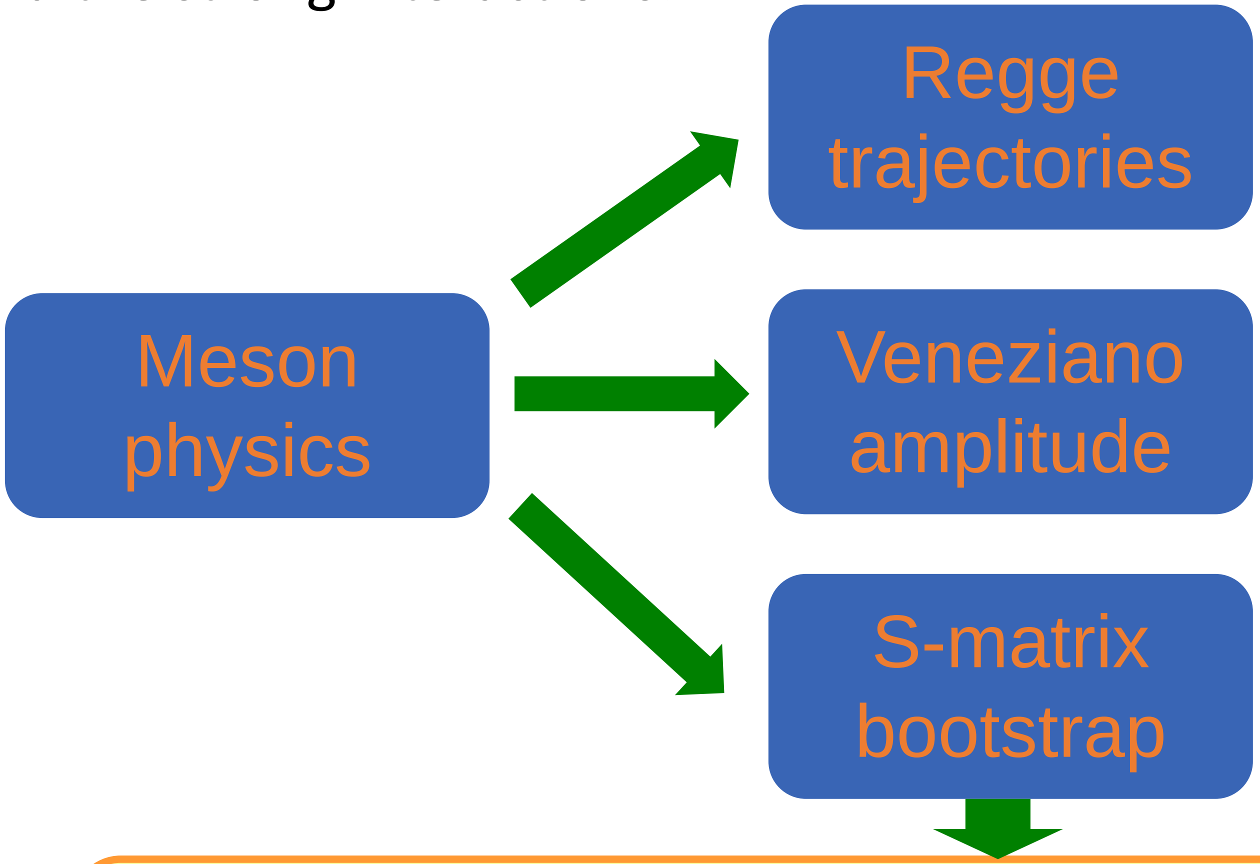
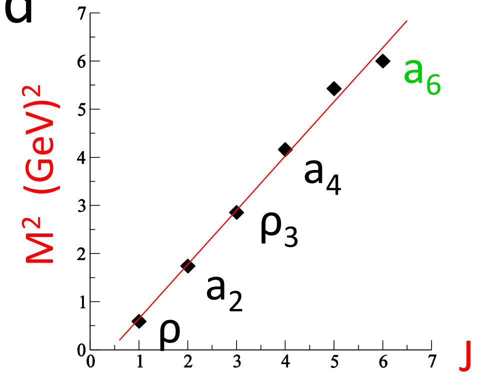
Lecture II
The S-matrix bootstrap in 2d

New S-matrix bootstrap, series of papers by
Paulos, Penedones, Toledo, van Rees, Vieira

Based on
- e-Print: 2012.15576, w/ Harish Murali (Purdue).
- e-Print: 1909.06495, w/ Lucia Cordova (Ecole Normale), Yifei He (Ecole Normale), Pedro Vieira (Perimeter).
- e-Print: 1805.02812, *JHEP* 11 (2018) 093, Yifei He (Ecole Normale), Andy Irrgang (Purdue).

Motivation: Before the quark model, many ideas were developed to understand the strong interactions:





$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

The S-matrix satisfies constraints from symmetries, analyticity, unitarity and crossing that can be used to completely determine it in a self-consistent way.

Recent idea in the S-matrix bootstrap

In the allowed space of S-matrices one can consider a functional and define a theory by the S-matrix that maximizes such functional.

A standard example is the coupling between a particle and its bound states (whose spectrum is assumed fixed). There is a maximum coupling because increasing the coupling further adds more bound states.
Paulos, Penedones, Toledo, van Rees, Vieira

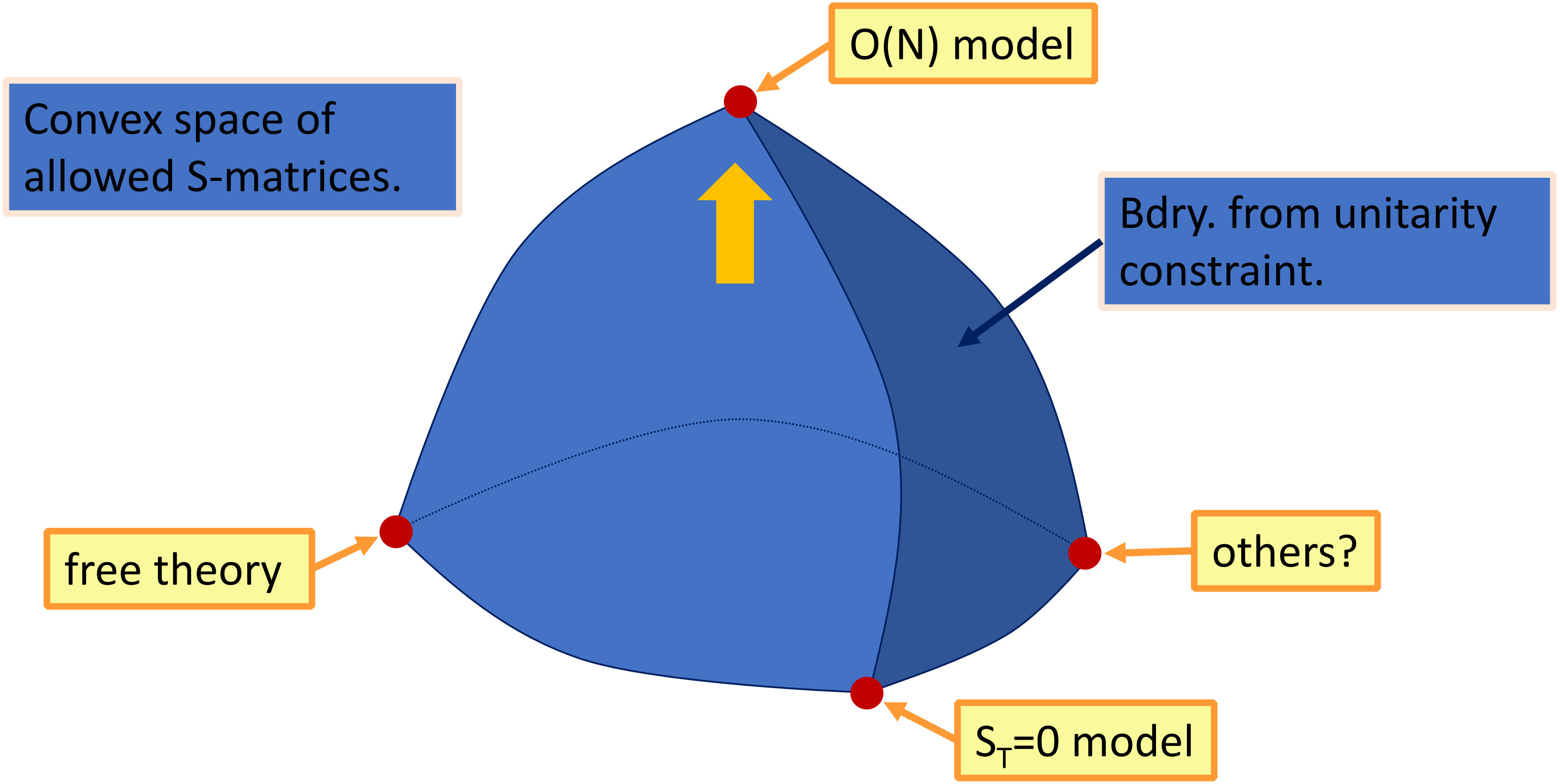
2d O(N) model has no bound states. We studied this model and found that many functionals lead to the same model. We argued that the space of allowed S-matrices is convex and has a vertex where the O(N) model sits. We expect this to be a somewhat generic picture.

Aside: Relation with conformal bootstrap.

Initially the idea was to put a massive field theory in AdS to define a conformal theory at the boundary. The conformal bootstrap at the boundary should lead to constraints in the bulk. Taking the radius of AdS to infinity, one should get constraints for massive theory in flat space. The result was that it was better to study the massive field theory in flat space directly.

What did we learn from the O(N)-model?.

The S-matrix in a subspace D (e.g. two particle states) satisfies unitarity, crossing, and symmetry properties. The allowed space is convex with interesting theories at its vertices.



Exact result using YBE (Zamolodchikov-Zamolodchikov)

$$Q(\theta) = \frac{\Gamma\left(\frac{\lambda - i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda - i\theta}{2\pi}\right)\Gamma\left(-\frac{i\theta}{2\pi}\right)}$$

$$s = 4m^2 \cosh^2 \frac{\theta}{2}$$

$$S_T = Q(\theta) Q(i\pi - \theta)$$

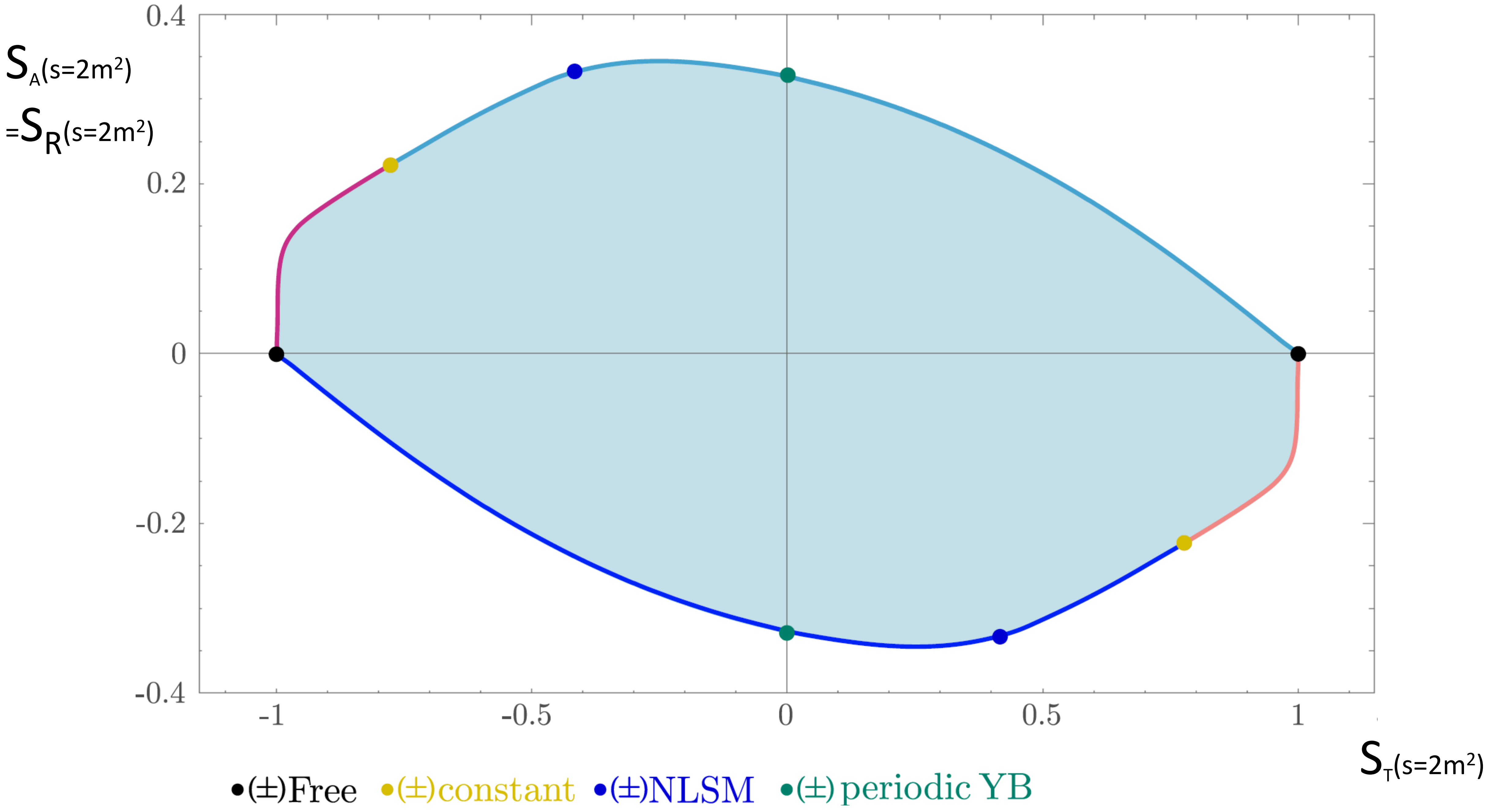
$$S_A = -\frac{i\lambda}{i\pi - \theta} S_T(s)$$

$$S_R = -\frac{i\lambda}{\theta} S_T(s)$$

$$\lambda = \frac{2\pi}{N-2}$$

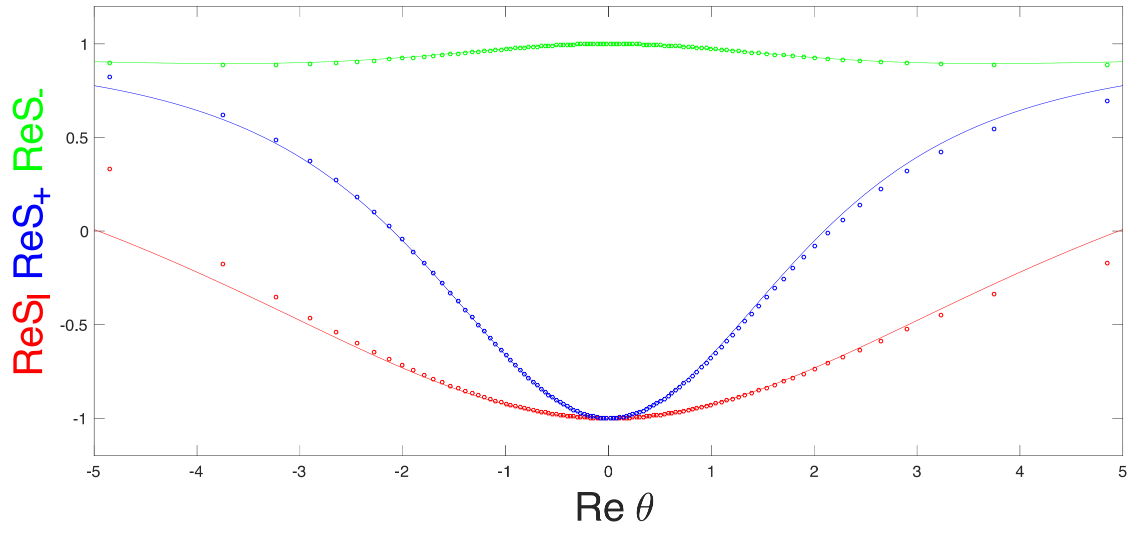
$$S_I = N S_A + S_T + S_R, \qquad S_\pm = S_T \pm S_R$$

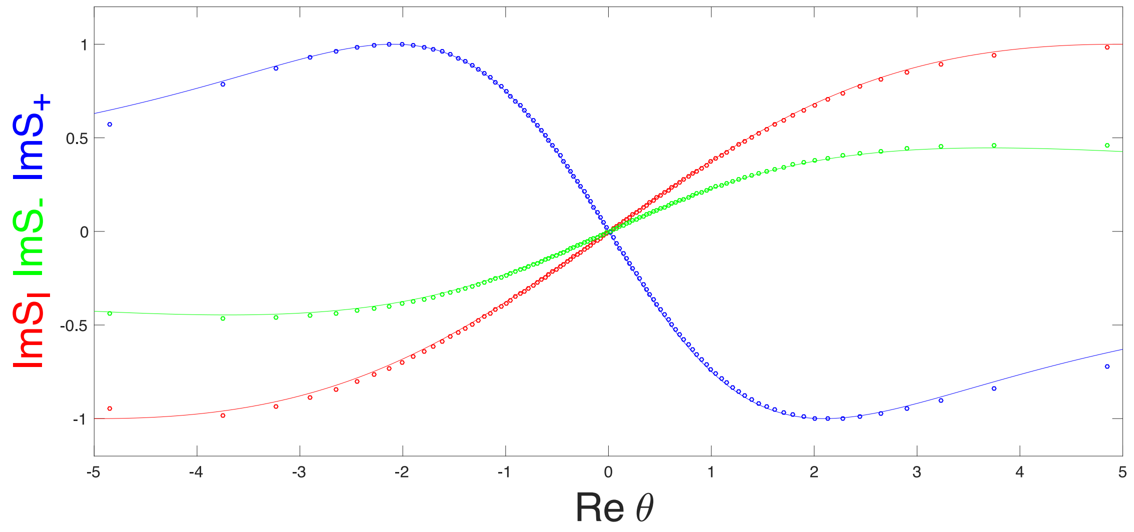
Allowed space of values for S



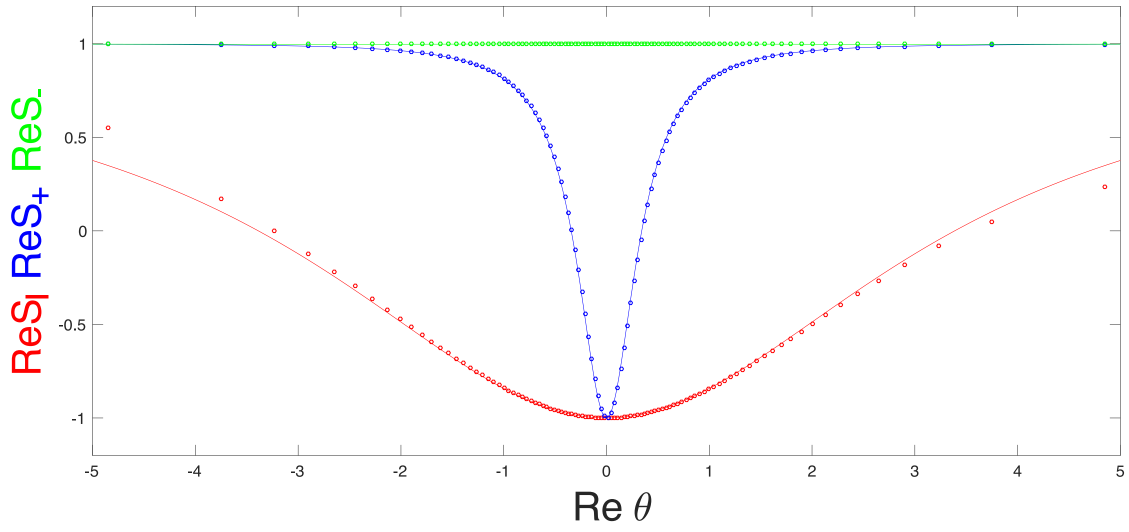
Maximization result vs integrable model on the physical line

N=4

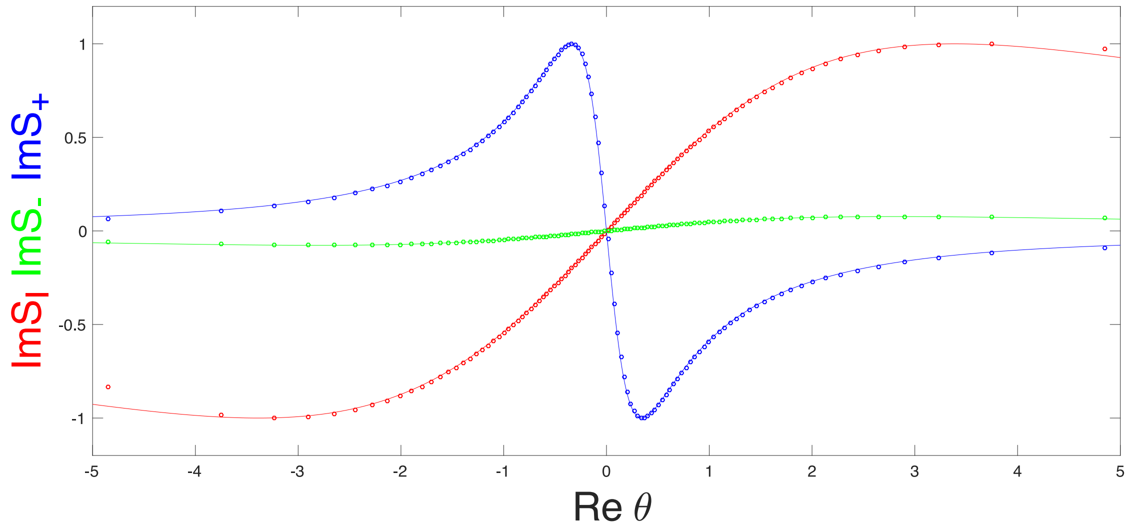




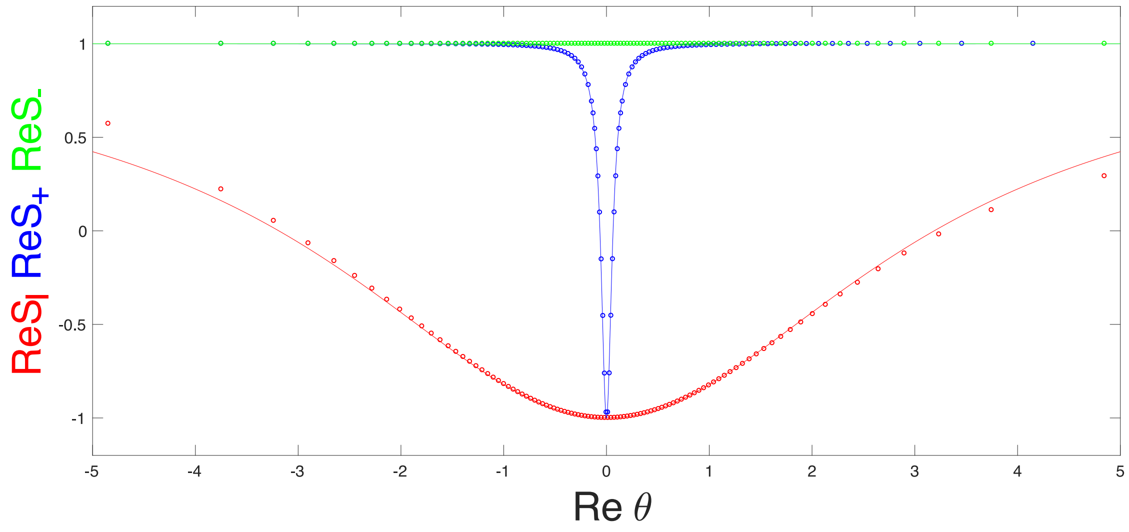
Maximization result vs integrable model on the physical line



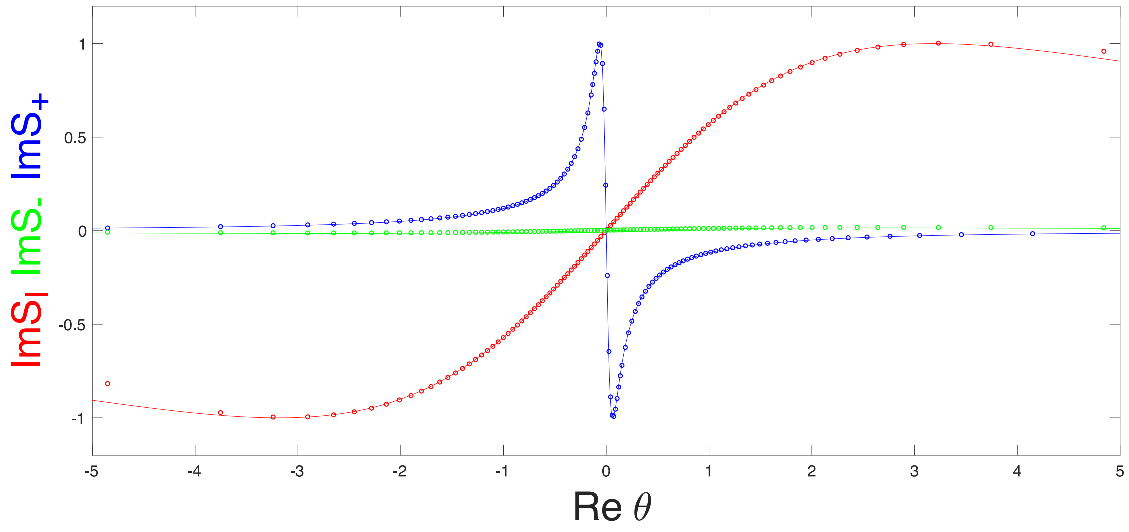
N=20



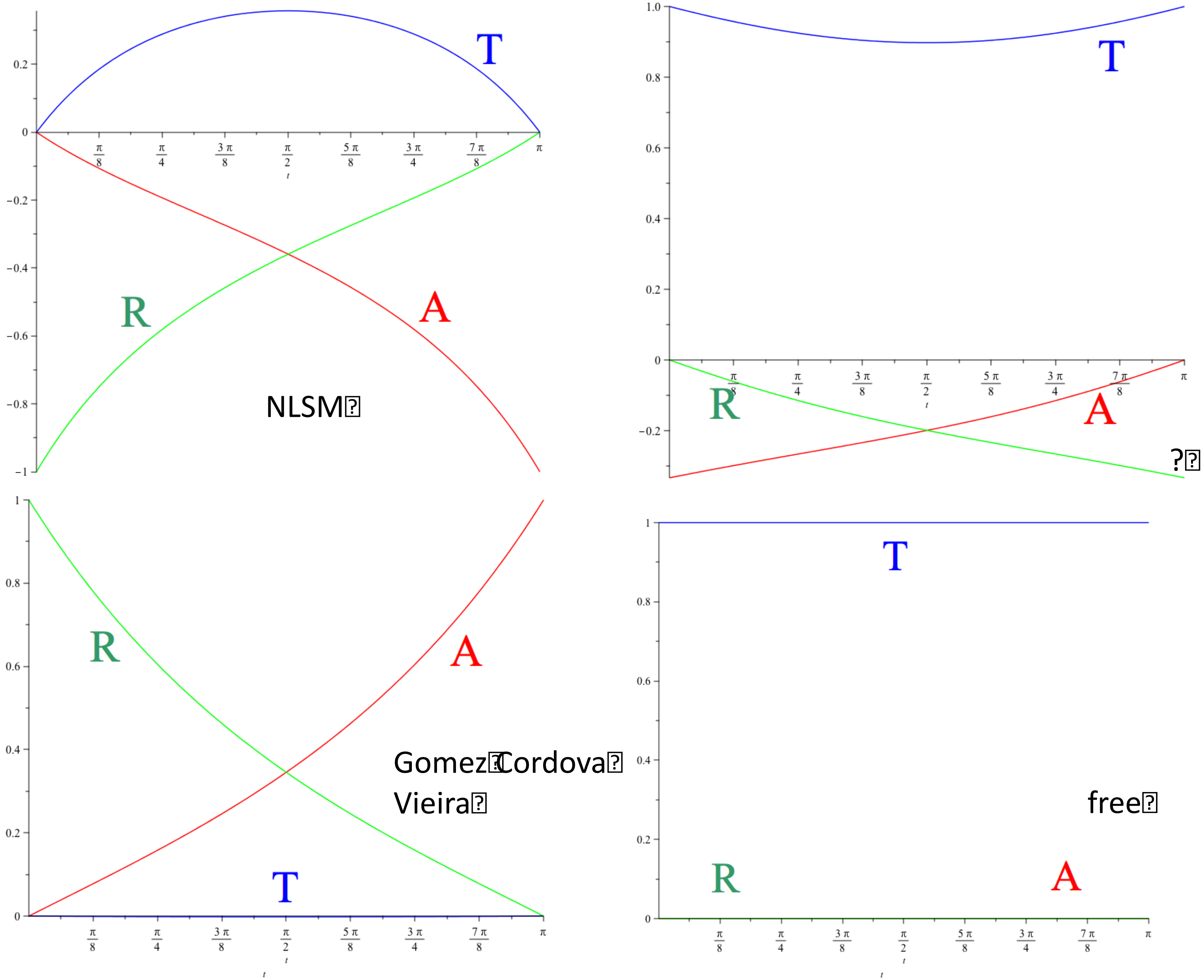
Maximization result vs integrable model on the physical line

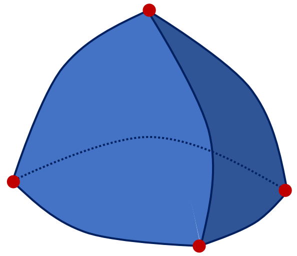


N=100



Mapping out the space





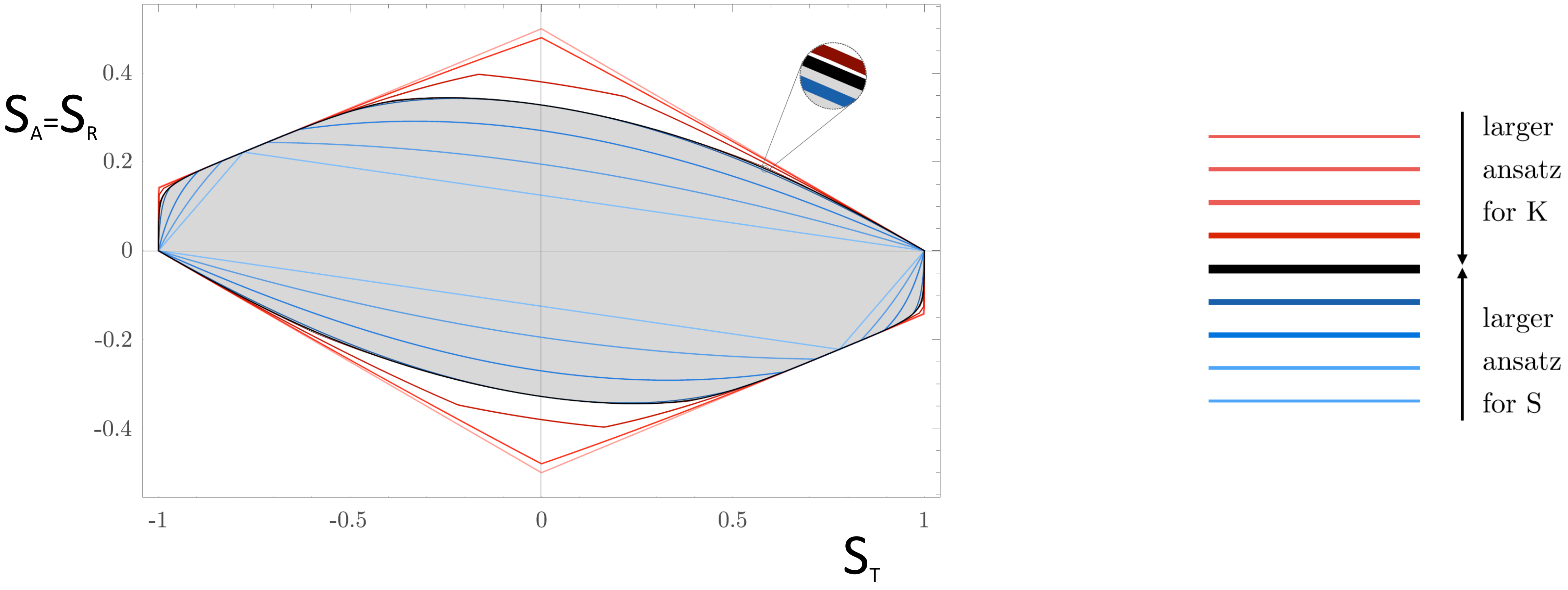
Addendum: Matlab program for the $O(6)$ model using the cvx package. It plots the real part of $S_{I,+,-}$ on the real axis of the θ plane.

```
% Matlab program, requires cvx package
% O(6) model with M=200 interpolating points. Iterative improvement.
N = 6;  M = 200;
% constructing the K matrix
vc = 1/M*cot(pi/2/M*[1:2*M-1]);
vc(2:2:end) = 0;
Km = toeplitz([0,vc],zeros(2*M,1));
Km = Km - Km.';
% Crossing symmetry matrix
C = [1/N N/2+1/2-1/N (1-N)/2; 1/N 1/2-1/N 1/2; -1/N 1/2+1/N 1/2];
K = kron(eye(3),Km(1:M,1:M))+kron(C,Km(1:M,(M+1):end));
for count = 1:10              % 10 iterations
 cvx_begin quiet
  variable ReS(3*M)
  ImS = K*ReS;
  ReS.*ReS + ImS.*ImS <= 1;
  % evaluating the functions at z0
  z0 = 0.3*i;
  zj = exp(1.i*pi/M*[1:M]);
  w1 = 1/2/M*(1-(z0./zj).^M).*(zj+z0)./(zj-z0);
  w2 = 1/2/M*(1-(z0./zj).^M).*(zj-z0)./(zj+z0);
  W0 =  kron(eye(3),real(w1)) + kron(C,real(w2));
  v1 = 1/2*(W0(2,:)+W0(3,:));
  v2 = (1/2/N*W0(1,:)+(1/4-1/2/N)*W0(2,:)-1/4*W0(3,:));
  if count==1    t = (v1-7.5*v2)*ReS;
  else               t = w0*ReS;
  end
  maximize(t);
 cvx_end
 V = ReS.*eye(3*M)+ImS.*K;  % infinitesimal variations around maximum
 w0 = ones(1,3*M)*V;        % new maximization functional
end
sigma = -atanh(cos(pi/M*[1:M-1]));
plot(sigma,ReS(1:M-1),'o',sigma,ReS(M+1:2*M-1),'+', ...
       sigma,ReS(2*M+1:3*M-1),'d')
```

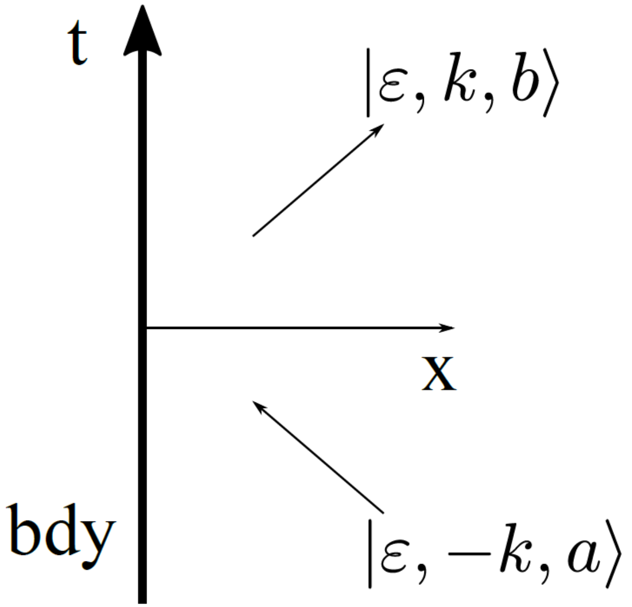
Program to obtain O(N) model S-matrices

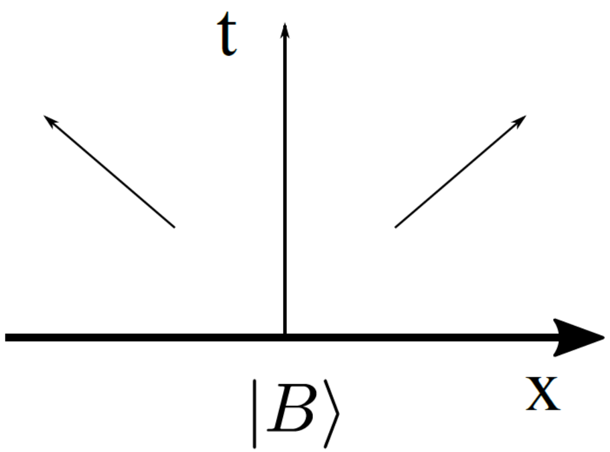
Dual problem

- Provides upper bounds that approach the maximum from above (as we improve the numerics)
- Requires less computational resources.
- In 4d, it focuses on a few partial waves, similar to experiment.



The 2d O(N) on a half line, reflection matrix





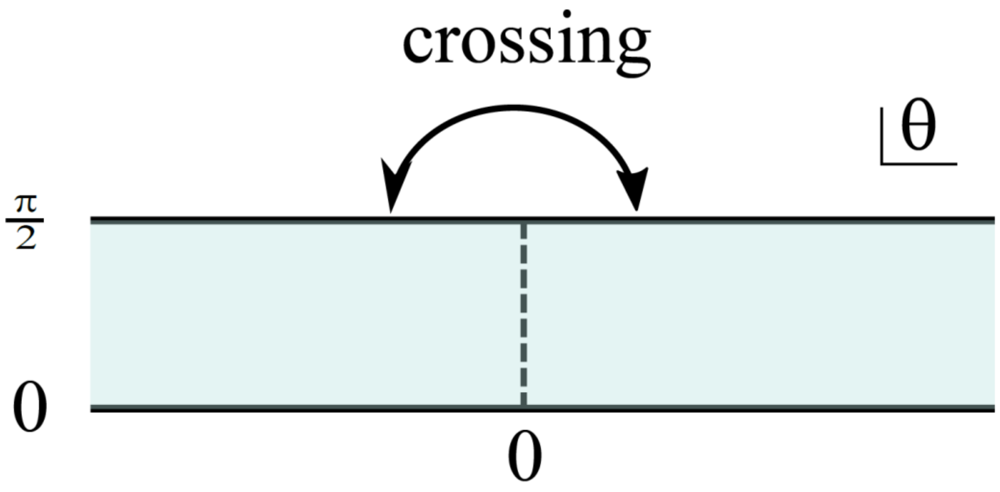
$$\varepsilon = m \cosh \theta$$

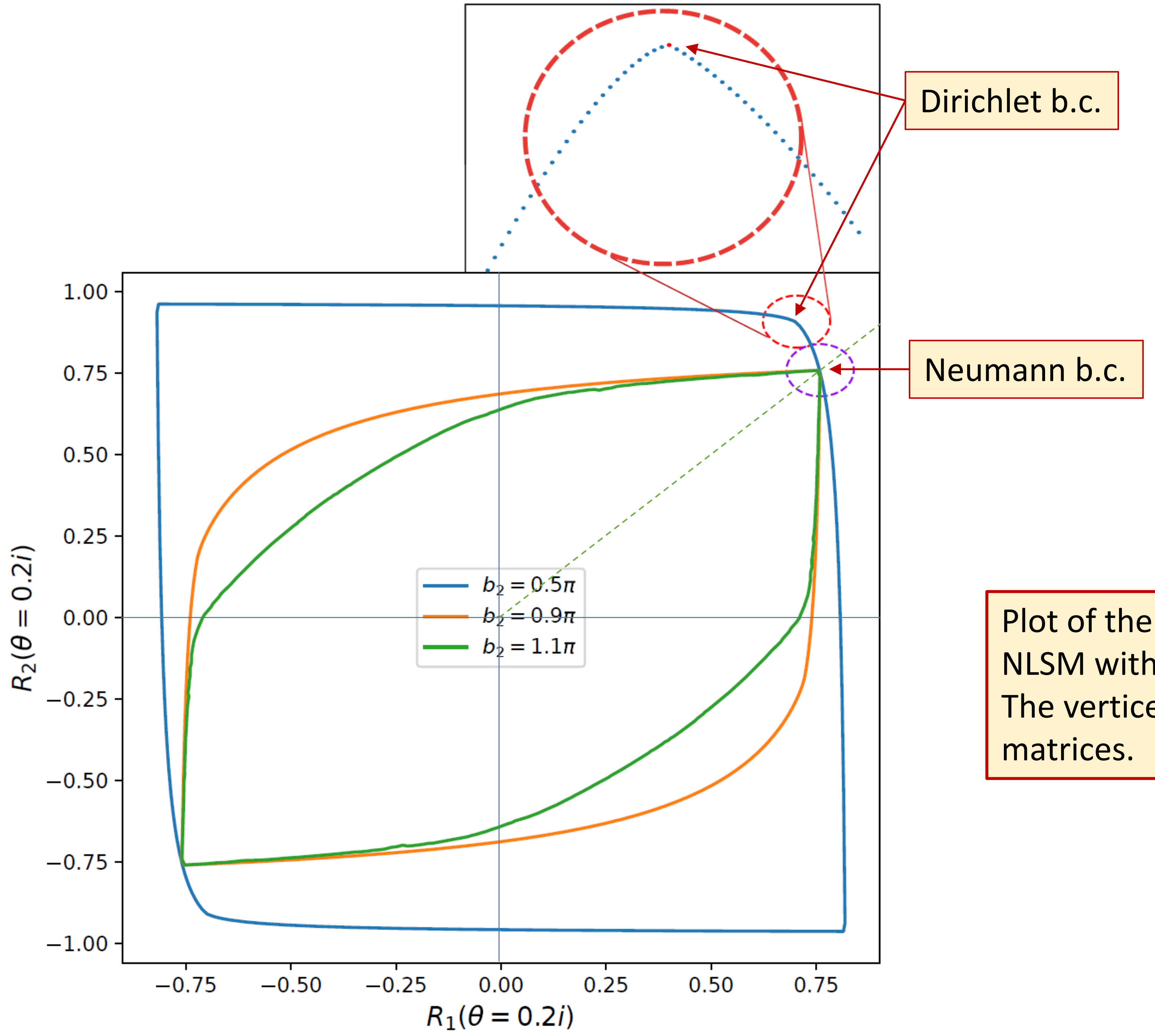
$$R_a^b(\theta) = {}_{\text{out}}\langle \varepsilon, k, b | \varepsilon, -k, a \rangle_{\text{in}}$$

$$|R_a^b(\theta)| \leq 1, \quad \theta \in \mathbb{R}$$

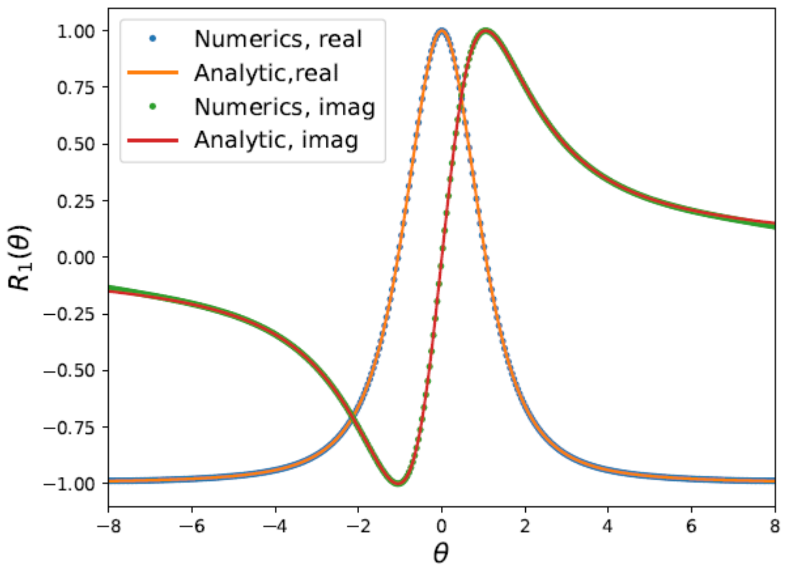
$$R_a^b\left(\frac{i\pi}{2} - \theta\right) = S_{cd}^{ab}(2\theta)\, R_d^c\left(\frac{i\pi}{2} + \theta\right)$$

$$R = \text{diag}\{\underbrace{R_1(\theta), \ldots R_1(\theta)}_{k}, \underbrace{R_2(\theta) \ldots R_2(\theta)}_{N-k}\} \;,$$

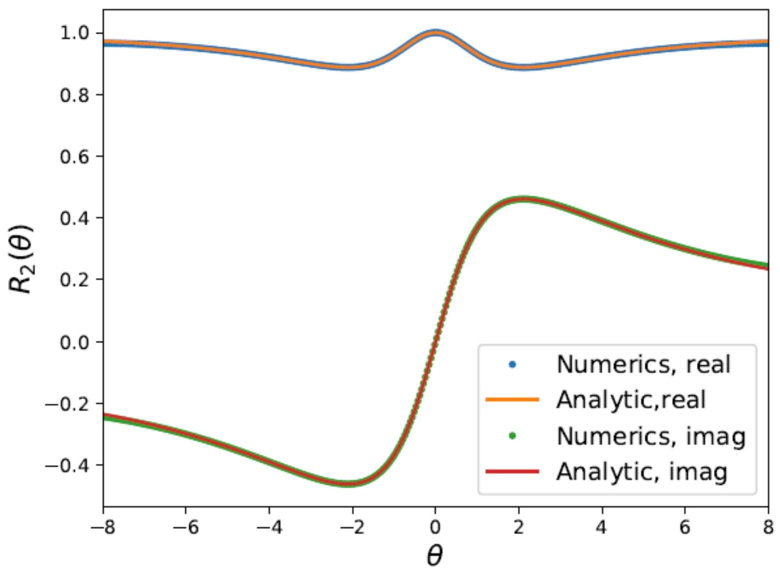




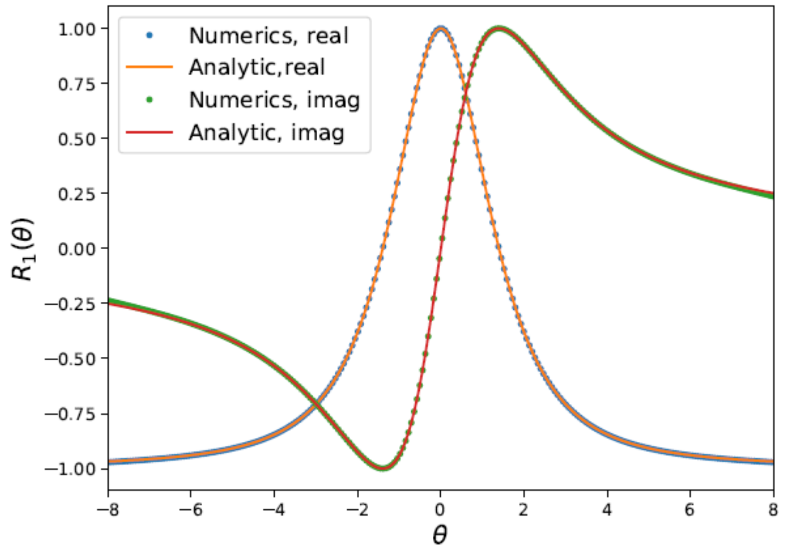
Plot of the allowed region (R_1,R_2) for the NLSM with N=6, k=1.
The vertices correspond to integrable R-matrices.



(a) $R_1(\theta)$, Dirichlet



(b) $R_2(\theta)$, Dirichlet



(c) $R_1(\theta)$, Neumann

Plot of the functions R_1,R_2 on the real axis (physical region) for the two vertices of the previous figure.