Two applications of the bootstrap in QCD

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"Rigidity" of a theory or mathematical structure

$\rightarrow$ Determined by few parameters $\rightarrow$ Can sometimes be determined from general constraints

·) crystal

·) analytic function

·) integrable theory

Interesting constraint : Positivity. example

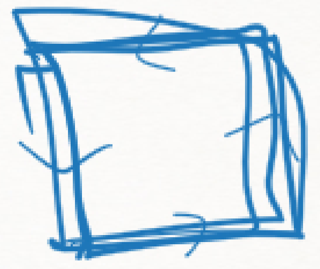
$$A_{ij} = \langle x_i \, x_j \rangle \qquad \xi_i A_{ij} \xi_j =$$

↖ random variables

$$\boxed{A \succeq 0} \qquad = \langle (\xi_i x_i)(\underbrace{\xi_j x_j}) \rangle \geq 0$$

Single plaquette model

$$Z = \int \underset{SU(N)}{dU} \; e^{-\frac{N}{2\lambda} \mathrm{Tr}(U + U^\dagger)}$$



$$\underbrace{W_n} = \frac{1}{N}\frac{1}{Z}\int dU \, \mathrm{Tr}(U^n)\, e^{-S}$$

$$W_1 = u = \begin{cases} 1 - \lambda/2 & \lambda < 1 \\ 1/(2\lambda) & \lambda \geq 1 \end{cases}$$

3rd order

Gross-Witten

Wadia '80

$$A \qquad \mathrm{Tr}(A^\dagger A) = \sum_{ij} |a_{ij}|^2 \geq 0$$

$$A = \sum_{n=0}^{L} c_n U^n \Rightarrow \mathrm{Tr} \sum_{n,m=0}^{L} c_m^* c_n (U^\dagger)^m U^n \geq 0$$

$$\uparrow$$

$$\frac{1}{N} \sum_{n,m=0}^{L} c_m^* c_n \langle \mathrm{Tr}\, U^{-m+n} \rangle \geq 0$$

$$W_{|n-m|}$$

$$\rho = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 & \cdots \\ w_1 & w_0 & w_1 & w_2 & \\ w_2 & w_1 & \ddots & w_1 & \\ w_3 & w_2 & w_n & \ddots & \end{pmatrix} \succeq 0$$

$$w_0 = \frac{1}{N} \operatorname{Tr} \mathbb{1} = 1$$

$$\begin{vmatrix} 1 & w_n \\ w_n & 1 \end{vmatrix} \qquad |w_n| \leq 1$$

<u>loop equation</u>

$$f = \int dU \, (U^n)_{ab} \, e^{-\frac{N}{2\lambda} \mathrm{Tr}(U+U^\dagger)}$$

$$U \longrightarrow (1+i\varepsilon) U \qquad \varepsilon \Rightarrow \text{hermitian}$$
$$\text{traceless}$$
$$U^\dagger \longrightarrow U^\dagger (1-i\varepsilon)$$

$$\delta_\varepsilon f = 0$$

$$(1+i\varepsilon)U^n + U(1+i\varepsilon)U^{n-1} + \cdots + U^{n-1}(1+i\varepsilon)U$$

$$i(\varepsilon U^n + U\varepsilon U^{n-1} + \cdots + U^{n-1}\varepsilon U) \longleftarrow$$

$$\delta_\varepsilon f = \underbrace{i\int dU\, e^{-S}}\Big[\big(\varepsilon U^n + U\varepsilon U^{n-1} + \dots + U^{n-1}\varepsilon U\big)_{ab}$$

$$+\, U^n_{ab}\Big(-\frac{N}{2\lambda}\,\mathrm{Tr}\big(\varepsilon U - \varepsilon U^{+}\big)\Big)\Big] = 0$$

$$\varepsilon_{cd}\, B^{cd} = 0 \qquad B^{cd} = \frac{1}{N}\,\mathrm{Tr}B\,\delta^{cd}$$

$$\Big\langle \cancel{\varepsilon_{cd}}\Big(\delta_{ac}\,U^n_{db} + U_{ac}\,U^{n-1}_{cd} + \dots + U^{n-1}_{ac}\,U_{db}$$

$$-\frac{N}{2\lambda}\,U^n_{ab}\big(U_{dc} - U^{+}_{dc}\big)\Big)\Big\rangle = 0 \quad (N\to\infty)$$

$$\underbrace{N\langle \mathrm{Tr}\, U^n\rangle} + \langle \mathrm{Tr} U\rangle\langle \mathrm{Tr}\, U^{n-1}\rangle + \dots \langle \mathrm{Tr}\, U^{n-1}\rangle\langle \mathrm{Tr}\, U\rangle$$

$$-\frac{\cancel{N}}{2\lambda}\left(\mathrm{Tr}\, U^{n+1} - \mathrm{Tr}\, U^{n-1}\right) = 0$$

$$w_n + w_1 w_{n-1} + \dots + w_{n-1} w_1 - \frac{1}{2\lambda}\left(w_{n+1} - w_{n-1}\right) = 0$$

$$\boxed{w_{n+1} - w_{n-1} = 2\lambda w_1 + 2\lambda \sum_{p=0}^{n} w_p w_{n-p}}$$

$$\begin{pmatrix} w_0 & w_1 & w_2 & \cdots \\ w_1 & \ddots & & w_2 \\ & \ddots & \ddots & w_1 \\ & & w_n & w_0 \end{pmatrix} \geq 0$$

i) Henry Lin

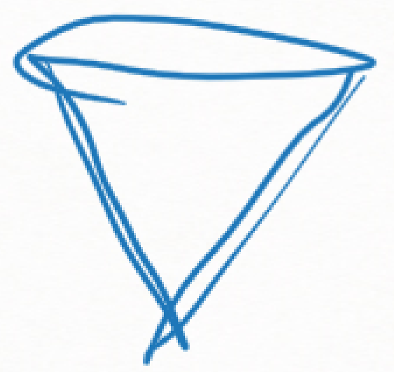
ii) P. Anderson & M.K.

Positivity

$$A_{ij} = \langle x_i x_j \rangle \qquad \xi_i \langle x_i x_j \rangle \xi_j = \langle (\xi_i x_i)^2 \rangle \geq 0$$

$$\underline{A \succeq 0} \qquad \boxed{\omega_a} \qquad A = A^{(0)} + A^{(1)}_a \omega_a$$

$$F = \sum_a \alpha_a \omega_a$$

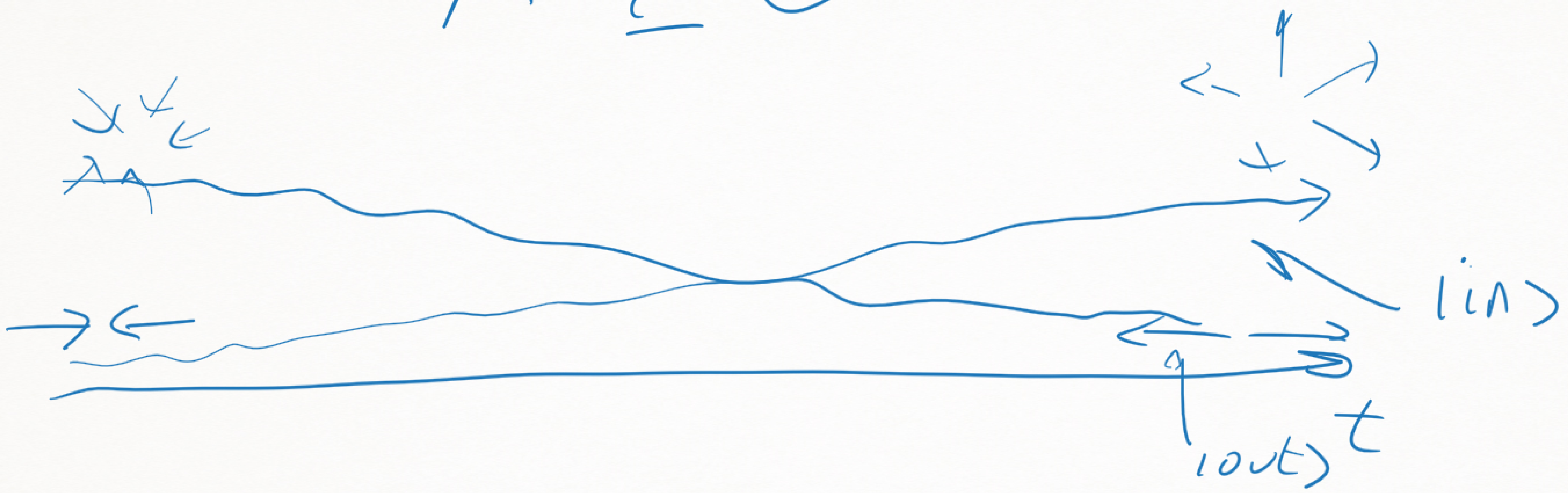


- MOSEK ←
- Gurobi
- sdpa → $\boxed{\text{sdpb}}$

$\{|\psi_i\rangle\}$ $\qquad A_{ij} = \langle \psi_i | \psi_j \rangle$

$$\xi_i^* A_{ij} \xi_j = \| \xi_i | \psi_i \rangle \|^2 \geq 0$$

$$A \geq 0$$



$\langle out | in \rangle =$

S

$$
\begin{array}{c|cc}
 & |in\rangle & |out\rangle \\
\hline
\langle in| & \mathbb{1} & S^\dagger \\
\langle out| & S & \mathbb{1}
\end{array}
\quad \succeq 0
$$

$$
\begin{pmatrix} 1 & S \\ S^\dagger & 1 \end{pmatrix} \succeq 0
$$

$$
\underline{|S|^2 \le 1}
$$

$$
\begin{array}{c|ccc}
 & |in\rangle & |out\rangle & \mathcal{O}|0\rangle \\
\hline
\langle in| & \mathbb{1} & S^\dagger & \langle in|\mathcal{O}|0\rangle \\
\langle out| & S & \mathbb{1} & \langle out|\mathcal{O}|0\rangle \\
\langle 0|\mathcal{O}^\dagger & & & \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle
\end{array}
\quad \succeq 0
$$

form-factor

$$S = \frac{1}{2g_0^2}\int d^2x\, (\partial_\mu \hat{n})^2 \qquad \hat{n}^2 = 1$$

$\mu = 0, 1$

N

$O(N)$ $\quad N$ massive scalar $\quad m = \mu\, e^{-\frac{2\pi}{g^2 N}}$

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a, p_1 $\longrightarrow$ $\quad \longleftarrow$ b, p_2

$(\varepsilon_1, \vec{p}_1) \quad (\varepsilon_1, -\vec{p}_1)$

$d \longleftarrow \quad \longrightarrow c$

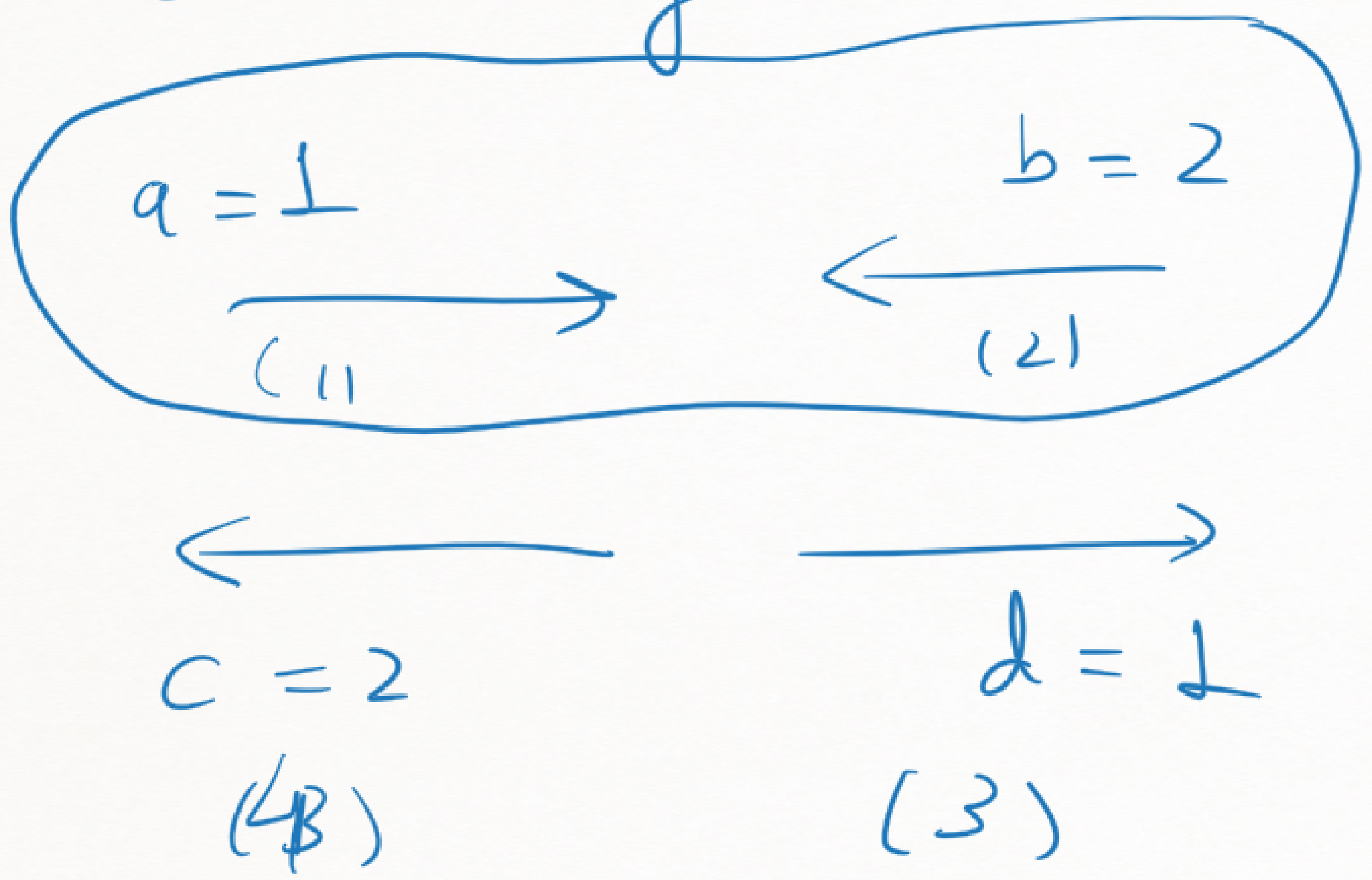
$(\varepsilon_1, -\vec{p}_1) \quad (\varepsilon_1, \vec{p}_1)$

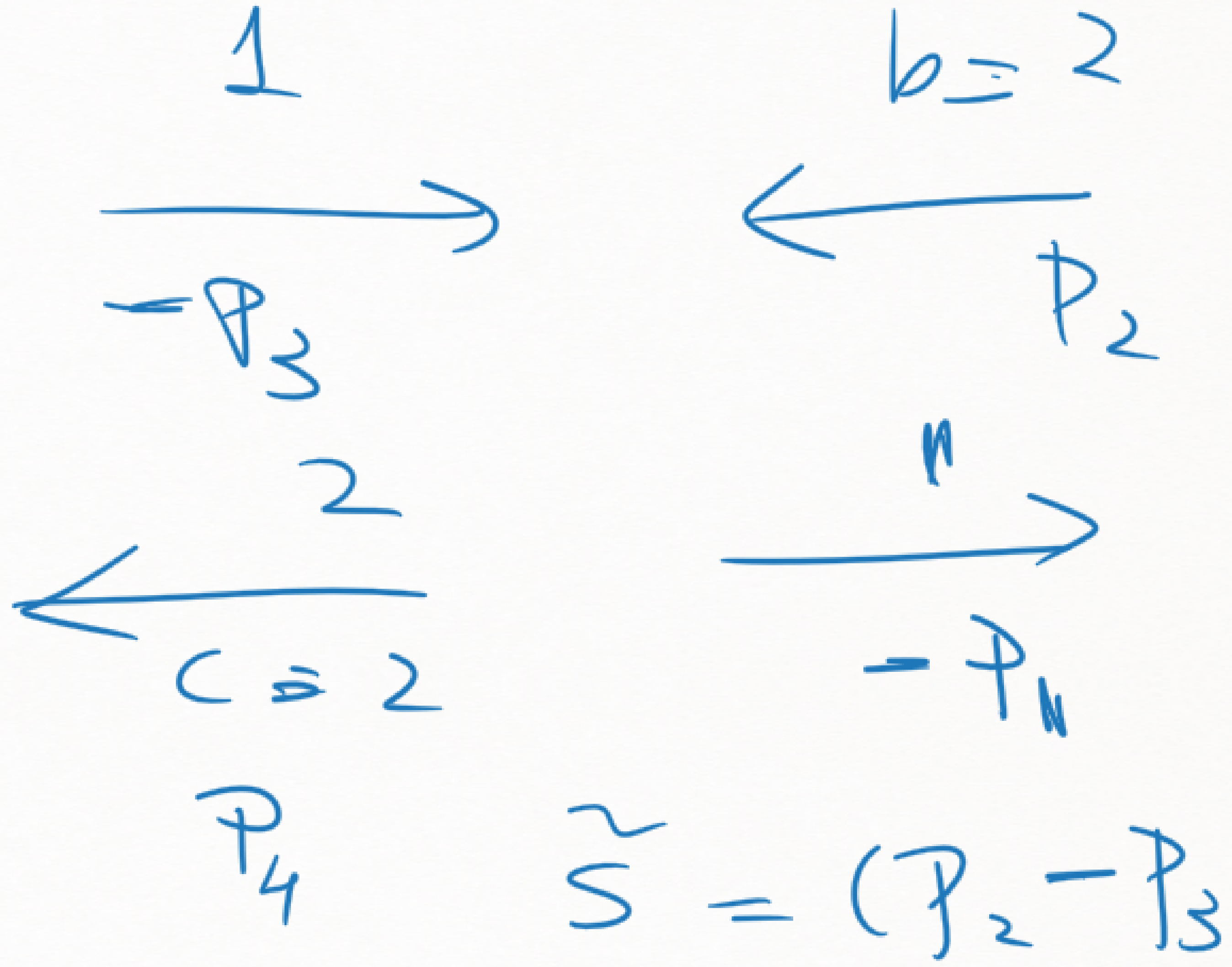
$p_4 \quad p_3$

$$S_{ab\,cd} = S_T\, \delta_{ac}\delta_{bd} + S_R\, \delta_{bc}\delta_{ad} + S_A\, \delta_{ab}\delta_{cd} + (c \leftrightarrow d)(p_3 \leftrightarrow p_4)$$

$$s = (p_1 + p_2)^2 = 4\varepsilon_1^2, \quad u = (p_1 - p_3)^2 = 0, \quad t = (p_2 - p_3)^2 = -4p_1^2 = 4m^2 - s$$

Crossing





$$\tilde{s} = (p_2 - p_3)^2 = 4m^2 - s$$

$$S_T(s) = S_T(\tilde{s} = 4m^2 - s)$$

$$S_A(s) = S_R(s)$$

$ab \to +, -, I$

$$|I=0\rangle = \frac{1}{\sqrt{N}}\left(|11\rangle + |22\rangle + \cdots - |NN\rangle\right)$$

$$\langle I=0|S|I=0\rangle = S_T + S_R + NSA = S_I$$

$$\underline{|S_I|^2 \le 1} \qquad \underline{|S_+|^2 \le 1}$$

$$S_{\pm} = S_T \pm S_R \qquad \underline{|S_-|^2 \le 1}$$

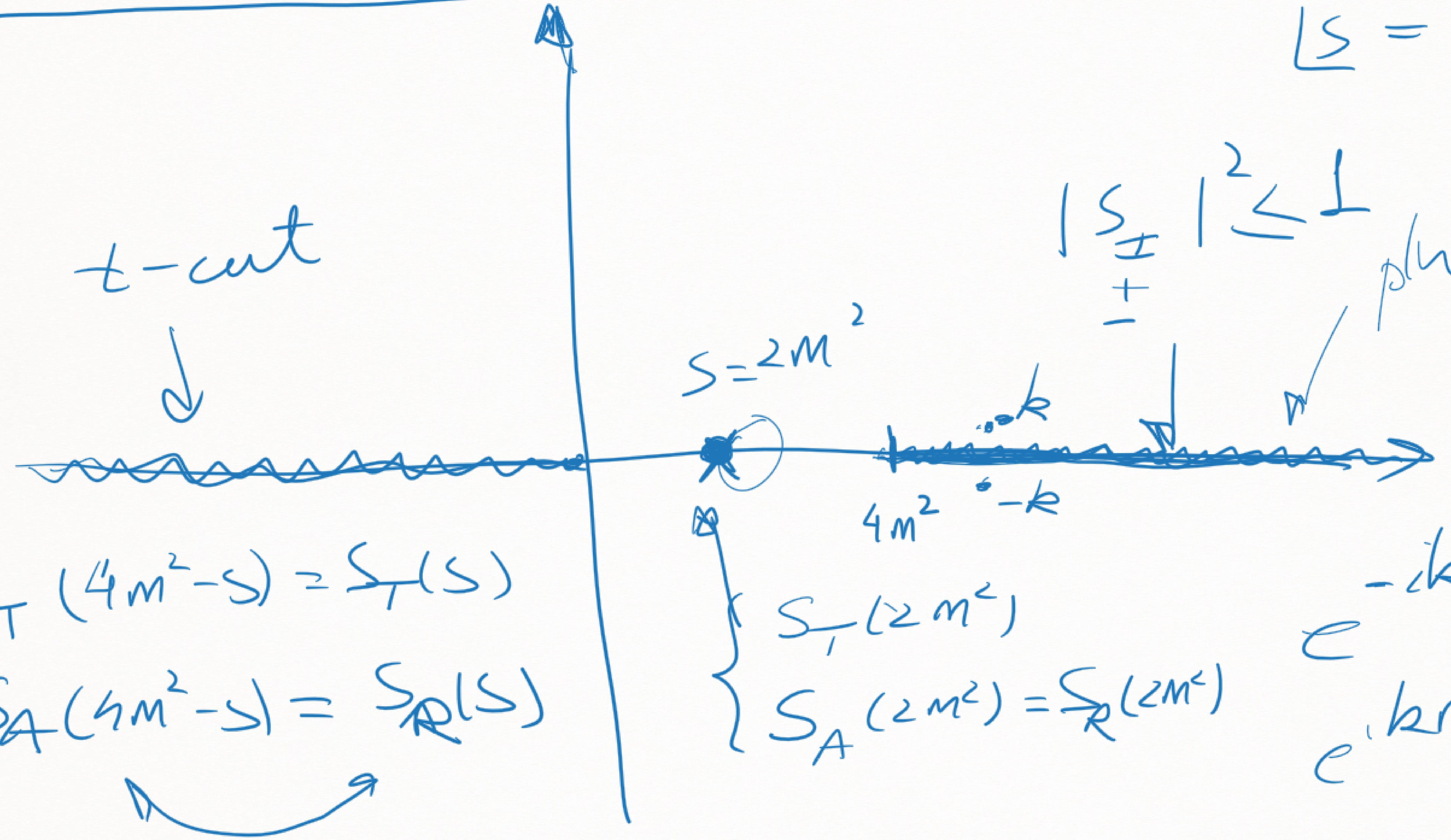
$$\boxed{S_A(s),\ S_T(s),\ S_R(s)}$$

$Ls = 4\varepsilon^2$

$|S_I|^2 \le 1$

$\pm$

physical line.



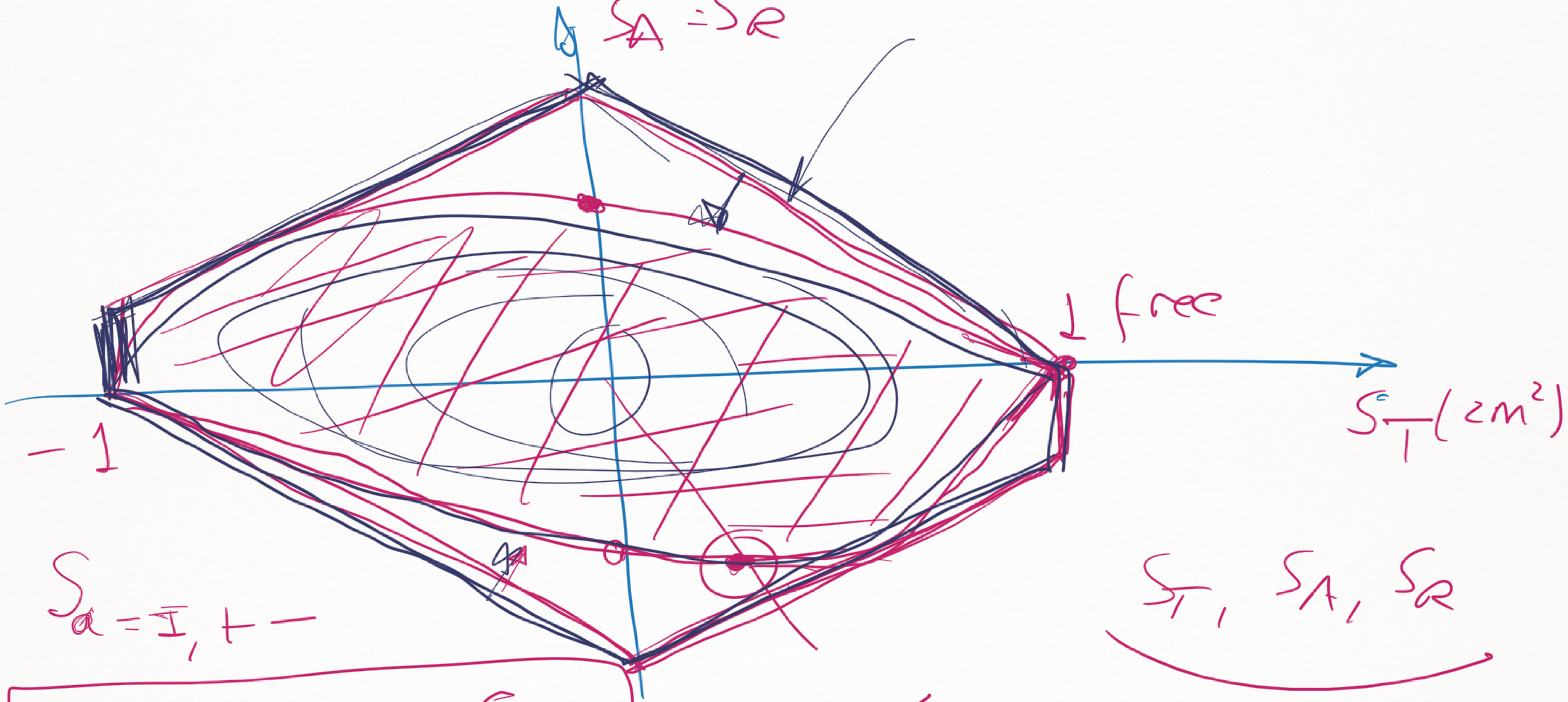
$k = \sqrt{\frac{s-4m^2}{4}}$

$$\begin{cases} S_T(4m^2-s) = S_T(s) \\ S_A(4m^2-s) = S_R(s) \end{cases}$$

$$\begin{cases} S_T(2m^2) \\ S_A(2m^2) = S_R(2m^2) \end{cases}$$

$$e^{-ikr} + e^{2i\delta} e^{ikr}$$

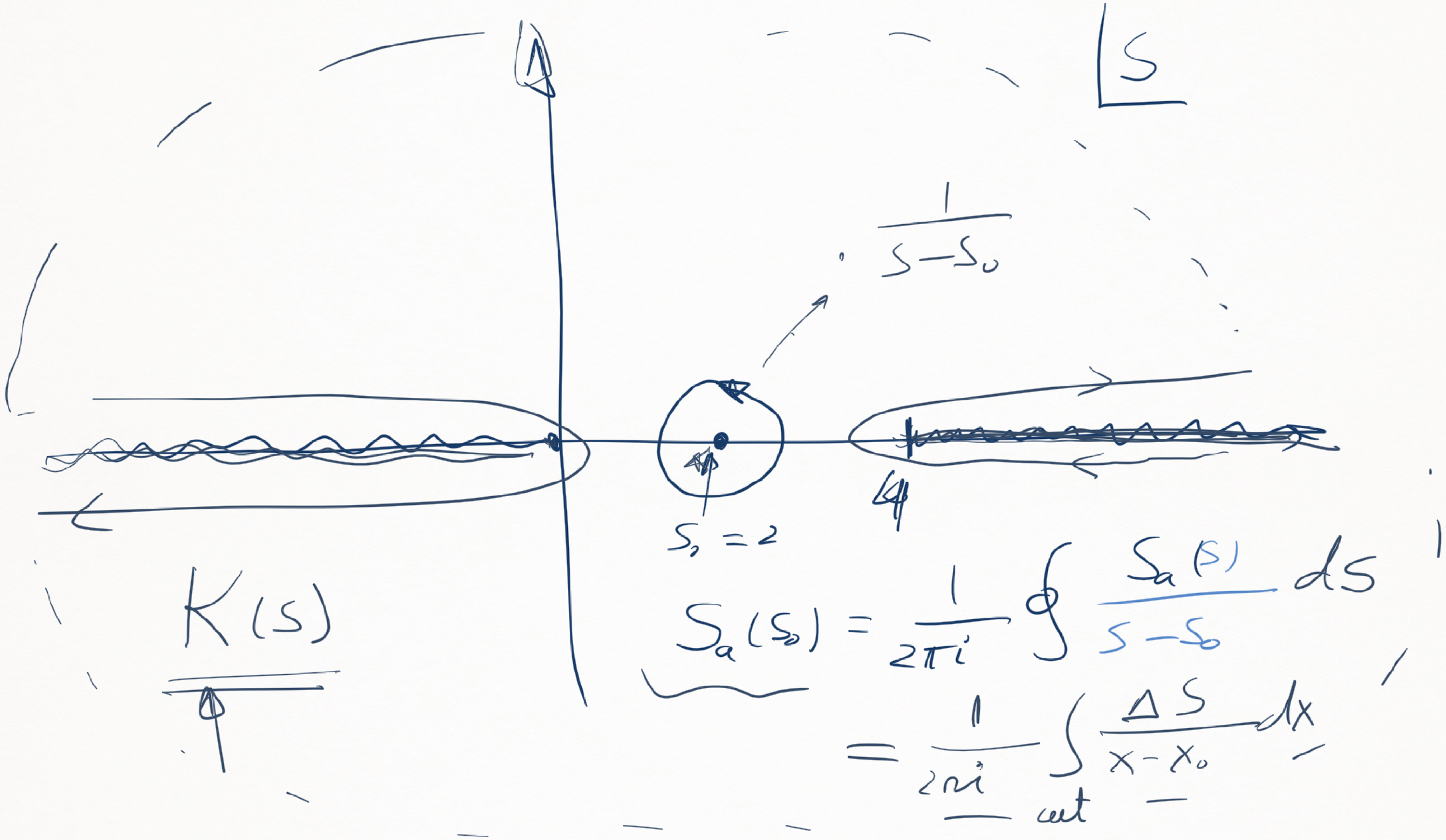
$$e^{ikr} + e^{2i\delta} e^{-ikr}$$



$S_a = I, t$ —

$$\boxed{S_a\,(4m^2 - s) = C_{ab}\, S_b(s)}$$

$S_T,\ S_A,\ S_R$



$$S_a(s_0) = \frac{1}{2\pi i}\oint \frac{S_a(s)}{s-s_0}\, ds$$

$$= \frac{1}{2\pi i}\int_{cut} \frac{\Delta S}{x-x_0}\, dx$$

max F

$$F = \sum V_a S_a(s=2) = \frac{1}{2\pi i}\oint K_a(s)\, S_a(s)$$

pole as $s=2$
residue V_a

$$= \frac{1}{2\pi i}\int_4^{\infty}\left(K_a(s^+)\, S_a(s^+) - K_a(s^-)\, S_a(s^-)\right)$$

$$+ \int_{-\infty}^{0}$$

$$\boxed{s \to 4-s}$$

$$K_a(s) = -K_b(4-s).$$

$$C_{ba}$$

$$K_a(s) = \left(K_b(s^*)\right)^*$$

$$F = \frac{2}{\pi}\int_4^{\infty} \mathrm{Im}\left(K_a S_a\right)$$

$$F = \sum V_a S_a(s=2) = \frac{2}{\pi}\int_4^\infty \mathrm{Im}(K_a S_a)\, dx \le$$

$$\le \frac{2}{\pi}\int_4^\infty |K_a|\underbrace{|S_a|}_{\le 1}\, dx \le \frac{2}{\pi}\int_4^\infty |K_a|\, dx$$

$$a = I, +, -$$

$$F \le \min_{\{K_a\}} \frac{2}{\pi}\int_4^\infty |K_a|\, dx$$

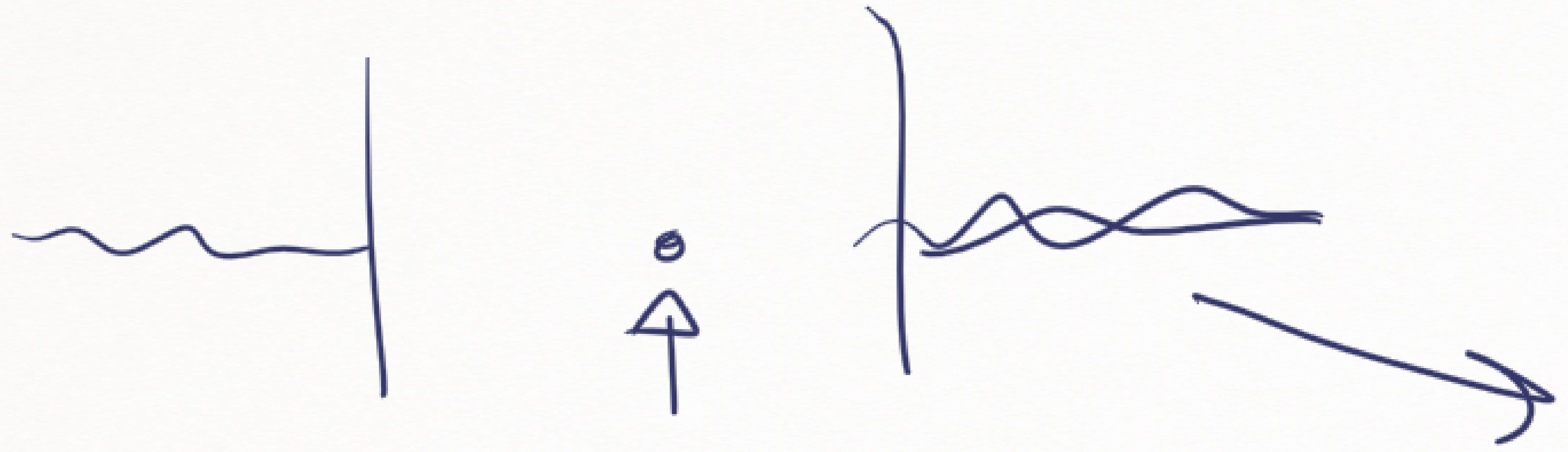
$$\boxed{K_a = \frac{V_a}{s-2}\,\frac{2}{\sqrt{s}\sqrt{4-s}}}$$

$$F \le \sum_a |V_a|$$

$$\boxed{S_a = \frac{i\overline{K_a}}{|K_a|}}$$

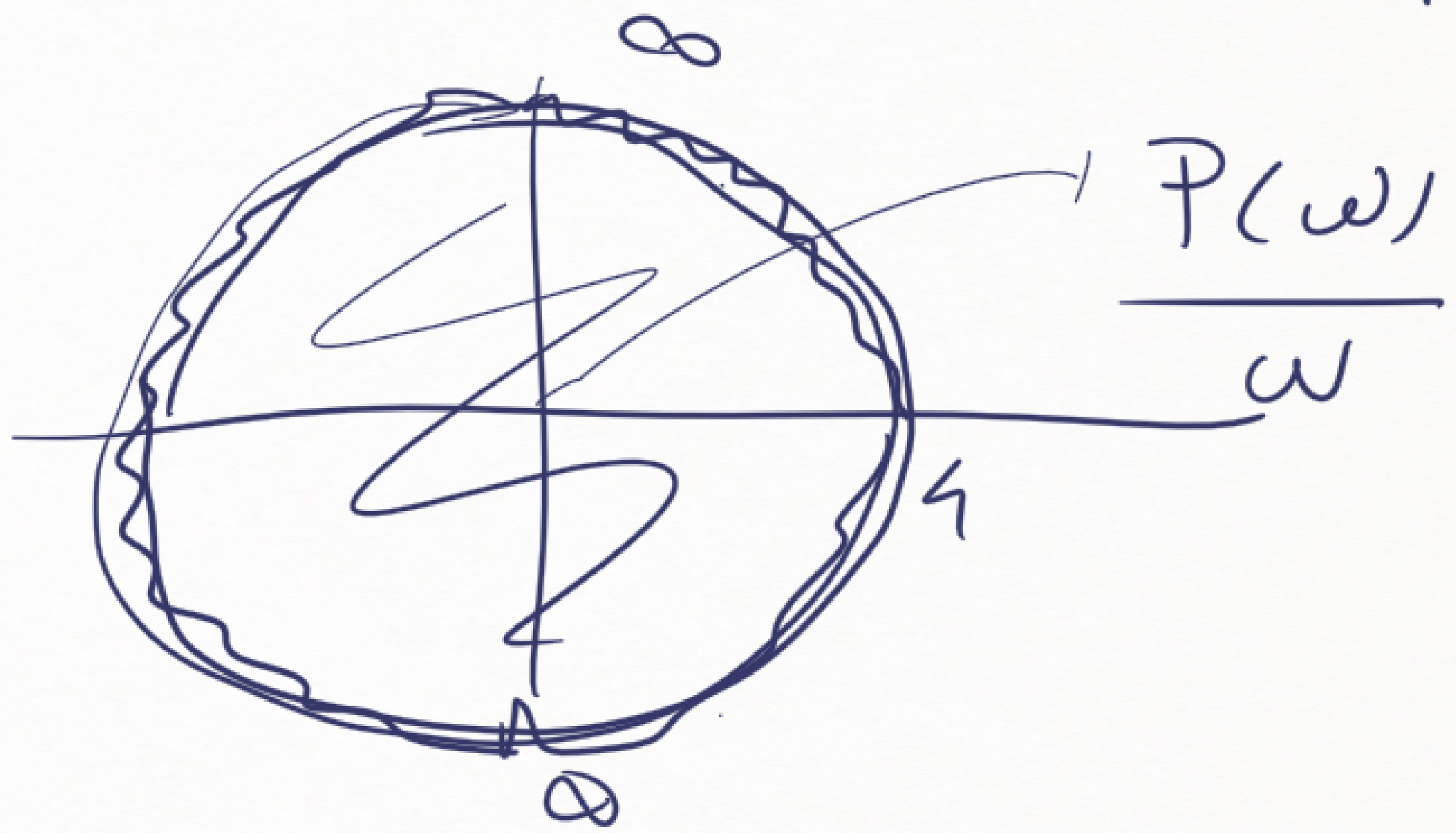
minimizer

$$\nu = t - u$$



$\omega \to -\omega$

$$\omega = \frac{\sqrt{s} - \sqrt{4-s}}{\sqrt{s} + \sqrt{4-s}}$$



- SNOWMASS paper (w/ Penedones, van Rees)

  2203.02421

- Aninda Sinha (Bangalore)

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Review

Positivity

$$D\begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{pmatrix} S_D & S_1 \\ S_2 & S_3 \end{pmatrix} \end{array} \begin{pmatrix} S_D^+ & S_2^+ \\ S_1^+ & S_3^+ \end{pmatrix} = \begin{pmatrix} S_D S_D^+ + S_1 S_1^+ & \varepsilon \\ \text{"} & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_D S_D^+ + S_1 S_1^+ = \mathbb{1} \qquad \begin{pmatrix} \mathbb{1} & S_D \\ S_D^+ & \mathbb{1} \end{pmatrix} \succeq 0$$

$$SS^+ = \mathbb{1}$$

$$\boxed{S_D S_D^+ - \mathbb{1}} = \underbrace{-S_1 S_1^+}_{} \leq 0$$

$A_1 \geq 0 \qquad A_2 \geq 0$

$0 \leq \alpha \leq 1 \qquad \alpha A_1 + (1-\alpha) A_2 \geq 0$

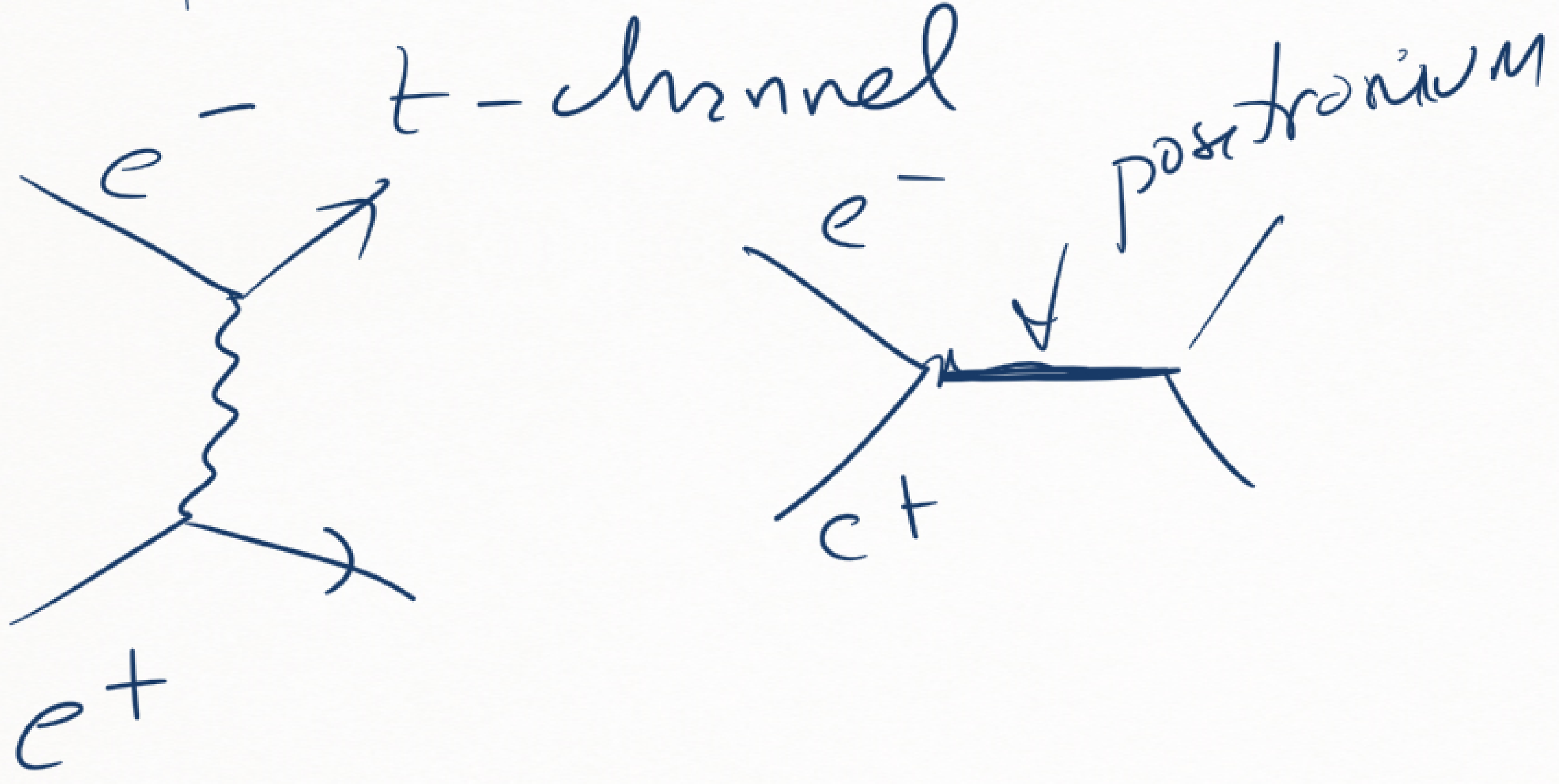
$\langle v| \qquad |v\rangle$

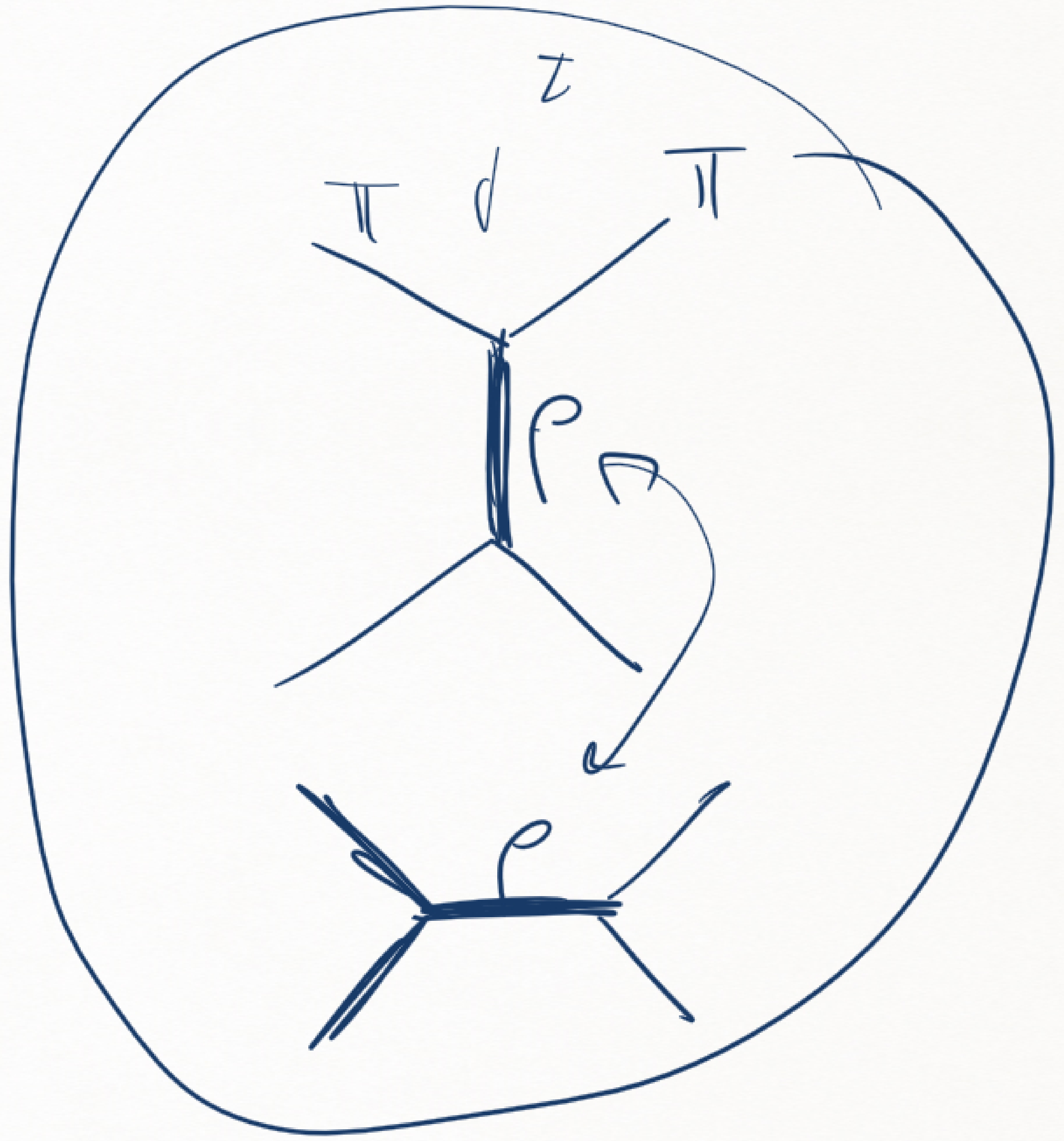
$$\alpha \underbrace{\langle v|A_1|v\rangle}_{\geq 0} + (1-\alpha) \underbrace{\langle v|A_2|v\rangle}_{\geq 0} \geq 0$$

space of positive semi-definite matrices is convex

$SU(3)$, 2 $\longrightarrow$ <u>theory of pions</u>

$SU(N)$, N_f $\longrightarrow$





$\pi^+ + \pi^-$

$\pi^0 + \pi^0 \longrightarrow \pi^0 + \pi^0$

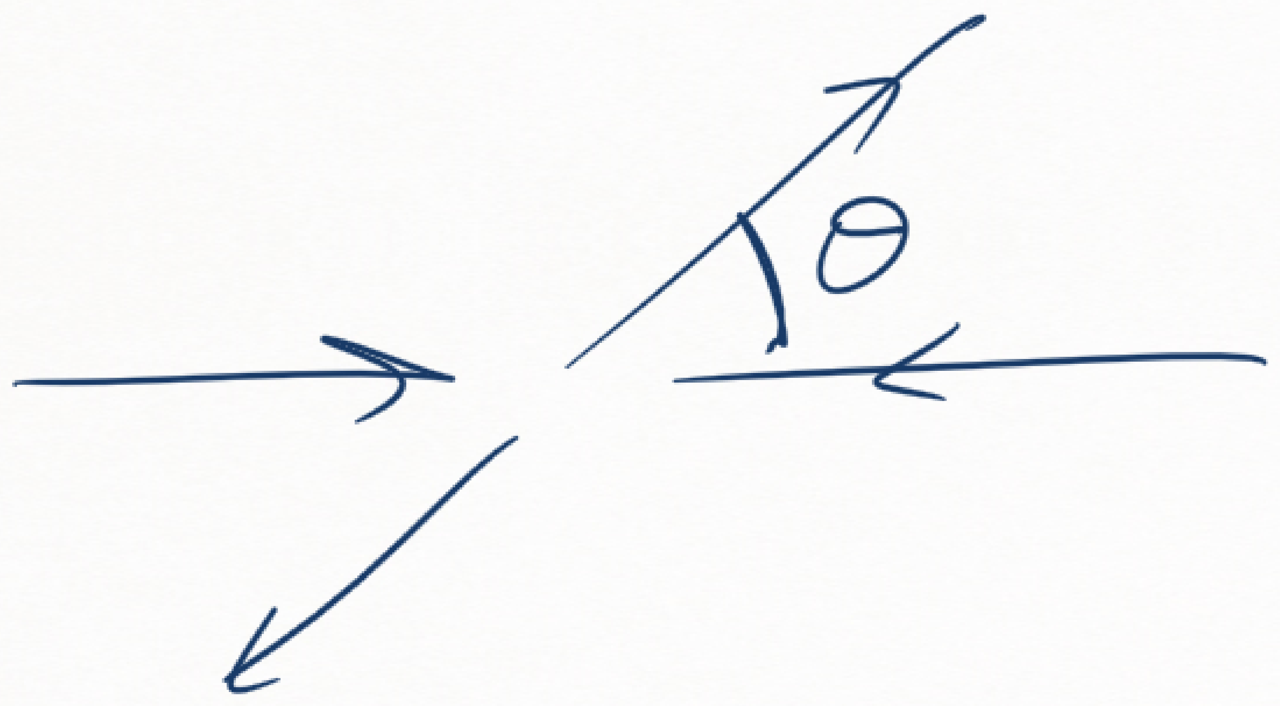
$p_1 \quad p_2 \qquad\quad p_3 \quad p_4$

$m^2 = 1$

$s = (p_1 + p_2)^2$

$t = (p_1 - p_3)^2 \qquad s + t + u = 4$

$u = (p_1 - p_4^2)$



Λ_{QCD}

$m/\Lambda_{QCD} \qquad \underline{N}$

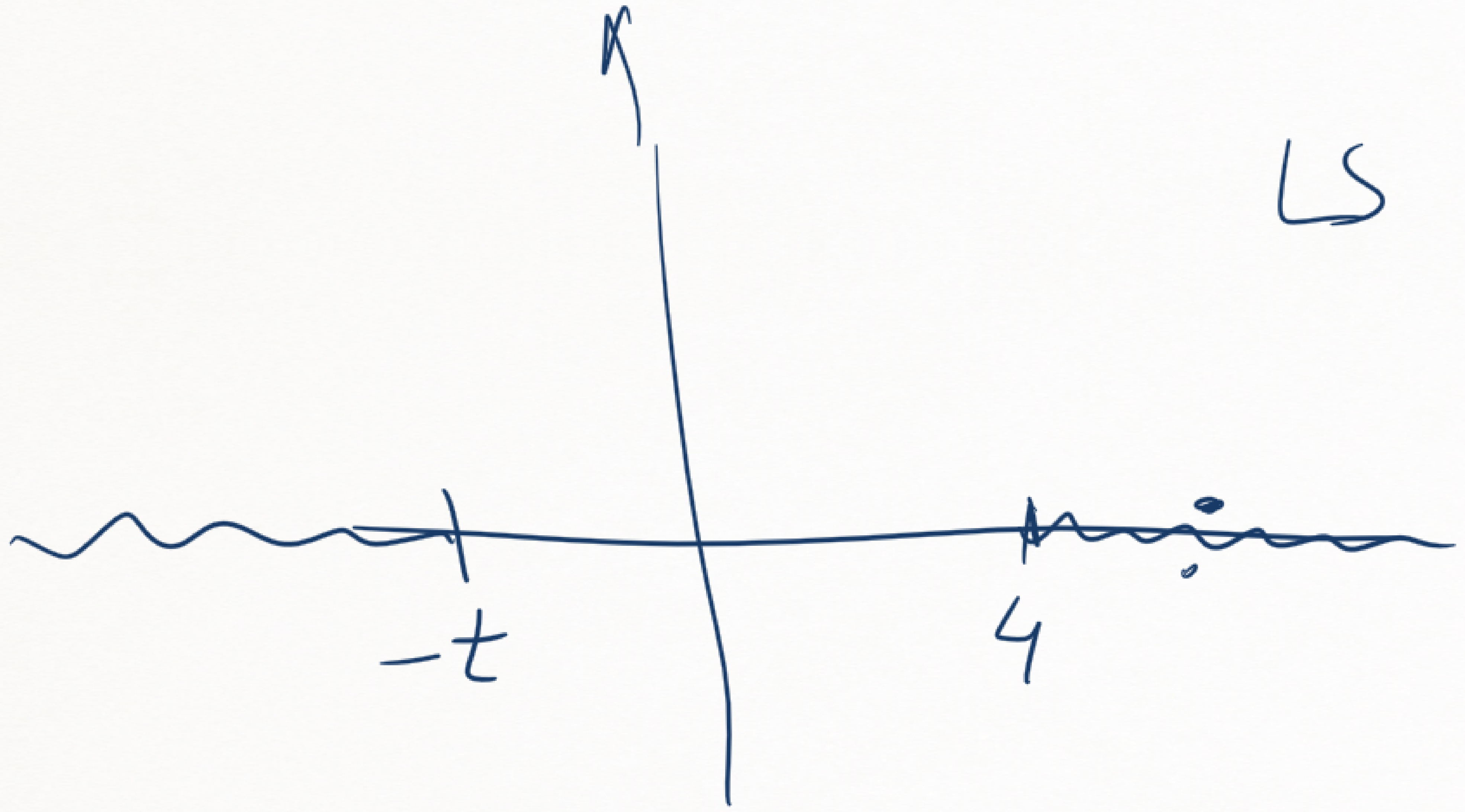
$S_\ell(s) = \underline{\eta_\ell}\, e^{2i\delta_\ell}$

$S_\ell = 1 + i\,\underline{h_\ell}(s) \qquad |S_\ell|^2 \leq 1$

$\boxed{|h_\ell(s)|^2 \leq 2\,\mathrm{Im}\, h_\ell(s)}$

$$F(s^+,t) = \sum_{l \text{ even}} \frac{2}{\pi}\sqrt{\frac{s}{s-4}}\,(2l+1)\,h_l(s)\times$$

$$\times\, P_l\Big(\underbrace{1+\frac{2t}{s-4}}_{\cos\theta}\Big)$$



$$s \leftrightarrow t \leftrightarrow u$$

$$\begin{cases} s>4 \\ t>4 \\ u>4 \end{cases} \qquad \boxed{s+t+u=4}$$

$$s+t<0$$

$$F(s,t,u) = f_0 + \frac{1}{\pi}\int_4^\infty dx\,\sigma(x)\left(\frac{1}{x-s} + \frac{1}{x-t} + \frac{1}{x-u}\right)$$

$$\frac{1}{x-s-i\epsilon} = \text{P.V.}\,\frac{1}{x-s} + i\pi\delta(x-s)$$

$$+\frac{1}{\pi^2}\int_4^\infty dx\int_4^\infty dy\;\rho(x,y)\left(\frac{1}{(x-s)(y-t)} + \frac{1}{(x-s)(y-u)} + \frac{1}{(x-t)(y-u)}\right)$$

Study the space of such functions

-) unitarity saturated $\Rightarrow$ ρ vanishes in certain region

•) " " at all energies $\Longrightarrow$ free theory

AKS theorem

$$\Longrightarrow \sum_{n=0}^{\infty} t^n P_n(z) = \underbrace{\frac{1}{\sqrt{1-2zt+t^2}}}\qquad |z| \leq \frac{1+t^2}{2t}$$

$$\sum_{\ell=0}^{\infty} a_\ell P_\ell(\operatorname{ch}\mu) < \infty \qquad \lim_{\ell\to\infty} |a_\ell|^{1/\ell} = e^{-\alpha}$$

$$|\mu| < \alpha$$

$$F = \frac{2}{\pi}\sqrt{\frac{s}{s-4}} \sum_{\ell \text{ even}} (2\ell+1)\, \underset{\mathrm{Im}}{h_\ell(s)}\, P_\ell(\underbrace{\mathrm{ch}\mu}_{\hookrightarrow 1+\frac{2t}{s-4}})$$

$$t = 4 \qquad \mathrm{ch}\mu = \mathrm{ch}\,\alpha_1(s) = \frac{s+4}{s-4}$$

$$\lim_{\ell\to\infty} |h_\ell(s)|^{1/\ell} \leq e^{-\alpha_1(s)}$$

$$\mathrm{Im}\, h_\ell(s) = \frac{1}{2}\underline{|h_\ell(s)|^2} \qquad |h_\ell(s)| \overset{\text{saturate}}{\sim} \underbrace{\left(\frac{\sqrt{s}-2}{\sqrt{s}+2}\right)^{\ell}}$$

$$\boxed{4 < s < 16}$$

$$\sum_{l=0}^{\infty} (2l+1)\, \underbrace{\operatorname{Im} h_l(s)}_{|\mu| < 2\alpha_1}\, P_l(\operatorname{ch}\mu) = \sigma(s) + \cancel{\int_4^{\infty} dy\, \rho(s,y) \left(\frac{1}{y-t} + \frac{1}{y-u} \right)}$$

$$1 + \frac{2t_2(s)}{s-4} = \operatorname{ch}(2\alpha_1(s))$$

$$\boxed{t < t_2(s)}$$

$$4 < s < 16$$

$$\rho(s,y) = 0$$

s saturating unitarity

$$\underbrace{y < t_2(s)} = 16 + \frac{64}{s-4}$$

suppose no particle production

$p(s, y) = 0$
$y > 4 \quad 4 < s < 16$

$p(s, y) = 0$
$\underline{s > 4}$ $\boxed{4 < y < 16}$

$$1 + \frac{2t}{s-4} = c\theta \qquad \underline{\theta = 0} \rightarrow t = 0$$

$$F(s, 0, 4-s) = F(s)$$ *forward amplitude*

$$F(s) = F(4-s)$$