Two applications of the bootstrap in QCD

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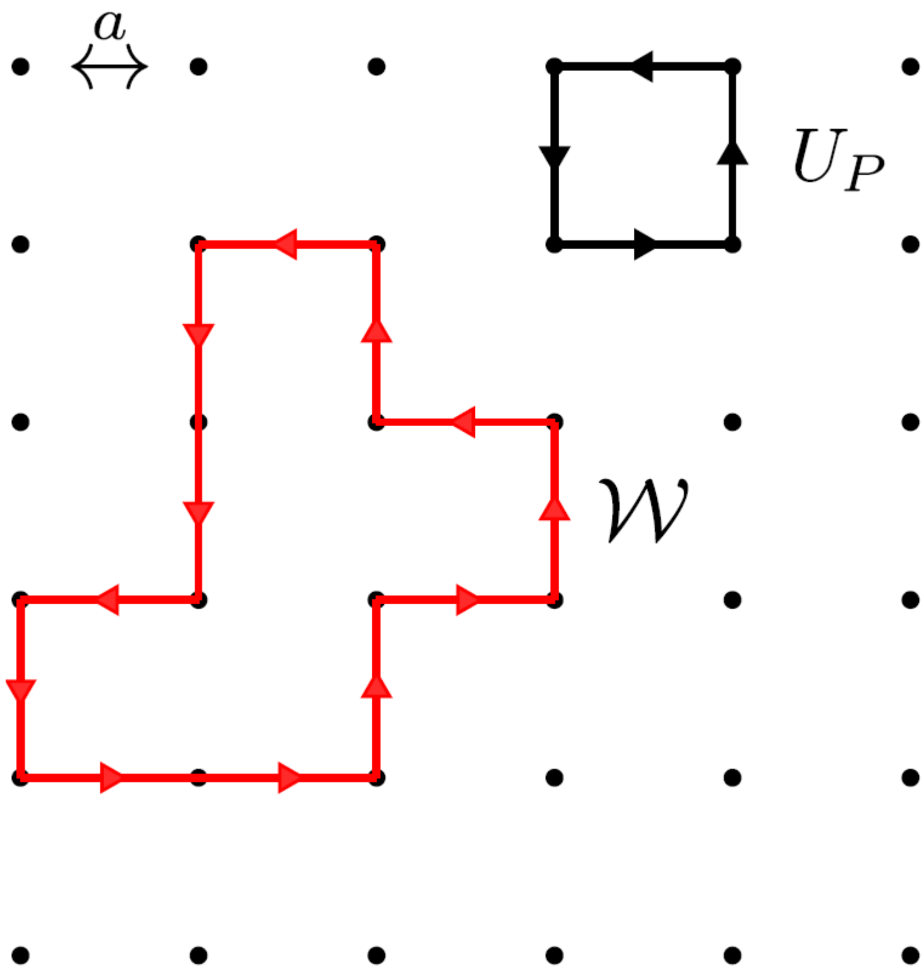
Lect I:
Loop equation and bootstrap methods in Lattice gauge theories

Based on **1612.08140** by P.D.Anderson and M.K.
See also **2203.11360** by Kazakov and Zechuan Zheng
2002.08387 by Henry W. Lin

Motivation

Can one define gauge theories purely in terms of gauge invariant quantities?
AdS/CFT gives one possibility in terms of a dual string theory.
More directly:
Wilson loops → Loop equation (Migdal-Makeenko)

Lattice gauge theory, pure YM, large-N, cubic lattice



Action

$$S = -\frac{N}{2\lambda}\sum_P \mathrm{Tr}U_P$$

$$Z = \int \prod_{\vec{x},\mu} dU_\mu(\vec{x})\ e^{-S}$$

λ : ‘t Hooft coupling
is like temperature

Phase Transition (large-N)

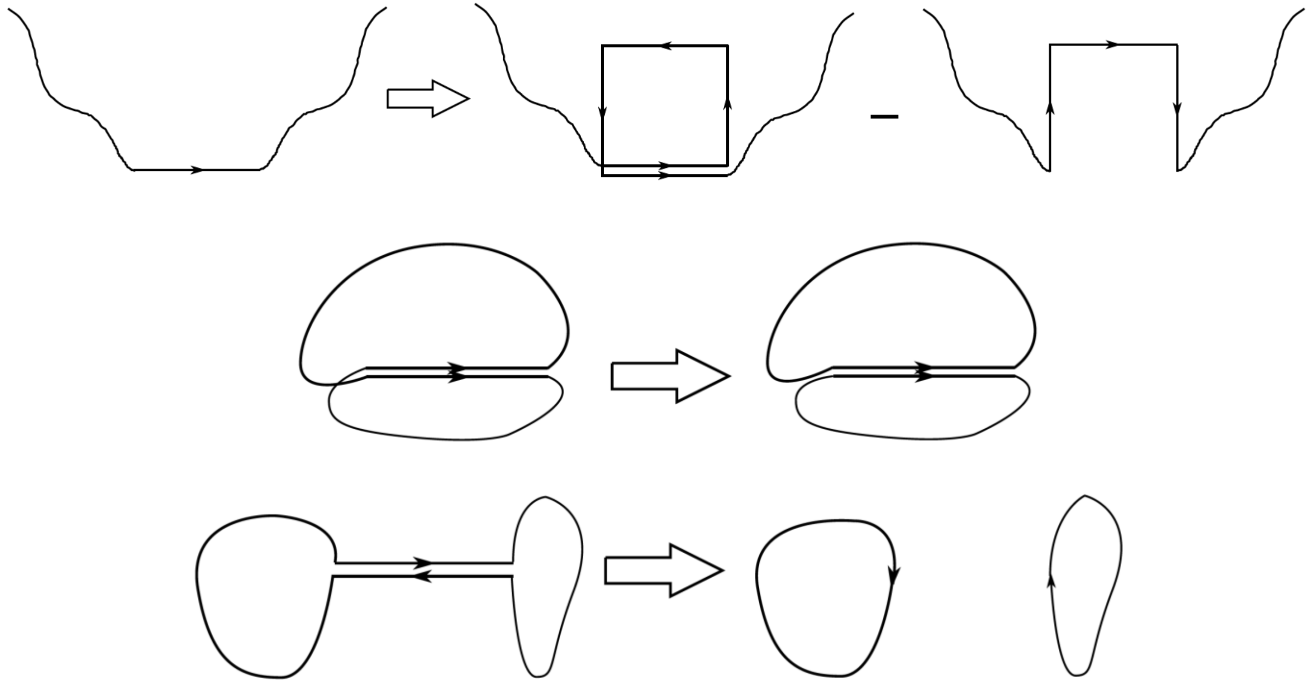
d=2 $\lambda_c = 1$, third order (Gross-Witten, Wadia, ‘80)

d=3 $\lambda_c \sim 1.2$, third order (Teper ‘06, numerical)

d=4 $\lambda_c = 1.3904$, first order (Campostrini ’99, numerical)

Loop equation (Migdal-Makeenko, Eguchi, Foerster,…)

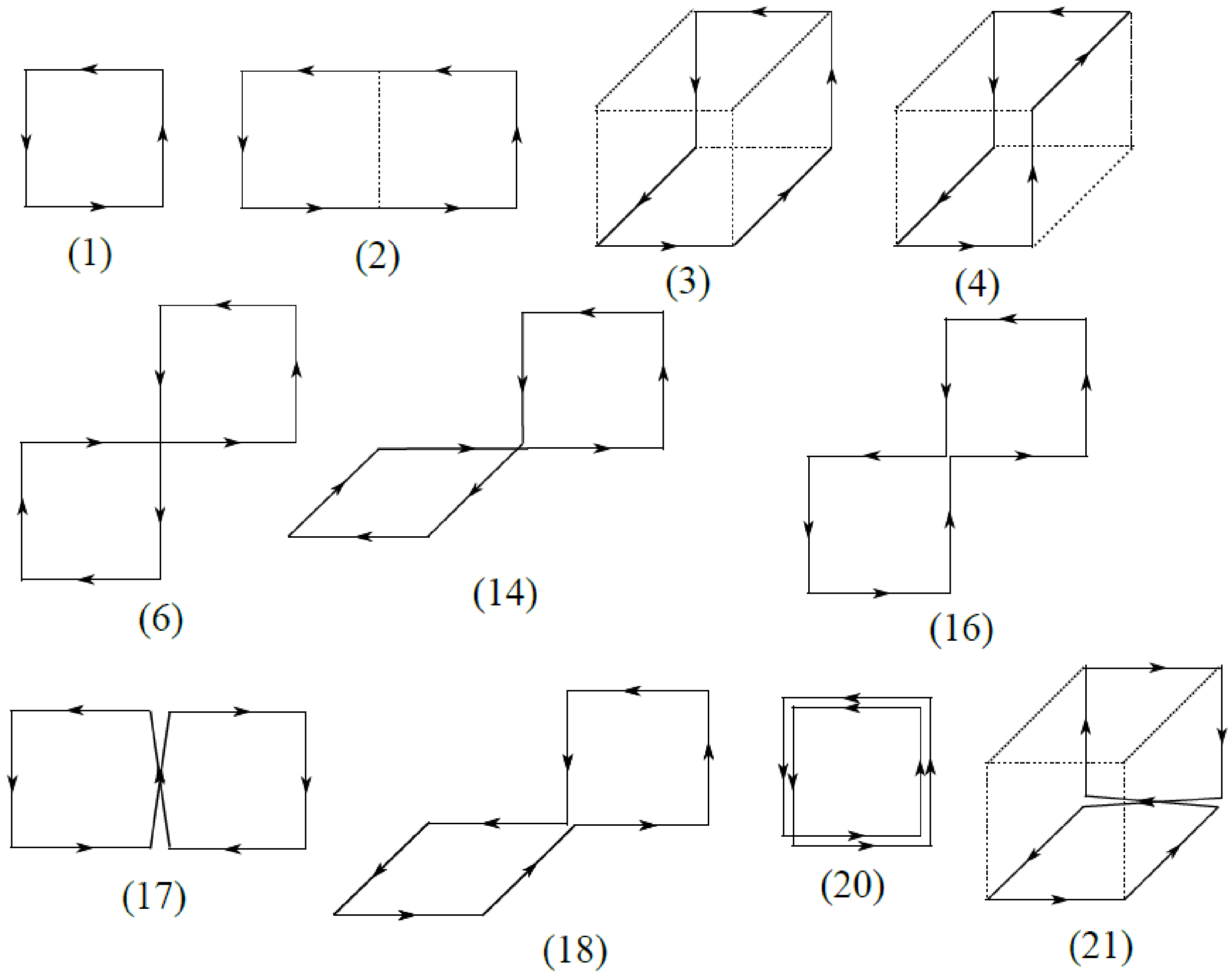
Graphic form of the equation:



Algebraic form of the equations (sum over links) :

$$\mathbb{K}_{i\to j}\mathcal{W}_j + 2\lambda\mathcal{W}_i + 2\lambda\mathbb{C}_{i\to jk}\mathcal{W}_j\mathcal{W}_k = \delta_{i1}$$

$$-\frac{1}{NL}S*\mathcal{W} + \mathcal{W} + \frac{1}{L}\sum_i \sigma_i\mathcal{W}_{1i}\mathcal{W}_{2i} = 0$$



L	# WL(4d)
10	268
12	5,324
14	142,105
16	4,483,136
18	152,322,746

$$-\mathcal{W}_0 - \mathcal{W}_2 - 4\mathcal{W}_3 + \mathcal{W}_{17} + \mathcal{W}_{20} + 4\mathcal{W}_{21} + 2\lambda\mathcal{W}_1 \quad = \quad 0$$

$$\mathcal{W}_2 + \mathcal{W}_6 + 4\mathcal{W}_{14} - \mathcal{W}_{16} - \mathcal{W}_{17} - 4\mathcal{W}_{18} = 0$$

In trying to solve the equations one faces a problem:

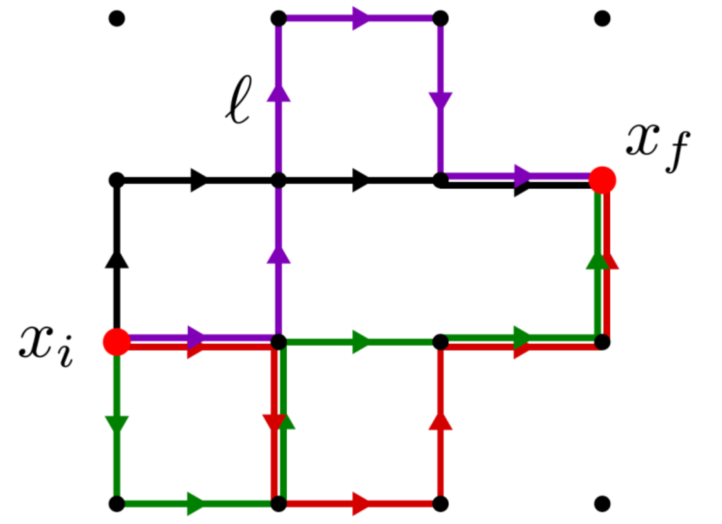
- To define the equations properly we have to cut the set of loops, e.g. length ≤ L, and then consider $L \to \infty$.
- The equations for length L have loops of length L+4. The number of loops increases exponentially with L.
- The limit $L \to \infty$ does not seem well defined, except at strong coupling where we set the unknown loops to 0.

We argue that including a certain set of positivity constraints gives a well defined limit $L \to \infty$ at any coupling. The reason is that the constraints put bounds on the energy density that improve as $L \to \infty$

They are more relevant at weak coupling.

Positivity constraints

$$A = \sum_{\ell=1}^{L} c_\ell U^{(\ell)}$$



$$\mathrm{Tr}\, A^\dagger A \geq 0 \quad \Rightarrow \quad \sum_{\ell\ell'} c_\ell^* c_{\ell'} \mathrm{Tr}\left[\left(U^{(\ell)}\right)^\dagger U^{(\ell')}\right] \geq 0 \quad \forall c_\ell$$

$$\hat{\rho}^{(L)}_{\ell\ell'} = \frac{1}{L}\left\langle \mathrm{Tr}\left[\left(U^{(\ell)}\right)^\dagger U^{(\ell')}\right]\right\rangle \succeq 0$$

$$\hat{\rho}^{(L)}_{\ell\ell} = \frac{1}{L} \Rightarrow Tr\left(\hat{\rho}^{(L)}\right) = 1$$

Closed loop:
goes along ℓ'
comes back along ℓ

ρ can be thought as a reduced density matrix obtained by tracing over color indices

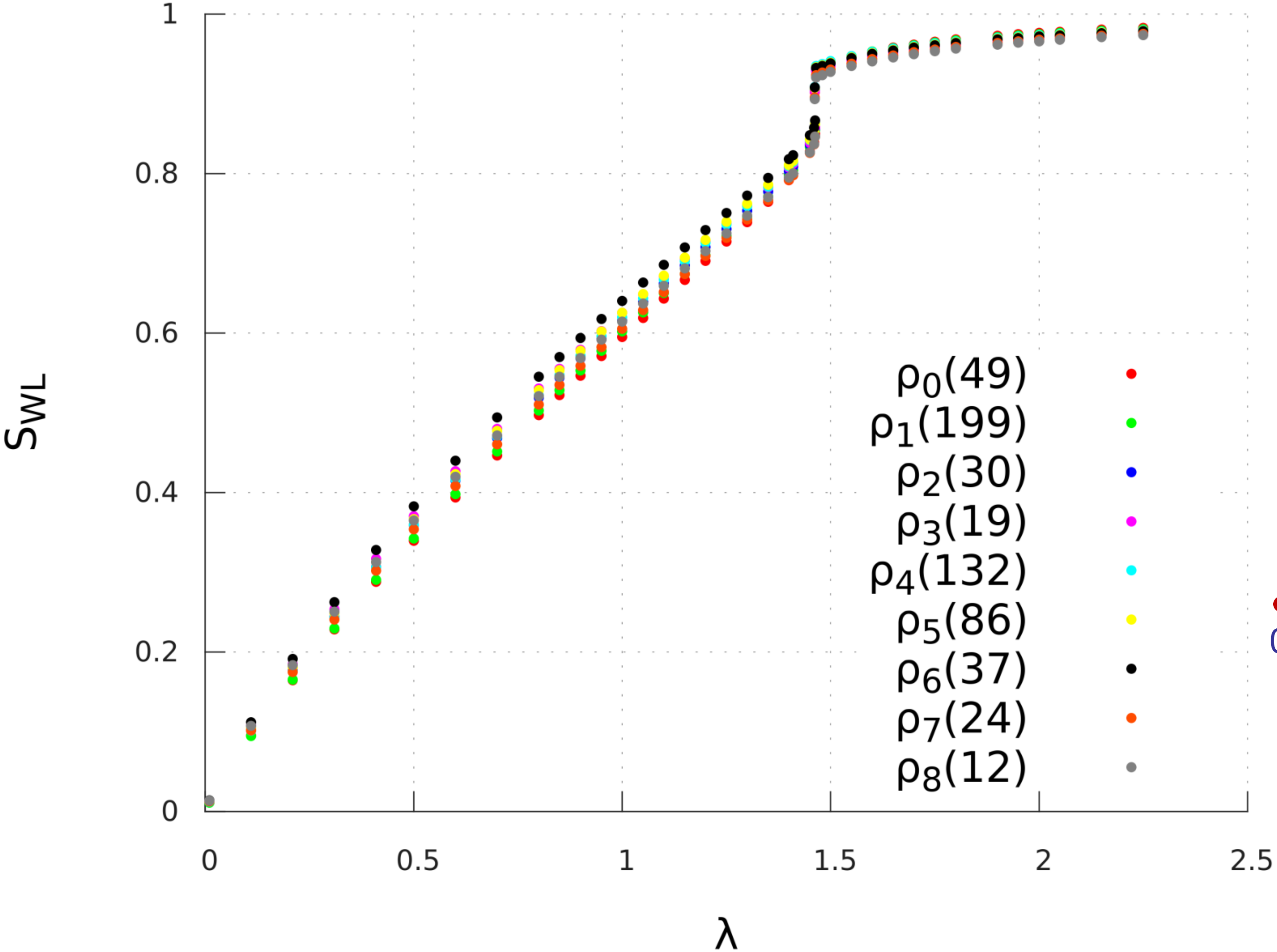
$$\hat{\rho}^{(L)}_{\ell\ell'} = \frac{1}{NL}\left\langle \mathrm{Tr}\left[\left(U^{(\ell)}_{ab}\right)^* U^{(\ell')}_{ab}\right]\right\rangle$$

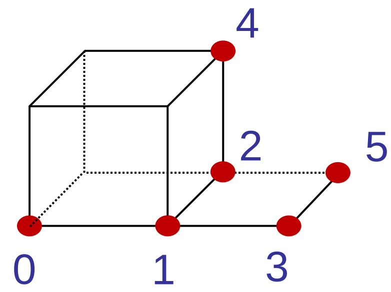
Its entropy computes the information loss due to tracing:

$$S_{WL} = -\mathrm{Tr}\,\hat{\rho}^{(L)}\log_L \hat{\rho}^{(L)}$$

When λ=0 all loops are 1, S=0, when λ→∞, all loops are zero, ρ=**I**, S is maximal. Behaves as system entropy.

Numerically S_{WL} is approx. independent of the choice of ρ





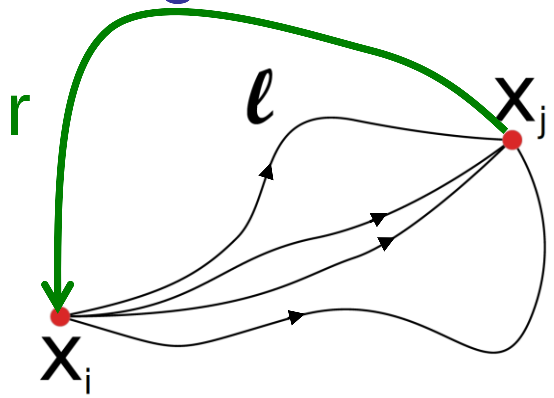
If ρ has a zero eigenvector c_0 (boundary of the domain):

$$c_{0\ell}^* \rho_{\ell\ell'} c_{0\ell'} = 0 \quad \Rightarrow \quad \langle \mathrm{Tr} A_0 A_0^\dagger \rangle = 0, \quad A_0 = \sum_\ell c_{0\ell} U^{(\ell)}$$

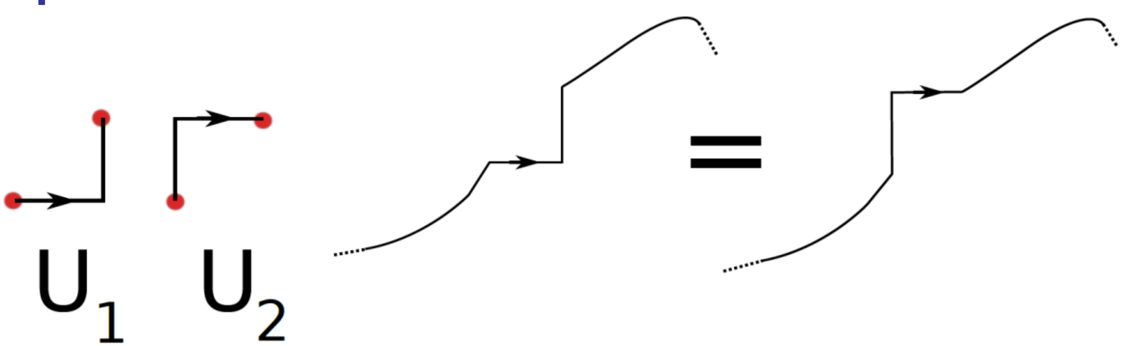
Thus $A_0 = 0$

Closing with an arbitrary path r we get linear equations

$$\sum_\ell c_{0\ell} \langle \mathrm{Tr}(U_r^\dagger U^{(\ell)}) \rangle = 0$$



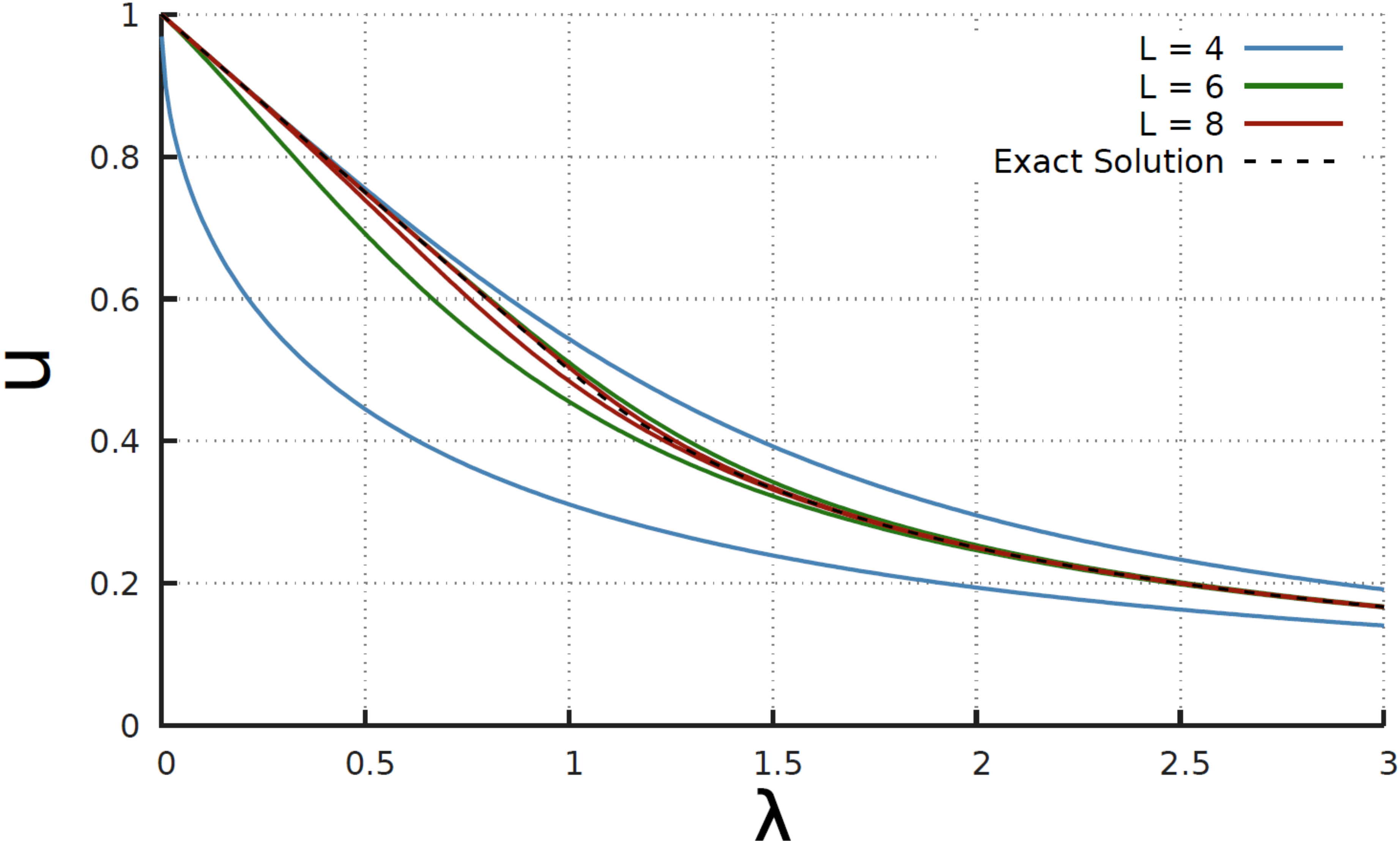
valid for arbitrary long loops. In particular, if u=1 then all loops are 1.



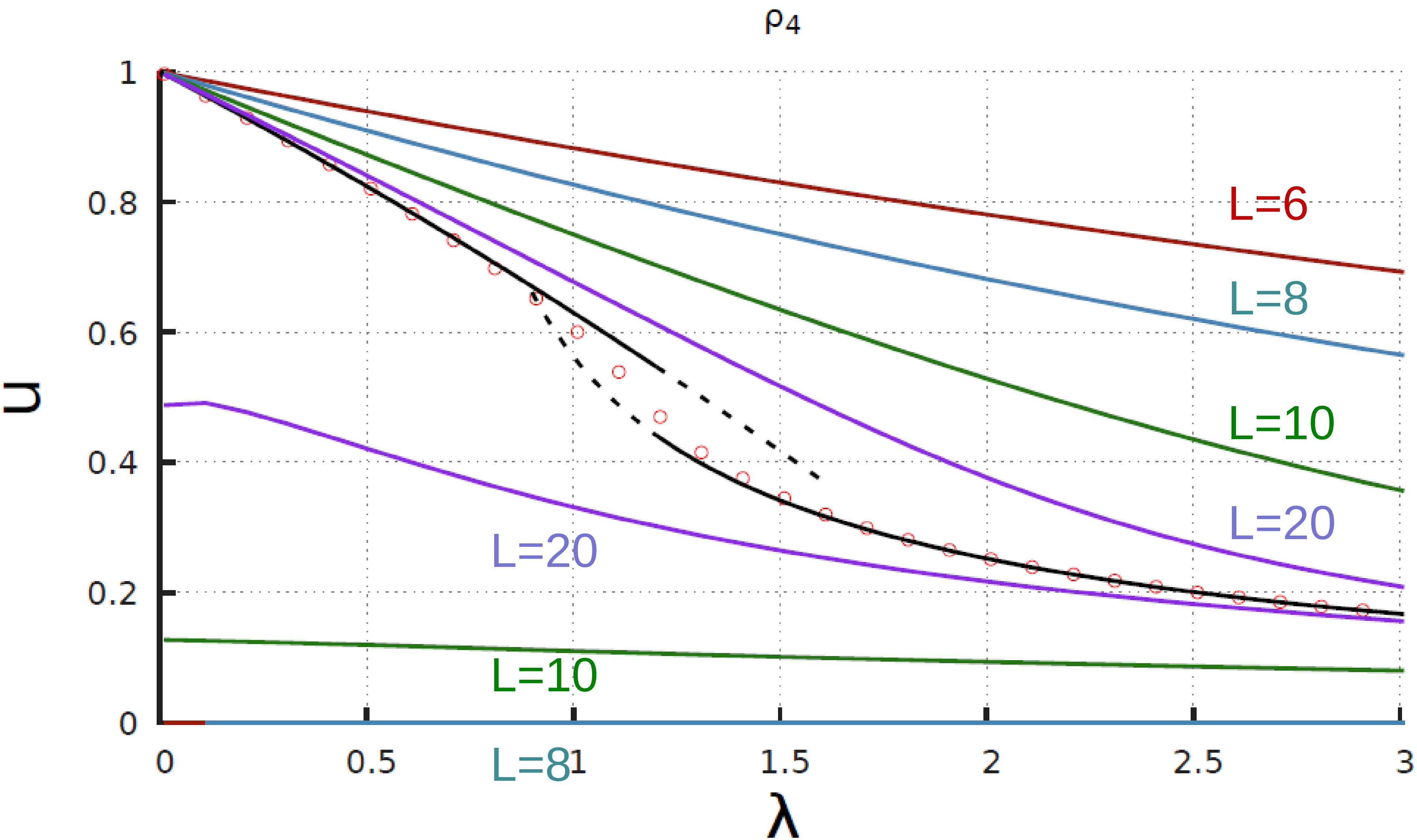
$$\rho = \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}$$

2d case $\lambda_c = 1$, third order(Gross-Witten '80, analytical)

$$\frac{E}{V} = -\frac{d(d-1)}{2}\frac{N^2}{\lambda}u\,, \qquad u = \frac{1}{N}\langle TrU_P\rangle$$

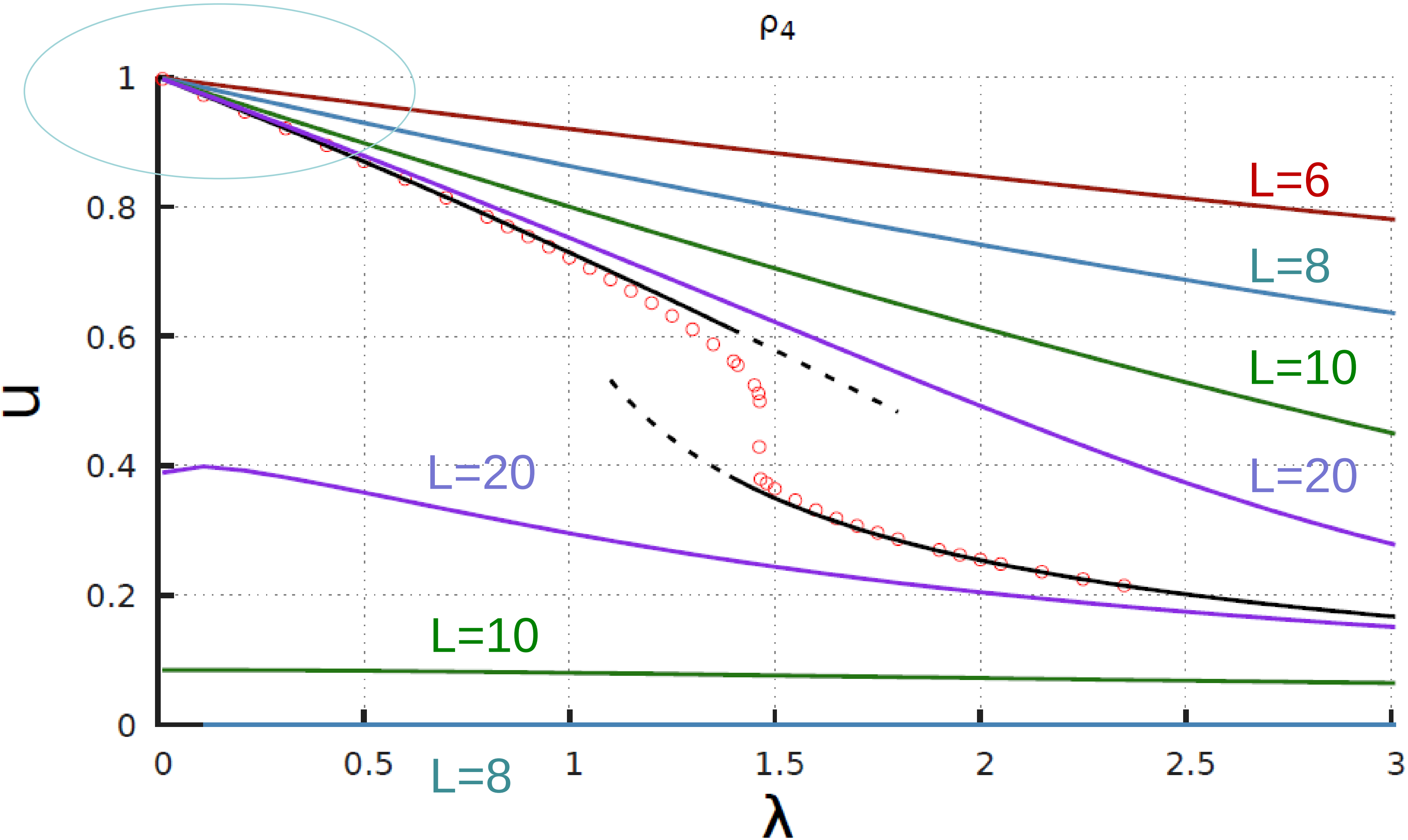


3D case $\lambda_c \sim 1.2$, third order (Teper '06)



L_{max}=20. Using matrix ρ_4 size 330x330 involving 5,299 variables.

4D case $\lambda_c = 1.3904$, first order (Campostrini '99)



L_{max}=20. Using matrix ρ_4 size 786x786 involving 11302 variables.

Gradient flow and the loop equation

Gradient flow (Luscher) introduces smeared operators that are easier to compute in the lattice (large loops)
Given a lattice configuration we flow the links using

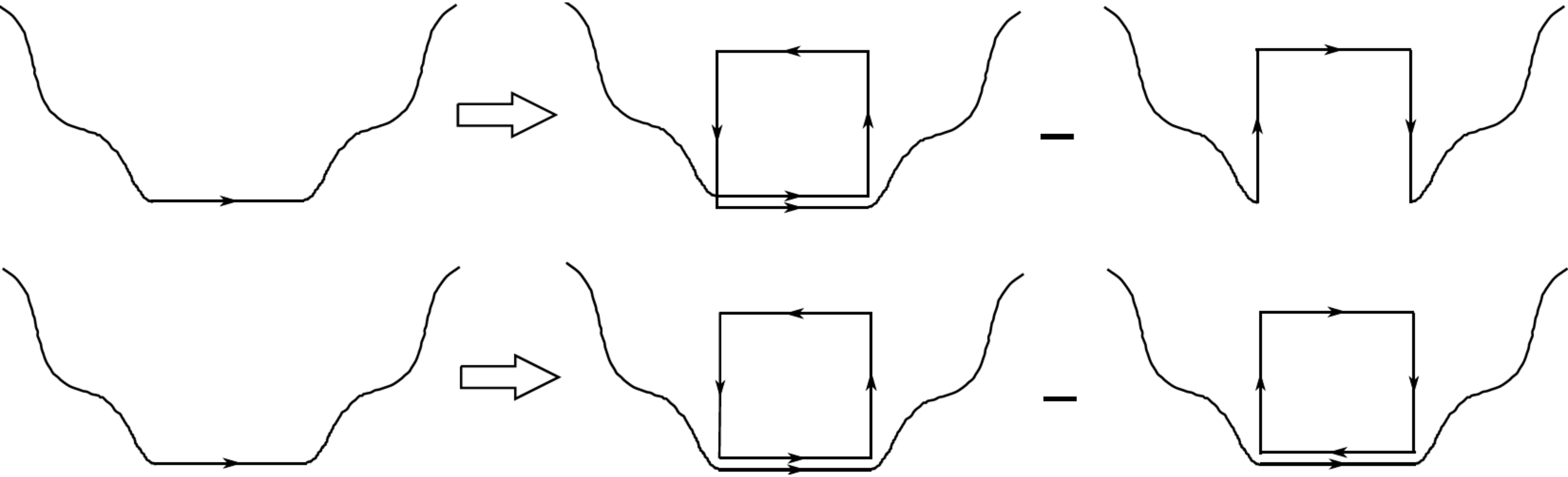
$$\partial_t U_{ac}(\vec{x},\mu) = -\frac{\lambda}{N}\,\partial_{\vec{x},\mu} S_W(U)_{ab} U_{bc}(\vec{x},\mu)$$

$$\partial_{\vec{x},\mu} S_W(U)_{ab} = -\frac{1}{2}\frac{\delta S_W}{\delta U_{bc}(\vec{x},\mu)} U_{ac}(\vec{x},\mu) + \frac{1}{2N}\delta_{ab}\,\frac{\delta S_W}{\delta U_{cd}(\vec{x},\mu)} U_{cd}(\vec{x},\mu)$$

For Wilson loops

$$\begin{aligned} \partial_t \mathcal{W}_i &= \frac{1}{N}\sum_j \langle U_1 \dots \frac{\partial U_j}{\partial t} \dots U_n \rangle \\ &= -\frac{1}{2}\mathbb{K}_{i\to j}\mathcal{W}_j + \frac{1}{2}\tilde{\mathbb{K}}_{i\to j}\mathcal{W}_j \end{aligned}$$

Graphically



Then, in the large-N limit

$$\partial_t \mathcal{W}_i = -\frac{1}{2}\mathbb{K}_{i\to j}\mathcal{W}_j \quad \Rightarrow \quad \mathcal{W}(t) = e^{-\frac{1}{2}t\mathbb{K}}\mathcal{W}(t=0)$$

The flowed Wilson loops obey a flowed loop equation

$$\mathbb{K}_{i\to j}\mathcal{W}_j(t) + 2\lambda\mathcal{W}_i(t) + 2\lambda C(t)_{i\to jk}\mathcal{W}_j(t)\mathcal{W}_k(t) = b_i(t)$$

where

$$\begin{aligned} b_i(t) &= \left(e^{-\frac{1}{2}t\mathbb{K}}\right)_{i1} \\ C(t)_{i\to jk} &= \left(e^{-\frac{1}{2}t\mathbb{K}}\right)_{ii'} C_{i'\to j'k'} \left(e^{\frac{1}{2}t\mathbb{K}}\right)_{j'j} \left(e^{\frac{1}{2}t\mathbb{K}}\right)_{k'k} \end{aligned}$$

What about the positivity constraints? Since the flowed Wilson loops are computed with unitary (flowed) links:

$$\rho_{ij}(t) = \rho_{ij,k}\mathcal{W}_k(t), \qquad \rho(t) \succeq 0, \quad \forall t$$

More constraints?

Conclusions

-) We constructed a matrix $\rho \succeq 0$ with WLs as entries and use it to correctly formulate the problem of solving the loop equations (especially at small coupling).

-) This numerically reproduces (in 2,3,4d) the $\lambda \to 0$ result

$$\mathcal{W}_1 = u = 1 - \frac{\lambda}{d} + \mathcal{O}(\lambda^2)$$

-) In the weak coupling phase ρ saturates the bounds, it has zero eigenvalues whose number increases as $\lambda \to 0$ (relevant for the continuum limit?).

-) We defined an off-shell Wilson loop entropy as the entropy associated with ρ (~ indep. of particular ρ).