Run-and-tumble particle with bounded rates

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Run-and-tumble particle on an infinite line

- Particle described by position $-\infty < x < \infty$ and internal state $+/-$
- Right-mover moves rightwards with speed $v_+(x)$ but changes its state at rate $\gamma_+(x)$; similar rule for left-mover
- Prob dist of right- and left-movers obey:

$$
\begin{aligned}
\frac{\partial P_+(x,t)}{\partial t} &= -[v_+(x)P_+(x,t)]' + \gamma_-(x)P_-(x,t) - \gamma_+(x)P_+(x,t) \\
\frac{\partial P_-(x,t)}{\partial t} &= -[-v_-(x)P_-(x,t)]' + \gamma_+(x)P_+(x,t) - \gamma_-(x)P_-(x,t)
\end{aligned}
$$

Homogeneous rates

- If the rates are uniform in space, no stationary state
- Mean displacement increases linearly in time
- Mean-squared displacement behaves as:

  - At short times,

$$\sigma^2 \sim \begin{cases} t^2 & , \text{ initially both R, L movers} \\ t^3 & , \text{ initially only R or L} \end{cases}$$

  - At long times,

$$\sigma^2 \sim \begin{cases} t & , \text{ both tumbling rates nonzero} \\ cnst & , \text{ either tumbling rate zero} \end{cases}$$

Stationary state

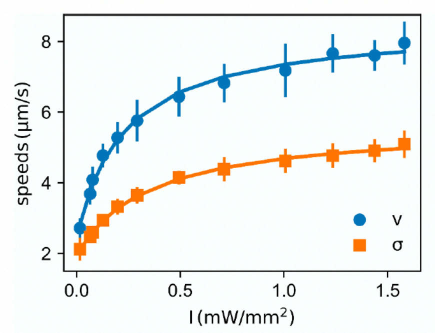
- Nontrivial steady state possible for space-dependent rates
- For arbitrary speed and tumbling rates, stationary state distribution exists and is given by (Schnitzer 1993)

$$
\begin{aligned}
P(x) &= P_+(x) + P_-(x) \\
&\propto \left[\frac{1}{v_+(x)} + \frac{1}{v_-(x)}\right] e^{\int_0^x \left(\frac{\gamma_-(y)}{v_-(y)} - \frac{\gamma_+(y)}{v_+(y)}\right) dy}
\end{aligned}
$$

provided the normalization constant is finite

Choice of rates

- However, typically rates chosen are not bounded above which is unphysical
- E.g., mean speed of photokinetic bacteria in response to light intensity saturates



(Frangipane et al. 2018)

Bounded rates

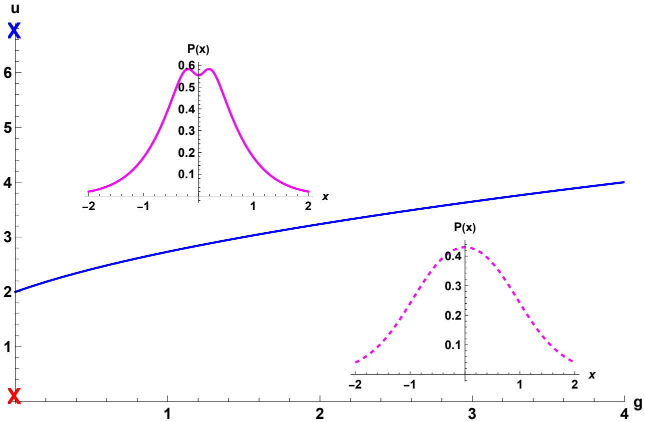
- Our model

$$\begin{aligned} v_\pm(x) &= v_0 \mp v_1 \tanh(ux) \ , \ \ v_0 \geq v_1 > 0 \\ \gamma_\pm(x) &= \gamma_0 \pm \gamma_1 \tanh(gx) \ , \ \ \gamma_0 \geq \gamma_1 > 0 \end{aligned}$$

- For non-negative u, g : stationary state exists for all $u, g \geq 0$ except $u = g = 0$

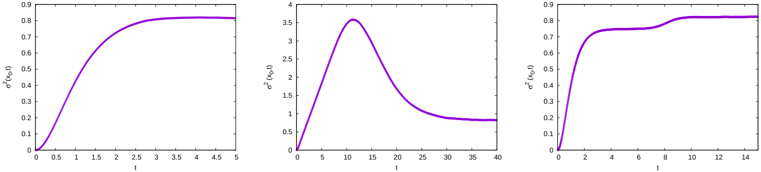
Stationary state

- Exact expression in terms of hypergeometric function
- Tails exponential/double-exponential; uni- or bimodal



Dynamics of mean-squared displacement

- Start with particle at $x_0 \geq 0$ (say)
- Dynamics depend on initial position



$x_0 \ll u^{-1}, g^{-1}$ | $x_0 \gg u^{-1}, g^{-1}$, tumbl rate $\not\to 0$ | $x_0 \gg u^{-1}, g^{-1}$, tumbl rate $\to 0$

- Consequence of bounded rates

Limiting case: exact Green's function

- Uniform speed, step-function tumbling rate ($u = 0, g \to \infty$)
- Prob dist for $x_0 = 0$ and $P_+(x, 0) = P_-(x, 0)$ (Singh et al. 2020)
- Exact Green's function derived for arbitrary x_0

$$G''(x,s)+\frac{2\gamma_1}{v_0}[2\Theta(x)-1]G'(x,s)+[\frac{4\gamma_1}{v_0}\delta(x)-\frac{s(2\gamma_0+s)}{v_0^2}]G(x,s)=\delta(x-a)$$

- Continuous G, discontinuity in G' at $x = a$ and $x = 0$

Dynamics of variance

- For step-fn tumbling: variance increases or saturates until

$$\langle x(t^*) \rangle_{homo} = 0$$

For general case: simulations consistent with analog

- For step-fn tumbling: Exponential relxn to steady state

$$\sigma^2(t) \sim \begin{cases} e^{-(\gamma_0 - \sqrt{\gamma_0^2 - \gamma_1^2})t} & , \ \gamma_0 \neq \gamma_1 \text{ [nonzero tumbl rate]} \\ e^{-2\gamma_0 t} & , \ \gamma_0 = \gamma_1 \text{ [either tumbl rate zero]} \end{cases}$$

For general case: Relaxation time scale not known

Summary and open questions

- Considered a RTP with bounded rates as physically sensible. Different qualitative shape of $\sigma^2(t)$ seen and generically expected for such rates.

- Explore other such choices with stationary state?
- Green's function for uniform tumbling and $|x|$ potential?
- Experiments?