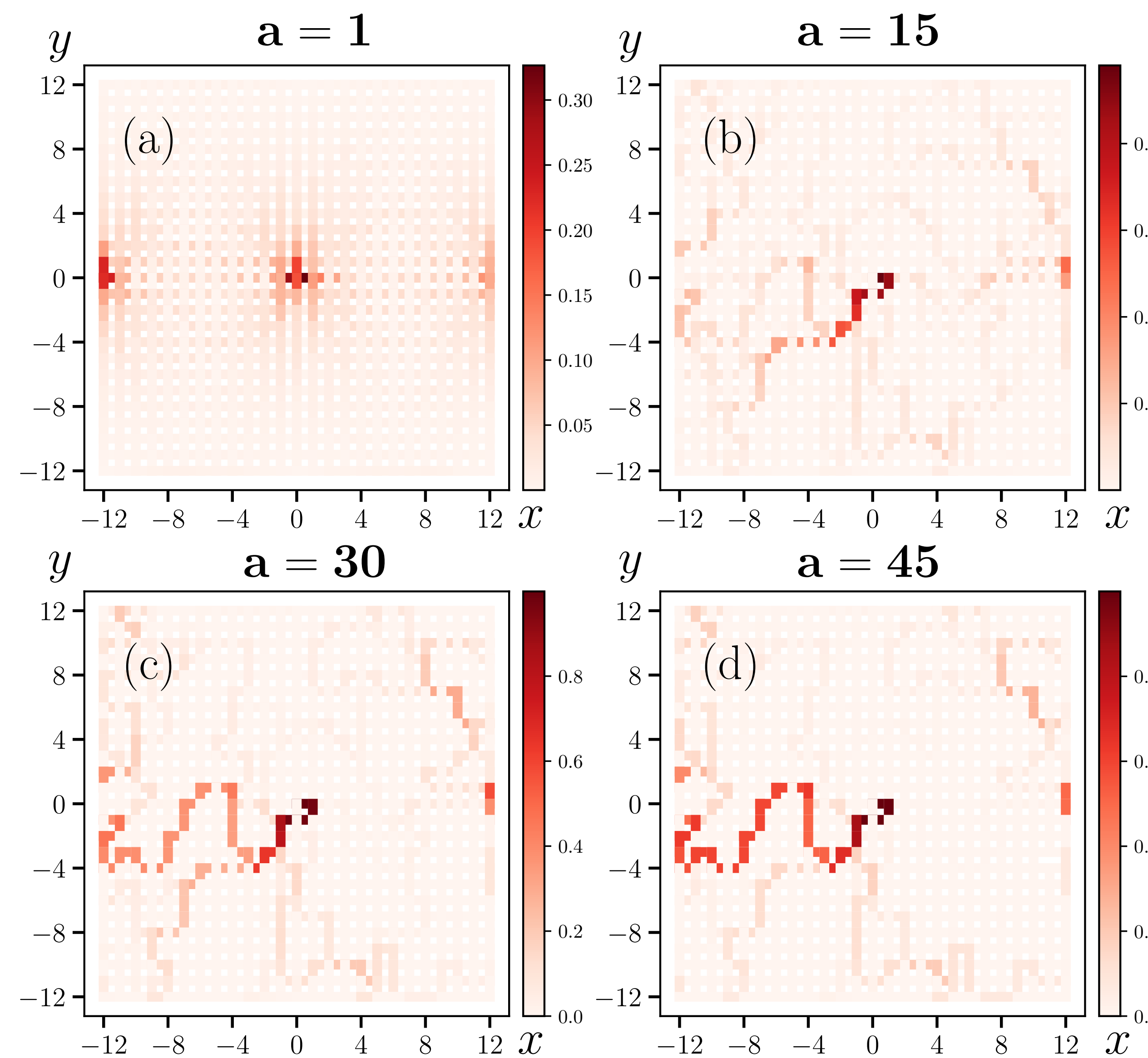
$$R_{\vec{r}_i,\vec{r}_j} = e^{a x_{\vec{r}_i,\vec{r}_j}}$$

(x is uniformly distributed)

- *Convenient Representation*

$$R_{\vec{r}_i,\vec{r}_j} = (1-\zeta_{\vec{r}_i,\vec{r}_j})^{-1}$$

$$f(\zeta) = a^{-1}(1-\zeta)^{-1} \quad \text{for} \;\; 0<\zeta<1-e^{-a}$$



STEADY STATE SOLUTION

- *We solve Kirchhoff's Laws*

$$|I\rangle_{\vec{r}} = \sum_{\hat{\Delta}} \frac{|V\rangle_{\vec{r}} - |V\rangle_{\vec{r}+\hat{\Delta}}}{R_{\vec{r},\vec{r}+\hat{\Delta}}}.$$

$$|I\rangle_{\vec{r}} = \sum_{\hat{\Delta}} \left(|V\rangle_{\vec{r}} - |V\rangle_{\vec{r}+\hat{\Delta}}\right)\left(1 - \lambda\zeta_{\vec{r},\vec{r}+\hat{\Delta}}\right)$$

- *In terms of the Decorated Lattice Laplacian*

$$\mathsf{L}|V\rangle + |I\rangle = 0$$

$$(\mathsf{L}^{(1)})_{ij} = \begin{cases} \sum_{\hat{\Delta}} \zeta_{\vec{r}_i,\vec{r}_i+\hat{\Delta}} & \text{if } i = j \\ -\zeta_{\vec{r}_i,\vec{r}_j} & \text{if } \vec{r}_j = \vec{r}_i + \hat{\Delta} \\ 0 & \text{otherwise.} \end{cases}$$

DISORDER PERTURBATION EXPANSION

- *We perform a perturbation expansion in the strength of the disorder*

$$\mathsf{L} = \mathsf{L}^{(0)} + \lambda \mathsf{L}^{(1)}$$

$$|V\rangle = |V\rangle^{(0)} + \lambda |V\rangle^{(1)} + \lambda^2 |V\rangle^{(2)} + \mathcal{O}(\lambda^3)$$

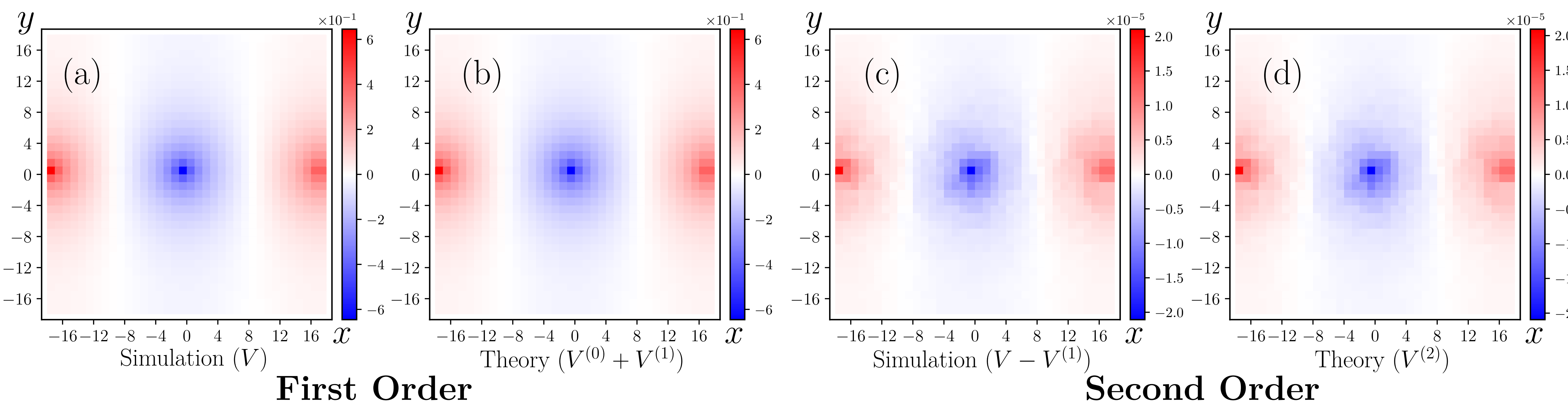
$$|V\rangle = \mathsf{G}^{(0)}[\mathbf{1} + \lambda \mathsf{L}^{(1)}\mathsf{G}^{(0)} + \lambda^2 (\mathsf{L}^{(1)}\mathsf{G}^{(0)})^2 + \mathcal{O}(\lambda^3)]|I\rangle,$$

- *This basically gives you a Dyson series*

$$\mathsf{G} = \mathsf{G}^{(0)} + \mathsf{G}^{(0)}\mathsf{L}^{(1)}\mathsf{G}^{(0)} + \mathsf{G}^{(0)}\mathsf{L}^{(1)}\mathsf{G}^{(0)}\mathsf{L}^{(1)}\mathsf{G}^{(0)} + \ldots .$$

DISORDER PERTURBATION EXPANSION

- *The expansion **converges well for small disorder***



DYADIC BOND DISORDER

$$\mathsf{L}^{(0)} = -\sum_{m=1}^{N_b} |b_m\rangle \langle b_m| \,,$$

$$\mathsf{L}^{(1)} = \sum_{m=1}^{N_b} \zeta_m |b_m\rangle \langle b_m| \,.$$

$$\left|\tilde{b}_m\right\rangle \equiv \sqrt{\zeta_m}\, |b_m\rangle$$

- *The Sherman-Morrison Formula*

$$(\mathsf{A} + |u\rangle \langle v|)^{-1} = \mathsf{A}^{-1} - \frac{\mathsf{A}^{-1} |u\rangle \langle v| \mathsf{A}^{-1}}{1 + \langle v|\mathsf{A}^{-1}|u\rangle}.$$

Perturbation of infinite networks of resistors

József Cserti, Gyula Dávid, and Attila Piróth

American Journal of Physics **70**, 153 (2002); doi: 10.1119/1.1419104

RECURSIVE DYADIC BOND DISORDER

$$\tilde{\beta}_i := 1 - \tilde{\mathcal{G}}^0_{ii},$$
$$\tilde{\mathcal{G}}^0_{pq} := \left\langle \tilde{b}_p \middle| \mathsf{G}^{[0]} \middle| \tilde{b}_q \right\rangle$$

- ***SINGLE* Bond Disorder**

$$\mathsf{L}^{[1]} = \mathsf{L}^{[0]} + \left|\tilde{b}_1\right\rangle\left\langle\tilde{b}_1\right|$$

$$\mathsf{G}^{[1]} = \mathsf{G}^{[0]} + \left(\frac{1}{\tilde{\beta}_1}\right)\mathsf{G}^{[0]}\left|\tilde{b}_1\right\rangle\left\langle\tilde{b}_1\right|\mathsf{G}^{[0]}$$

- ***TWO* Bond Disorder**

$$\mathsf{L}^{[2]} = \mathsf{L}^{[1]} + \left|\tilde{b}_2\right\rangle\left\langle\tilde{b}_2\right|$$

$$\mathsf{G}^{[2]} = \mathsf{G}^{[0]} + \left(\frac{\tilde{\beta}_2}{\tilde{\beta}_1\tilde{\beta}_2 - (\tilde{\mathcal{G}}^0_{12})^2}\right)\mathsf{G}^{[0]}\left|\tilde{b}_1\right\rangle\left\langle\tilde{b}_1\right|\mathsf{G}^{[0]} +$$
$$\left(\frac{\tilde{\beta}_1}{\tilde{\beta}_1\tilde{\beta}_2 - (\tilde{\mathcal{G}}^0_{12})^2}\right)\mathsf{G}^{[0]}\left|\tilde{b}_2\right\rangle\left\langle\tilde{b}_2\right|\mathsf{G}^{[0]} +$$
$$\left(\frac{\tilde{\mathcal{G}}^0_{12}}{\tilde{\beta}_1\tilde{\beta}_2 - (\tilde{\mathcal{G}}^0_{12})^2}\right)\left[\mathsf{G}^{[0]}\left|\tilde{b}_1\right\rangle\left\langle\tilde{b}_2\right|\mathsf{G}^{[0]} + \mathsf{G}^{[0]}\left|\tilde{b}_2\right\rangle\left\langle\tilde{b}_1\right|\mathsf{G}^{[0]}\right]$$

NON-PERTURBATIVE GREEN'S FUNCTIONS

- *Arbitrary Bond Disorder*

$$\mathsf{L}^{[n+1]} = \mathsf{L}^{[n]} + \left|\tilde{b}_n\right\rangle\left\langle\tilde{b}_n\right|$$

$$\mathsf{G}^{[n]} \equiv \mathsf{G}^{[0]} + \sum_{i,j}^{n} c_{ij}^{[n]} \mathsf{G}^{[0]} \left|\tilde{b}_i\right\rangle\left\langle\tilde{b}_j\right| \mathsf{G}^{[0]}$$

- *The coefficients can be determined exactly!*

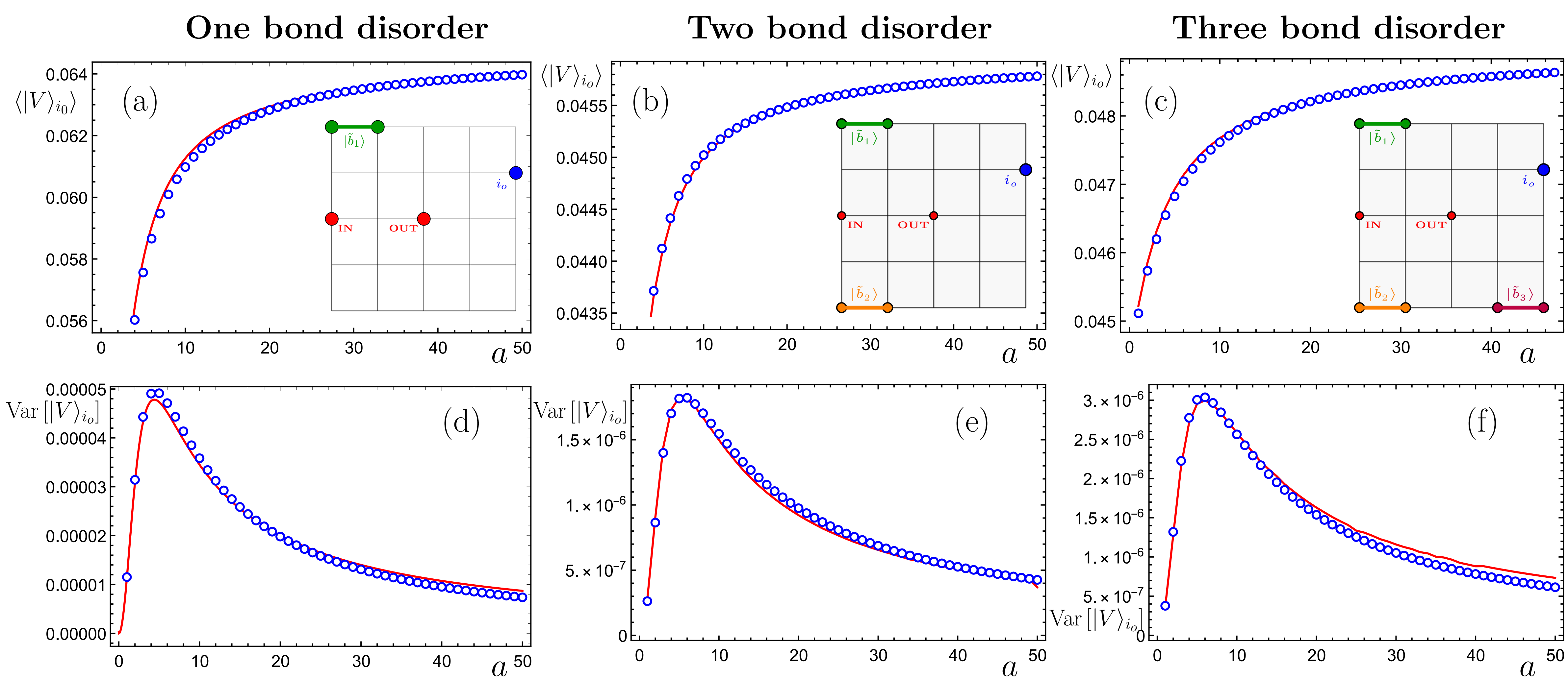
$$\tilde{\beta}_i c_{ij}^{[n]} - \sum_{k\neq i} \tilde{\mathcal{G}}_{ik}^{0} c_{kj}^{[n]} = \delta_{ij} \quad (1 \leq i,j,k \leq n)$$

$$\tilde{\mathcal{G}}^{[n]} \equiv \begin{pmatrix} \tilde{\beta}_1 & -\tilde{\mathcal{G}}_{12}^0 & -\tilde{\mathcal{G}}_{13}^0 & \cdots & -\tilde{\mathcal{G}}_{1n}^0 \\ -\tilde{\mathcal{G}}_{21}^0 & \tilde{\beta}_2 & -\tilde{\mathcal{G}}_{23}^0 & \cdots & -\tilde{\mathcal{G}}_{2n}^0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\tilde{\mathcal{G}}_{n1}^0 & -\tilde{\mathcal{G}}_{n2}^0 & -\tilde{\mathcal{G}}_{n3}^0 & \cdots & \tilde{\beta}_n \end{pmatrix}$$

$$c_{ij}^{[n]} = \frac{\det(\tilde{\mathcal{G}}^{[n]}(i,j))}{\det(\tilde{\mathcal{G}}^{[n]})}$$

$$c_{ii}^{[n]} = \frac{\det(\mathcal{G}^{[n-1]})}{\det(\mathcal{G}^{[n]})}$$

SIMULATIONS AT HIGH DISORDER



REFERENCE:

Green's functions for random resistor networks
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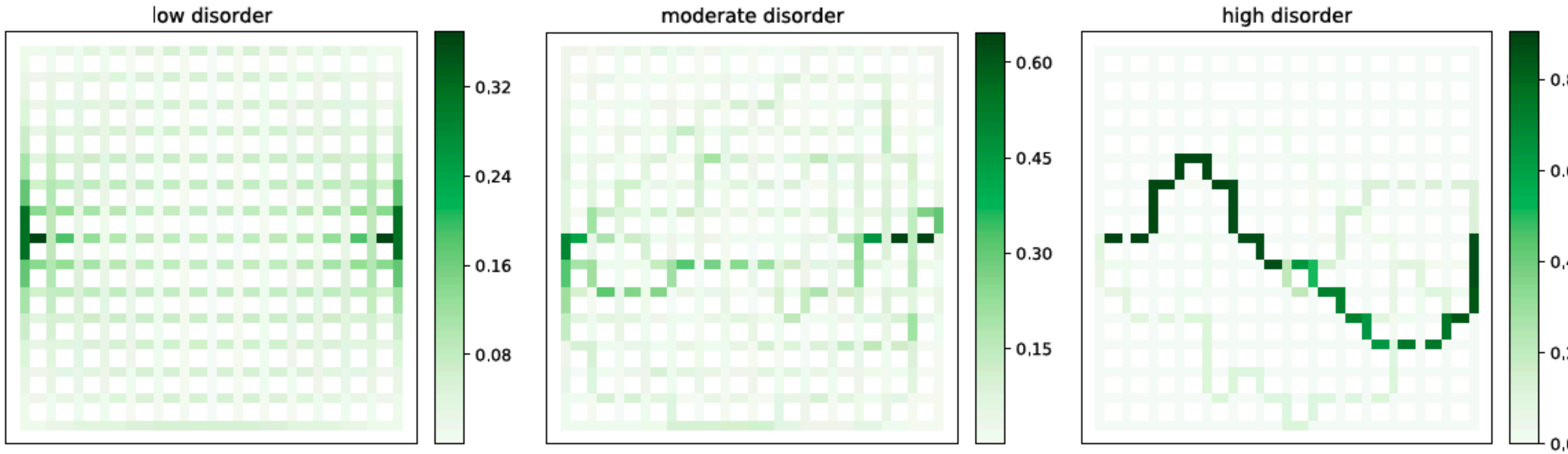
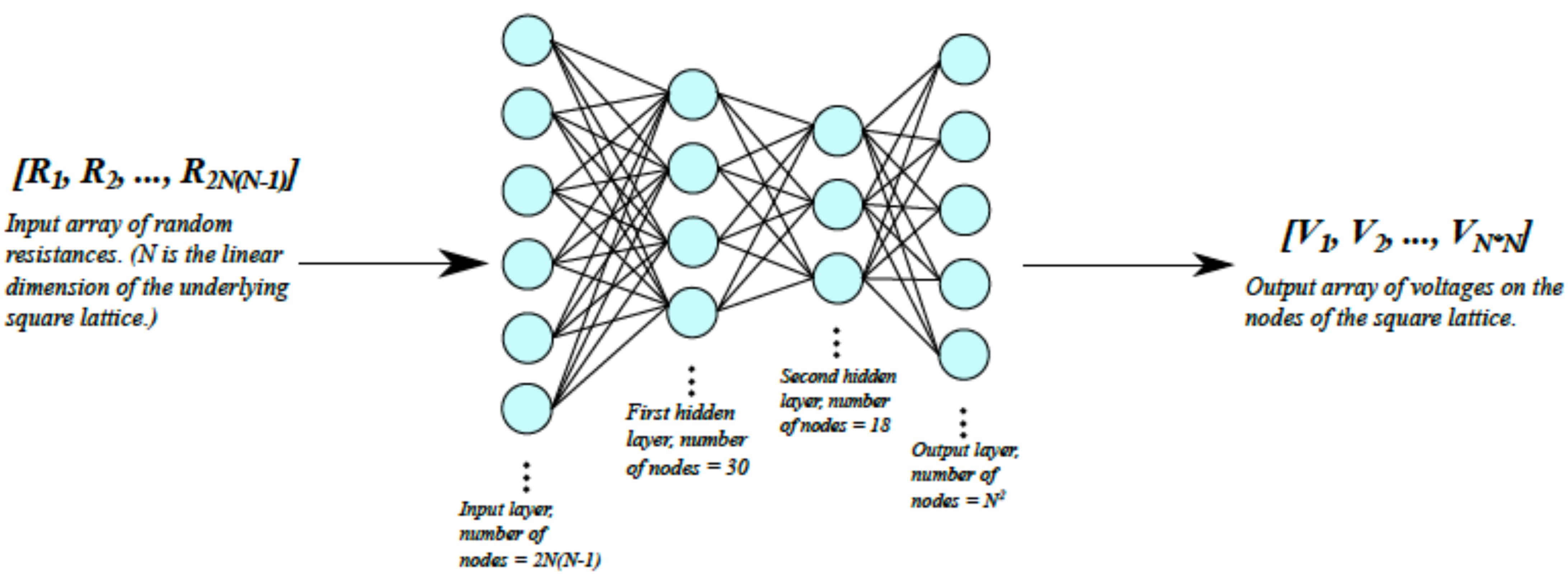
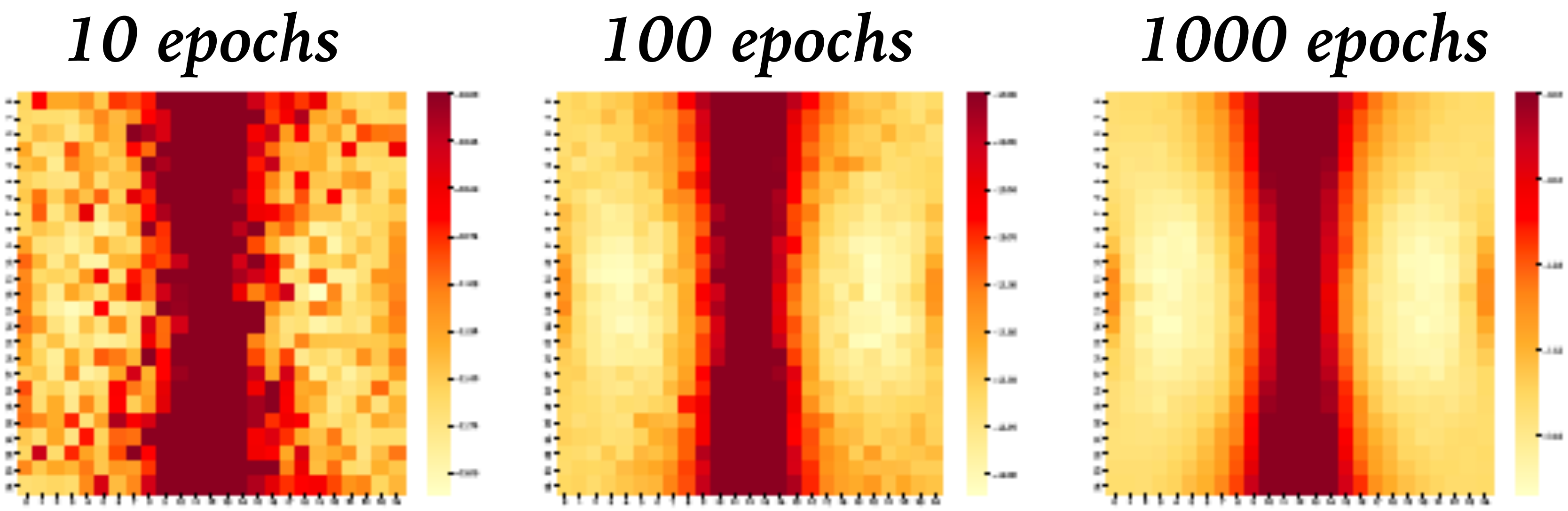
Sayak Bhattacharjee[1,*] and Kabir Ramola[2,†]

CAN A MACHINE LEARN KIRCHHOFF'S LAWS?

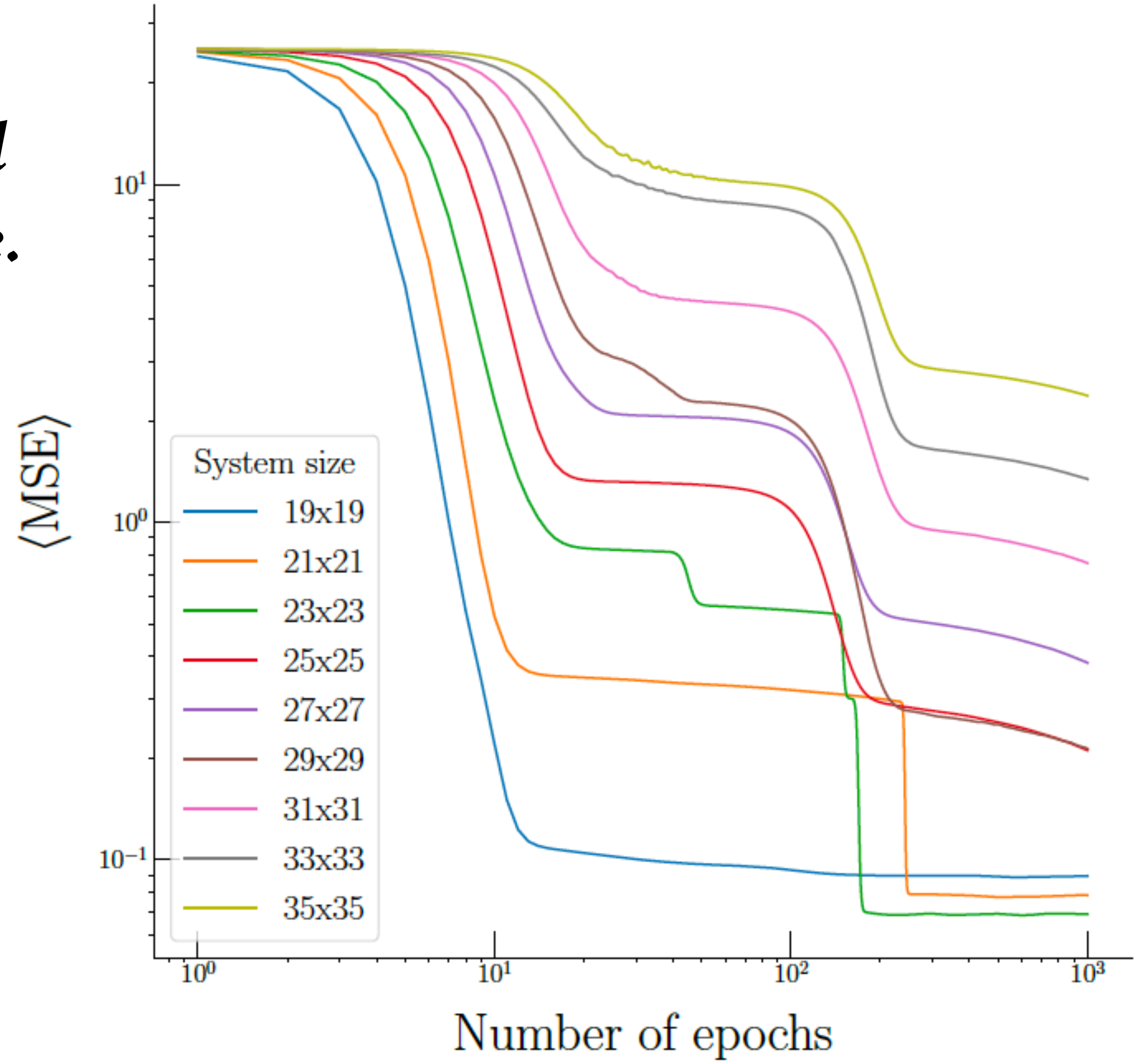




TWO-STEP RELAXATION



- *Mean Square Error shows interesting spatial dependence and a two-step relaxation profile.*



Thank You.