Random Forced Burgers Equation and KPZ Universality: Renormalization Approach

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Problems at the Interface of Mathematics and Physics

ICTS - TIFR

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$u_t + (u \cdot \nabla) u = \nu \Delta u + f^\omega(x,t)$ ← Random forced Burgers eq.

$u(x,t)$, $x \in \mathbb{R}^d$, $f^\omega(x,t) = -\nabla F^\omega(x,t)$

↑ random force

$\Updownarrow$ $\underline{u(x,t) = \nabla \varphi}$

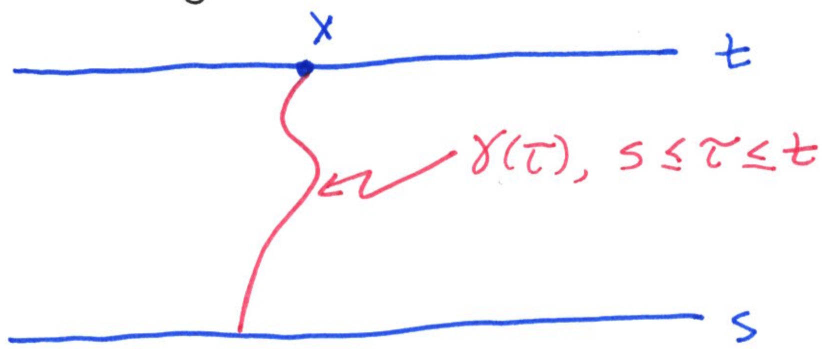
$\varphi_t + \frac{1}{2} |\nabla \varphi|^2 = \nu \Delta \varphi - F^\omega(x,t)$ ← Random Hamilton-Jacobi eq.

$\nu = 0$ → inviscid case

- $\langle F^\omega(x,t) F^\omega(y,s) \rangle = c(x-y)\, \delta(t-s)$
  Realizations of F^ω are smooth in x, and white in t

- $d=1$, $F^\omega(x,t)$ - space-time white noise
  KPZ equation (Kardar - Parisi - Zhang)

1. Cauchy problem (inviscid case, $\underline{\nu=0}$): $\varphi(x,s) = \Psi(x)$



Lax-Oleinik variational principle

$$\varphi^{\omega}(x,t) = \inf_{\gamma:\, \gamma(t)=x} \left[\Psi(\gamma(s)) + \int_s^t \left(\frac{\dot{\gamma}^2}{2} - F^{\omega}(\gamma(\tau),\tau) \right) d\tau \right]$$

$$u^{\omega}(x,t) = \nabla \varphi^{\omega}(x,t) = \dot{\bar{\gamma}}(x,t)$$

$$\bar{\gamma}^{\omega}_{x,t,s} = \operatorname*{argmin}_{\gamma:\, \gamma(t)=x} \left[\Psi(\gamma(s)) + \int_s^t \left(\frac{\dot{\gamma}^2}{2} - F^{\omega}(\gamma(\tau),\tau) \right) \right] \leftarrow \text{minimizer}$$

2. Cauchy problem (viscous case, $\nu > 0$)

$$\begin{cases} d\gamma^{u}(\tau) = u(\gamma(\tau),\tau) + \sqrt{2\nu}\, dW(\tau) \\ \gamma(t) = x \end{cases}$$

$u(x,\tau)$ - stochastic control
$dW(\tau)$ - white noise

stochastic diff. eq. (SDE), backward in time

$$\varphi_\nu^\omega(x,t) = \inf_u E_W\left[\Psi(\gamma^u(s)) + \int_s^t \left(\frac{u^2(\gamma(\tau),\tau)}{2} - F^\omega(\gamma^u(\tau),\tau)\right)\right]$$

$$u_\nu^\omega(x,t) = \nabla\varphi_\nu^\omega(x,t)$$ (Moreover, optimal stochastic control $\bar{u} = u_\nu^\omega$)

Directed Polymers

$$\varphi(x,t) = -2\nu \log Z(x,t)$$ ← Hop~~f~~f-Cole transformation

$$Z_t = \nu\Delta Z + \frac{F^\omega(x,t)}{2\nu} Z$$ ← Stochastic Heat eq.

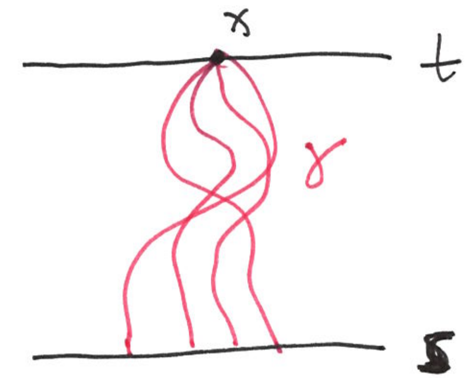
$$Z^\omega(x,t) = E_B\, e^{\frac{1}{2\nu}\left[\int_0^{t-s} F^\omega(x+\sqrt{2\nu}\,B(\tau),\, t-\tau)\,d\tau \;-\; \Psi(x+\sqrt{2\nu}\,B(t-s))\right]},$$

where $B(\tau)$ is a standard Brownian motion in $\mathbb{R}^d$

Feynman-Kac formula

$$P_{x,t}^\omega(B) = \frac{1}{Z_1^\omega(x,t)}\, e^{\frac{1}{2\nu}\int_0^{t-s} F^\omega(x+\sqrt{2\nu}\,B(\tau),\, t-\tau)\,d\tau}\, dB$$

$$\gamma(\tau) = x + \sqrt{2\nu}\,B(t-\tau)$$
$$s \le \tau \le t$$



Conjecture: "One force - one solution" (almost surely)

Starting from two different initial conditions Ψ_1, Ψ_2 the solutions φ_1, φ_2 approach each other as $t \to \infty$.

$t=0$

There exists a unique (global) solution

$$\underline{\underline{u}}^\omega = \nabla \varphi^\omega$$

$s=-\infty$

Theorems:

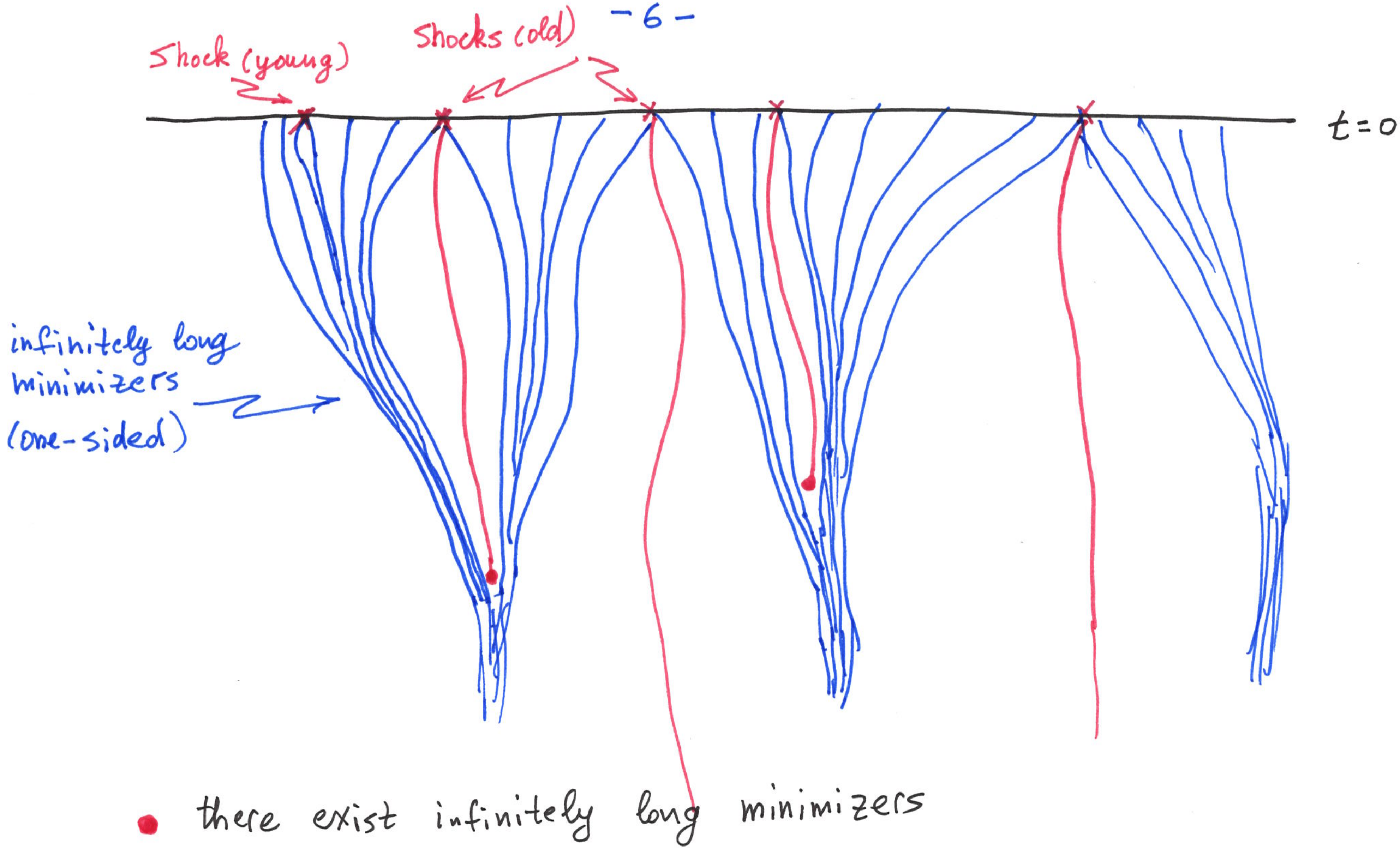
$d=1, \nu=0$ Bakhtin, Cator, Kh.

$d=1, \nu>0$ Bakhtin, Li

$d \geq 3, \nu>0$, small F^ω Imbrie, Spencer; Bolthausen; Sinai; Kifer; Hurth, Navarro, Kh.

↑

weak disorder (diffusive behavior of directed polymers)



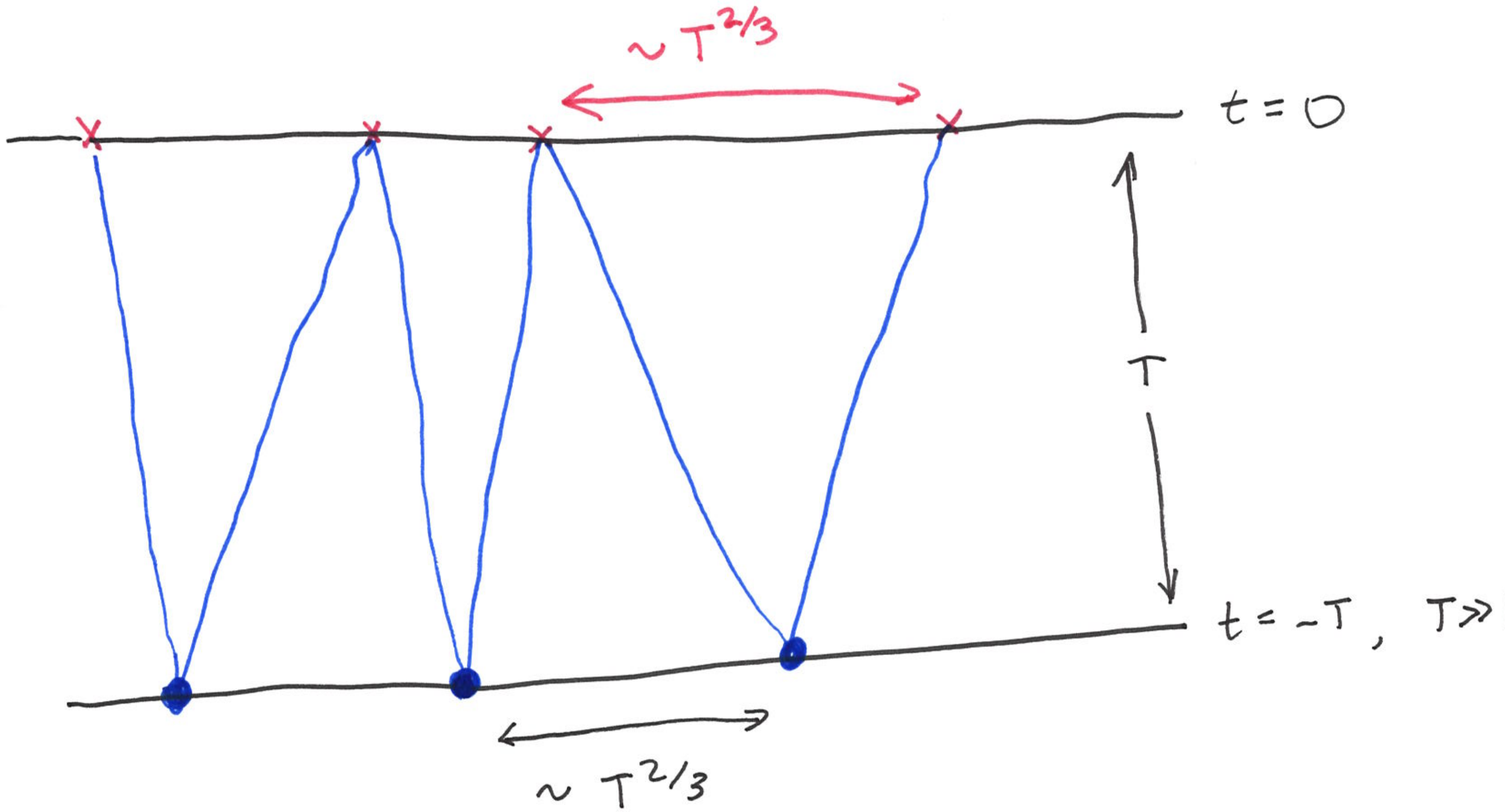
- there exist infinitely long minimizers
- such infinitely long minimizers are asymptotically close to each other as $t \to -\infty$

Positive viscosity:

- Polymer measures have limit as $s \to -\infty$
- Limiting polymer measures are asymptotic to each other as $\tau \to -\infty$

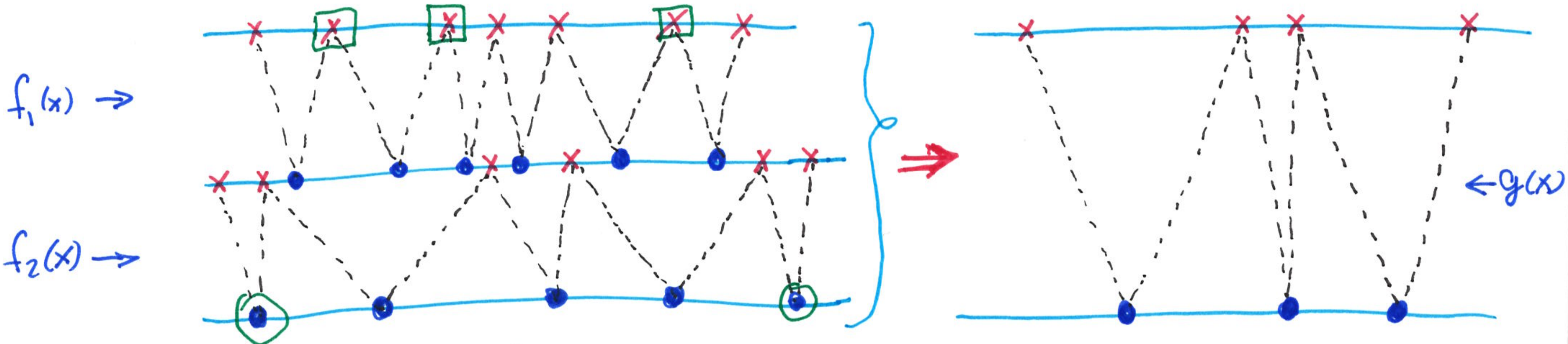
Weak disorder	$\longleftrightarrow$ strong disorder	
$d \geq 3$, small force	$d = 1, 2$ any force	Carmona - Hu
	$d \geq 3$ strong force	Comets - Shiga - Yoshida
Diffusive behavior	strongly non-diffusive behavior (localization)	

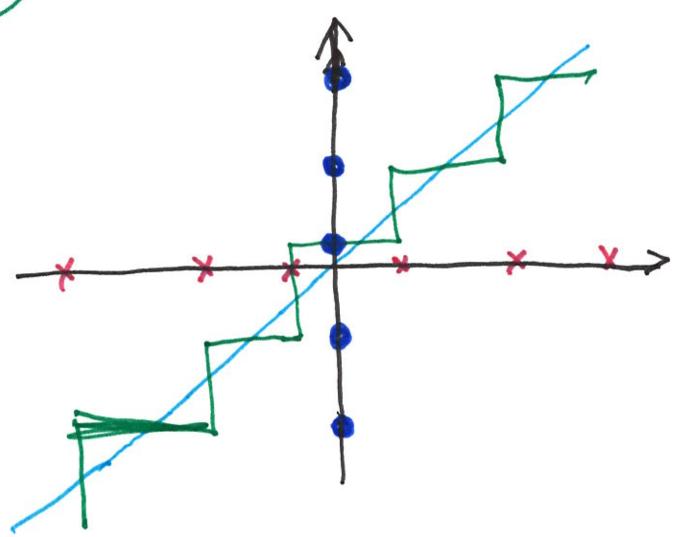
KPZ phenomenon ($d=1$)



Conjecture: After rescaling by $\alpha T^{2/3}$ point fields of red crosses and blue dots converge in distribution in a limit $T\to\infty$ to two correlated universal point fields

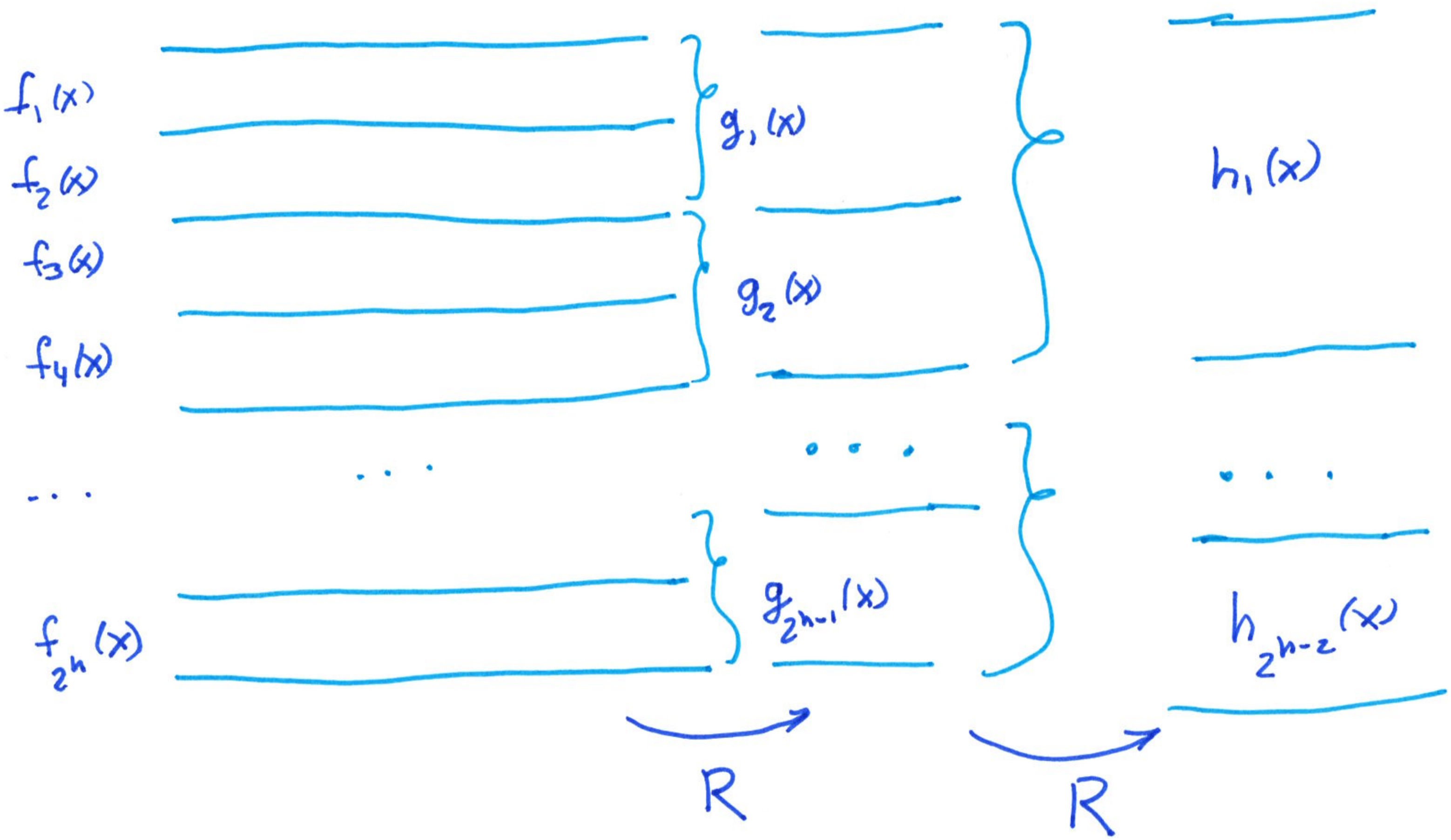
Renormalization Transformation:





← random monotone piecewise constant map

$$g = g(x) = f_2(f_1(x)) = f_2 \circ f_1$$



Fixed points: $\mathrm{Dist}\,(f_1, f_2, \ldots f_{2^n}, \ldots) = \mathrm{Dist}\left(\frac{1}{\alpha} g_1(\alpha x), \frac{1}{\alpha} g_2(\alpha x), \ldots \frac{1}{\alpha} g_{2^{n-1}}(\alpha x), \ldots\right)$

Conjecture: for all $\alpha > 1$ there exists a unique fixed point. Moreover, this fixed point is stable.

Uniqueness is up to a trivial rescaling: $\{f_i(x)\} \to \{\frac{1}{\beta} f_i(\beta x)\}$

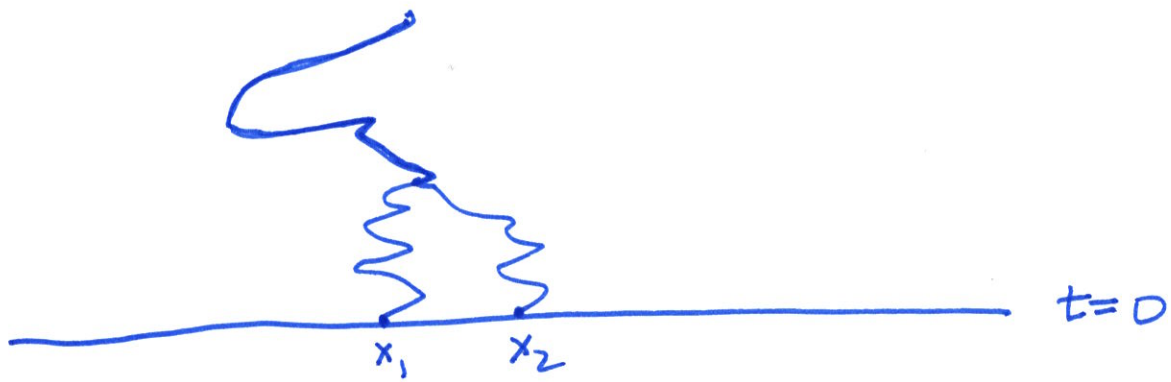
What about stability?

Suppose that for a stationary sequence $\{f_1, f_2, \dots f_{2^n}, \dots\}$ after n renormalizations the density of point fields decays as α^{-n}. Then, under renormalizations the distribution of $R^n\{f\}$ converges to the distribution of the fixed point corresponding to α.

In the KPZ case n steps of renormalization correspond to time $T = 2^n$ and density $\frac{1}{T^{2/3}} = \frac{1}{2^{2/3 n}}$. Hence, $\boxed{\alpha = 2^{2/3}}$

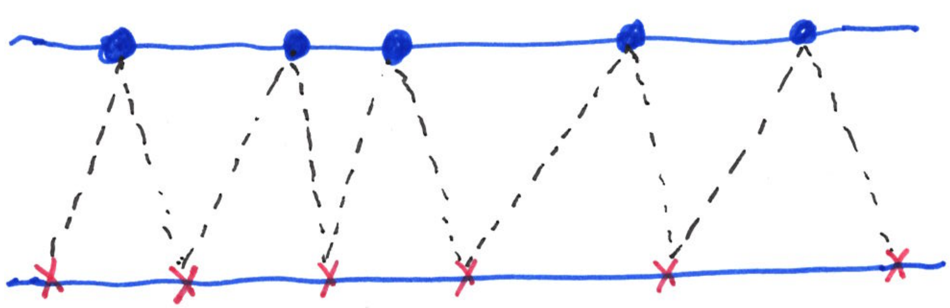
Fixed point in the case $\alpha = \sqrt{2} = 2^{1/2}$

Coalescing Brownian Motions (Arratia, 79)



- From every point $x \in \mathbb{R}^1$ one starts an independent Brownian Motion
- When any two Brownian Motions meet they continue as one Brownian Motion (coalesce)

Arratia proved that after an arbitrary small positive time $\tau > 0$ only a discrete set of points x is reached by all initial Brownian Motions. Thus, after time 1 we have:



$\longleftrightarrow \quad f(x)$

Time-1 map for coalescing Brownian Motions

Let $\{f_1, f_2, \ldots, f_{2^n}, \ldots\}$ be an iid (independent identically distributed) sequence with the distribution given by $f(x)$.

It is easy to see that $\{f_i,\ i=1,2,\ldots\}$ is a fixed point for R corresponding to $\alpha = 2^{1/2}$. This follows immediately from the scaling invariance of Brownian Motions.

Let $\{g_1, g_2, \ldots, g_{2^n}, \ldots\}$ be another iid sequence of monotone piecewise constant maps.

<u>Theorem</u> (L. Li, Kh.)

Under mild conditions $R^n\{g_i\} \xrightarrow[n\to\infty]{} \{f_i\}$ with exponential rate. More precisely, the Wasserstein distance between the probability distributions for $R^n\{g_i\}$ and $\{f_i\}$ is exponentially small.

Stability of fixed point $\{f_i\}$

Coalescing Fractional Brownian motions

$B_H(t)$ – fractional Brownian motion with Hurst index H.

Self-similarity: $B_H(at) = |a|^H B(t)$ (in distributional sense)

Problem: how to continue fractional Brownian motions when two motions coalesce?

One can try to choose one motion independently and continue it.

It seems to work (numerically) for $\frac{1}{2} < H < 1$
($H = \frac{1}{2}$ corresponds to standard Brownian motion)

Plan: construct fixed points for different H and compare it to the KPZ fixed point (corresponding to $H = 2/3$)