Dynamical Fluctuations in Riesz and Dyson gases

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Outline

Motivations

1. Macroscopic fluctuations in the Riesz gas

2. Large Deviations of the current in the Dyson gas

[3. Free expansion of a gas with long-range interactions]

Concluding remarks

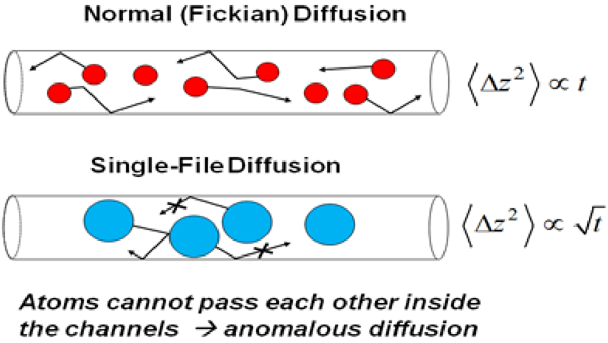
R. **Dandekar***, P. Krapivsky, KM: Phys. Rev. E* **107** *044129 (2023)*

R. **Dandekar***, P. Krapivsky, KM: Current fluctuations in the Dyson gas, to appear in Phys. Rev. E (arxiv 2409.06881)*

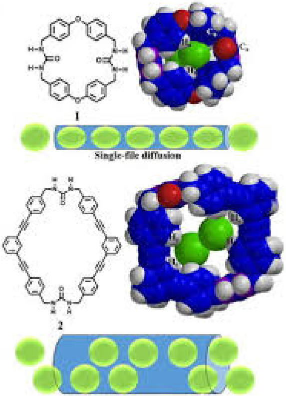
P. Krapivsky, KM: Expansion into the vacuum of stochastic gases with long-range interactions (to be submitted)

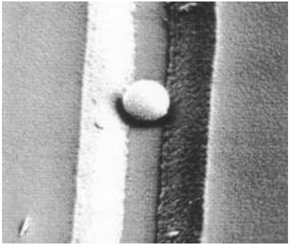
Motivation : Anomalous Single-file diffusion in 1d

Single-file diffusion is an important phenomena soft-condensed matter (for example, transport through cell membranes).



Experimental observations

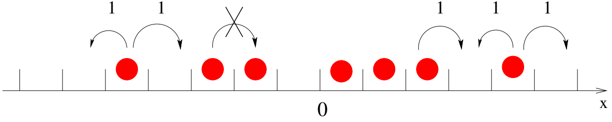




(C. Bechinger's group in Stuttgart)

Symmetric exclusion

Consider the Symmetric Exclusion Process, $(p = q = 1)$ on $\mathbb{Z}$ with a uniform finite density ρ of particles. This model was invented by F. Spitzer in 1970.
It is a pristine model for single-file diffusion is the Symmetric Exclusion Process, in which particles perform continuous-time random walks with hard-core (classical) exclusion interaction



Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

Because of the non-overtaking constraint, the tracer's position X_t and the current Q_t are tightly linked.

The Symmetric Exclusion Process (SEP) on Z.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays an anomalous diffusive behaviour: when $t \to \infty$, we have

$$\langle X_t^2 \rangle \simeq 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}} \qquad \text{(Arratia, 1983)}$$

The full distribution of X_t remained unknown for many years.

• An exact formula valid for any time has been obtained by using Integrable Probabilities (Bethe Ansatz) for the microscopic model.

• The large deviations of X_t can also be calculated by using fluctuating hydrodynamics and by solving exactly the MFT equations (Inverse Scattering) at the macroscopic level.

Exact Tracer distribution

At any finite time, the distribution of the tracer can be expressed as an infinite dimensional (Fredholm) determinant:

$$\langle e^{\lambda N(x,t)} \rangle = \det(1 + \omega K_{t,x}) W_0(\lambda)$$

where

$$K_{t,x}(\xi_1, \xi_2) = \frac{\xi_1^{|x|} e^{\epsilon(\xi_1)t}}{\xi_1 \xi_2 + 1 - 2\xi_2} \quad \text{with} \quad \epsilon(\xi) = \xi + \xi^{-1} - 2$$

The asymptotic analysis of this determinant yields concrete formulas for the cumulants of the tracer's position (revealing non-gaussian behaviour).

- Variance : $\langle X_t^2 \rangle = 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}}$ (Arratia)
- 4th order: $\frac{\langle X_t^4 \rangle_c}{\sqrt{4t}} = \frac{1-\rho}{\sqrt{\pi}\rho^3}[1 - (4 - (8 - 3\sqrt{2})\rho)(1-\rho) + \frac{12}{\pi}(1-\rho)^2]$

Fluctuating Hydrodynamics (MFT)

The coarse-grained evolution of the system, conditioned on a given value of the tracer's position X_t, can be recast as an *'optimal transport problem'*. This leads to a Hamiltonian equations coupling two fields $(q(x,t), p(x,t))$, where $q(x,t) = \rho(x,t)$ is the density and $p(x,t)$ is the control-field (conjugate momentum):

$$\partial_t q = \partial_x[D(q)\partial_x q] - \partial_x[\sigma(q)\partial_x p]$$

$$\partial_t p = -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2$$

with Hamiltonian $H = \sigma(q)(\partial_x p)^2/2 - D(q)(\partial_x q)(\partial_x p)$.

The only information of the microscopic scale relevant macroscopically is embodied in D and σ.

For SEP, we have $D(q) = 1$ and $\sigma(q) = 2q(1-q)$. Other details are 'blurred' in the continuous limit.

The MFT for SEP are integrable

The Hamiltonian equations for SEP are classically integrable in the Liouville sense and explicit formulas for the optimal fields (q^*, p^*) that describe the dynamical evolution that generates a given fluctuation (rare event) can be found at the hydrodynamic scale.

This, in turn, yields the Cumulant Generating Function (CGF) of the current. In the long time limit, $\langle e^{\lambda Q_T} \rangle \simeq e^{\sqrt{T}\mu(\lambda)}$, with

$$\mu(\lambda) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\omega^n}{n^{3/2}}$$

where $\omega = (e^{\lambda} - 1)\rho_-(1 - \rho_+) + (e^{-\lambda} - 1)\rho_+(1 - \rho_-)$

(see Derrida-Gershenfeld J. Stat. Phys. 2009; Imamura-M-Sasamoto PRL 2017 and CMP 2021 for a microscopic derivation).

Some questions

• For a driven diffusive gas, with transport coefficients $D(\rho)$ and $\sigma(\rho)$, the variance of a tracer is (Krapivsky-M-Sadhu PRL 113, 2014)

$$\langle X_t^2 \rangle = \frac{\sigma(\rho)}{\rho^2 \sqrt{\pi}} \sqrt{\frac{t}{D(\rho)}}$$

This formula depends on whether the initial conditions are fluctuating (*annealed*) or fixed (*quenched*). In the latter case, there is an additional $\sqrt{2}$ in the denominator. What about higher cumulants (Bénichou and Grabsch)?

• Can one generalize these results to non-diffusive single-file systems?

• For the Dyson gas, the tracer behaves as

$$\langle X_t^2 \rangle = \frac{\log t}{\pi^2 \rho^2} \qquad \textbf{(H. Spohn, 1986)}$$

Can this formula (and higher cumulants) be derived from hydrodynamics?

• More generally, how is the behaviour modified if the particles interact through a potential, possibly long-ranged (Riesz gas)?

• How does the tracer behave in the scaling limit? *(for SEP: Peligrad and Sethuraman proved fBm-1/4).*

DYNAMICAL FLUCTUATIONS

IN THE RIESZ GAS

Particles with long-range interactions

We consider a gas of particles on the line subject to a white noise interacting through a Riesz potential of strength g. In the over-damped limit, the particle positions x_i evolve according to

$$\dot{x}_i = g \sum_{j \neq i} \frac{x_i - x_j}{|x_i - x_j|^{2+s}} + \eta_i$$

with

$$\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{ij} \delta(t - t')$$

For $s > 1$: the gas is effectively short-ranged.
For $0 < s < 1$: the gas is long-ranged and the free energy functional is non-local.

Our goal is to generalize the MFT approach and investigate fluctuations of the integrated current and the position of a tagged particle in a set-up with uniform density ρ.

Coarse-grained description of the Riesz gas

For $s > 0$, the system is characterized by a single dimensionless parameter

$$G = \frac{g\rho^s}{D}$$

that measures the relative strength of interactions versus noise. This allows us to set $g = D = 1$.

We shall focus on the long-ranged case ($0 < s < 1$).

The coarse-grained density field of the particles satisfies the continuity equation

$$\partial_t q + \partial_x J = 0$$

The local current $J = J(x,t)$ contains the standard diffusion term, $-\partial_x q$, plus a deterministic contribution J_{Riesz} arising from the Riesz potential and a stochastic component due to the noise:

$$J = J_{\mathrm{Riesz}} - \partial_x q + \sqrt{2q}\,\eta$$

where $\eta(x,t)$ is a space-time white noise.

The Riesz current

For $0 < s < 1$, the Riesz current $J_{\rm Riesz}$ can be expressed as

$$J_{\rm Riesz} = q\mathcal{H}_s[q]$$

where the modified Hilbert transform is defined as (in the sense of principal values)

$$\mathcal{H}_s[q] = \int dy \, \frac{x-y}{|x-y|^{2+s}} \, q(y)$$

(For $\mathbf{s = 0}$, this is the π times the **usual Hilbert transform**).

Equivalently, the total deterministic current can be derived from

$$J_{\rm Riesz} - \partial_x q = -q \, \frac{\partial}{\partial x} \left(\frac{\delta}{\delta q} \mathcal{F}[q] \right)$$

with free energy

$$\mathcal{F}[q] = \frac{1}{2s} \int \int dx \, dy \, \frac{q(x)q(y)}{|x-y|^s} + \int dx \, q \ln q$$

Stochastic Hydrodynamics of the Riesz gas

When $0 < s < 1$, the fluctuating hydrodynamics of the 1d stochastic Riesz gas is governed by the stochastic PDE (Dean-Kawasaki eq.)

$$\partial_t q = -\partial_x \left(q\mathcal{H}_s[q] - \partial_x q + \sqrt{2q}\,\eta \right)$$

The strategy is the same as usual: express the transition probability as a path integral. The characteristic function of the total current Q_T that has flown through the origin during the time interval $(0, T)$,

$$Q(T) = \int_0^{\infty} dx\,[q(x,T) - q(x,0)]$$

is given by

$$\langle e^{\lambda \mathcal{Q}_T} \rangle = \int \int \mathcal{D}q\,\mathcal{D}p\; e^{\lambda \mathcal{Q}_T - \int_0^T \int dt\,dx\,S(q,p)}\,P[q(x,0)]$$

with the action

$$S(q,p) = p\partial_t q - q(\partial_x p)^2 - q(\partial_x p)\mathcal{H}_s[q]\,\partial_x p + \partial_x p\,\partial_x q$$

Non-Local equations for macroscopic fluctuations

In the long time limit, $T \to \infty$, the evolution of the system conditioned to a given value of the total current Q_T follows an optimal trajectory that satisfy the saddle-point equations

$$\partial_t q = \partial_x^2 q - -\partial_x \left(2q\partial_x p + q\mathcal{H}_s[q]\right)$$

$$\partial_t p = -\partial_x^2 p - (\partial_x p)^2 - \mathcal{H}_s[q]\partial_x p + \mathcal{H}_s[q\partial_x p]$$

This integro-differential system generalizes the original MFT equations to the Riesz gas with long-range interactions.

The boundary conditions involve the fugacity parameter:

$$p(x, T) = \lambda\theta(x) \quad \text{and} \quad p(x, T) = \lambda\theta(x) + \frac{\delta \mathcal{F}}{\delta q(x, 0)}$$

Perturbative solution of the MFT equations

A perturbative expansion w.r.t. λ allows us to find exact formulas for the variance of the current and a tagged particle position as well as for the two-time correlators.

$$\langle Q^2(T)\rangle = \frac{2^{\frac{1}{1+s}}\Gamma\left(\frac{1}{s+1}\right)}{s}\left[\frac{4^s\,\Gamma\left(1+\frac{s}{2}\right)}{\pi^{s+3/2}\,\Gamma\left(\frac{1-s}{2}\right)}\right]^{\frac{1}{s+1}}(\rho T)^{\frac{s}{s+1}}$$

The displacement of a tagged particle satisfies

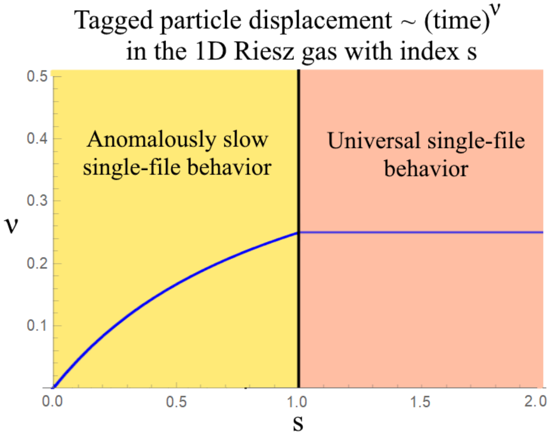
$$\langle X^2(T)\rangle = \frac{1}{\rho^2}\langle Q^2(T)\rangle$$

Two-time correlations are given by

$$\langle X(t_1)X(t_2)\rangle \propto t_1^{\frac{s}{1+s}} + t_2^{\frac{s}{1+s}} - |t_1 - t_2|^{\frac{s}{1+s}}$$

The particle behaves as a fractional Brownian with exponent $\nu = \frac{\mathbf{s}}{\mathbf{2(s+1)}}$.

For short range forces, $s > 1$, we have usual single-file behaviour with exponent $1/4$.



Limiting cases - Discussion

• For $s = 1$, the particles are interacting through a 3d Coulomb potential and are confined to a one dimensional-line. A heuristic analysis suggests

$$X^2 \sim \sqrt{\frac{T}{\log T}}$$

• We could not integrate the full MFT system for the Riesz gas and get higher cumulants (at the moment...).

• The scalings and the amplitude formulas we have obtained have been recently retrieved using a different approach by Touzo, Le Doussal and Schehr (2411.01355) who linearized the equations of motion around the equally spaced crystal configuration.

• For $s = 0$, we have a Dyson gas.

LARGE DEVIATIONS

OF THE CURRENT

IN THE DYSON GAS

Optimal fluctuation equations

When the interaction potential is logarithmic ($s = 0$) the particles perform a Dyson Brownian Motion:

$$\dot{x}_i = \sum_{j \neq i} \frac{1}{x_i - x_j} + \eta_i$$

This corresponds to the collective motion of the eigenvalues of time dependent random matrices: this is known to be a very rigid system and the tracer is logarithmically subdiffusive (Spohn, 86).

We wish to study the macroscopic fluctuations of this interacting particles system, including higher order cumulants, from a stochastic hydrodynamics viewpoint.

Optimal fluctuation equations

In the long time limit, the saddle-point equations, after neglecting the subdominant diffusive term, reduce to

$$\partial_t q = -\partial_x \left(2q\partial_x p + q\mathcal{H}[q]\right)$$

$$\partial_t p = \pi\mathcal{H}[q\partial_x p] - \pi\mathcal{H}[q]\partial_x p - (\partial_x p)^2$$

Here, $\mathcal{H}[q]$ is the standard Hilbert transform, defined as

$$\mathcal{H}[q] = \frac{1}{\pi}\int dy\, \frac{q(y)}{x-y}\, q(y)$$

The boundary conditions involve the fugacity parameter.

Contrarily to the case of the Riesz gas, the optimal saddle point equations for the Dyson gas can be analyzed non-perturbatively.

Analytic results

In the long time limit, the characteristic function of the time-integrated current for the Dyson gas, $Q_t = \int_0^\infty dx\,[q(x,t) - q(x,0)]$ by

$$K(\gamma) = \langle \mathrm{e}^{\gamma Q_t} \rangle = \frac{g^3 \rho^4 t^2}{D} \mu \left(\frac{D\gamma}{g^2 \rho^2 t} \right)$$

with

$$\boxed{\mu(\lambda) = \frac{\lambda^2}{\pi^2} \left(\frac{1}{2} \log \frac{\pi^6}{\lambda^2} + \frac{1}{2} \right)}$$

Inverting the Laplace transform, we obtain:

$$\boxed{\log[\mathrm{Prob}(Q_t)] = \frac{\pi^2 g Q_t^2/4}{D W_{-1}(-q)} \left[1 + \frac{1}{2 W_{-1}(-q)} \right]}$$

where W_{-1} is a real branch of the Lambert function and $q = Q_t/(2\pi g \rho^2 t)$.

Cumulants

From this expression, we retrieve the variance of the Dyson current and obtain higher cumulants:

$$\langle Q_t^2 \rangle \simeq \frac{2D}{\pi^2 g} \log(g\rho^2 t)$$

Fluctuations of the current are drastically reduced compared to single-file diffusion with local interactions.

Higher cumulants are given by

$$\langle Q_t^{2m} \rangle_c \simeq \frac{D(-1)^{m-1}(m-2)!}{\pi^2 g} \left(\frac{4D \log(g\rho^2 t)}{\pi^2 g} \right)^{m-1}$$

There are some close analogies with the distribution of the number of eogenvalues of a random matrix in an interval (Fogler-Shklovskii, Dyson 1995).

Strategy of the calculation

• Defining a velocity field, $v = \mathcal{H}[q] + 2\partial_x p$, the equations take a hydrodynamic form with pressure $P = -\pi^2 q^3/3$ (Matytsin, 1994):

$$\begin{aligned} \partial_t q + \partial_x (qv) &= 0 \\ \partial_t v + v(\partial_x v) &= \pi^2 q(\partial_x q) \end{aligned}$$

The main difference with the Matytsin problem is in the boundary conditions (non-local and mixed).

• Using the complex-valued function, $f = v + i\pi q$, the above system is mapped to the complex Burgers equation: $\partial_t f + f\partial_x f = 0$

• Our very peculiar boundary conditions imply a PT (parity and time reversal) invariance of $f(x,t)$: $f(x,t) = f(-x, 1-t)$.

• A closed functional equation is obtained for the density profile that leads to the CGF of the current.

Conclusions

• The ideas and techniques that have been fruitful to study simple interacting particle processes (stochastic hydrodynamics, MFT-type equations) can be extended to long-range gases to describe scaling behaviour and rare events.

• There are some (subtle?) divergence issues for the Riesz gas when one tries to extract perturbatively higher cumulants.

• The Dyson gas is likely to be exactly solvable. A satisfactory achievement would be have a full microscopic picture on a par with the macroscopic analysis.

• The deterministic expansion of N Riesz/Dyson particles initially concentrated in a single point is analytically tractable and displays interesting scaling features.