Simulation: Why, What and How?

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We shall start with . . .

https://www.youtube.com/watch?v=JgTfISjq4o4

Ken Mattingly (Picture Source: NASA)

Ken Mattingly is an American aeronautical engineer, test pilot and astronaut. A bunch of people woke him up in the movie clip.

Ken Mattingly

- He was chosen as the Command Module Pilot on the Apollo 13 mission in 1970.
- Three days prior to launch, he was removed from the mission due to exposure to German measles (which he never contracted) and was replaced by a member of the backup crew.
- As a result, he statyed back on Earth and missed the dramatic in-flight explosion that crippled the spacecraft.
- However, this was a blessing in disguise because Ken Mattingly played a very important role in helping the crew solve the problem of power conservation during re-entry through **simulations**.

What is simulation?

According to Wikipedia . . .

"A simulation is the imitation of the operation of a real-world process or system over time. Simulations require the use of models; the model represents the key characteristics or behaviors of the selected system or process, whereas the simulation represents the evolution of the model over time. Often, computers are used to execute the simulation."

Why is it important?

- Helps us understand the evolution of various real-world processes.
- In many cases, it also helps us improve the system and its dynamics.
- Simulations are useful in science, engineering, banking, insurance, etc.

An example (Image Credit: Jon Evans)

In 1827, while examining pollen grains suspended in water under a microscope, Scottish botanist Robert Brown observed minute particles, ejected from the pollen grains, executing a continuous jittery motion.

Although Brown did not provide a theory to explain the motion the phenomenon is now known as Brownian motion.

Brownian motion

In 1905, theoretical physicist Albert Einstein published a paper where he modelled the motion of the pollen particles as being moved by individual water molecules.

Brownian motion arises naturally in physics, chemistry, biology and even in the analysis of financial market!

`https://en.wikipedia.org/wiki/Brownian_motion`.

Question: How to simulate this jittery motion?

Easy Examples

A much simpler question

Question: Suppose you have to choose one person at random (with equal probability) out of two people, say A and B. How can you do that?

Toss a fair coin. If head appears, choose A and if tail appears, choose B.

Question: What if the coin is not fair?

A slightly modified question

Question: Suppose you have to choose one person at random (with equal probability) out of two people, say A and B. How can you do that using a possibly unfair coin?

Basically I am asking the following question: *using a possibly unfair coin, how can one simulate an event whose probability is 50%?*

Acceptance-rejection method

Question: Suppose you have to choose one person at random (with equal probability) out of two people, say A and B. How can you do that using a possibly unfair coin?

The Method:

1. Toss the coin two times.
2. If HH or TT appear, then reject this outcome and go back to Step 1.
3. If HT appears, then choose A and if TH appears, the choose B.

We assume tacitly that the two tosses are *independent*. Then HT and TH are the only "accepted outcomes" and they are equally likely. That's why, this method works.

A slightly modified question

Question: Suppose you have to choose one person at random (with equal probability) out of three people, say A, B and C. How can you do that using a possibly unfair coin?

Basically I am asking the following question: *using a possibly unfair coin, how can one simulate an event whose probability is* $1/3$*?*

Acceptance-rejection method

Question: Suppose you have to choose one person at random (with equal probability) out of three people, say A, B and C. How can you do that using a possibly unfair coin?

The Method:

1. Toss the coin three times.
2. If *HHH* or *TTT* appear, then reject this outcome and go back to Step 1.
3. Otherwise, do the following:
   - if *THH* or *HTT* appears, then choose A,
   - if *HTH* or *THT* appears, then choose B,
   - if *HHT* or *TTH* appears, then choose C.

QUESTIONS?
COMMENTS?
CONCERNS?

Acceptance-rejection method

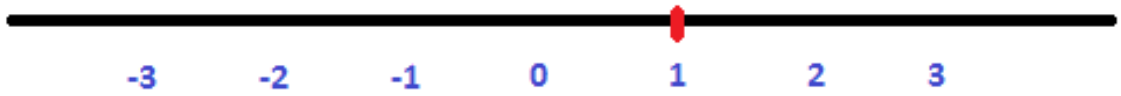
- Acceptance-rejection method can be used to choose 1 person uniformly at random from a group of any number of people.

- In fact, this is a very useful and popular simulation method in many many complicated situations as well.

Random Walks

What is a random walk?

What is a random walk?



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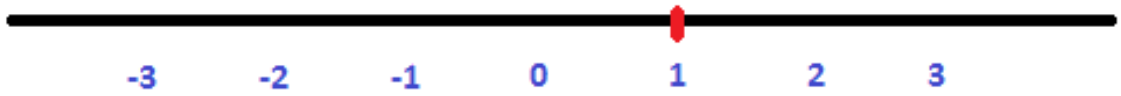

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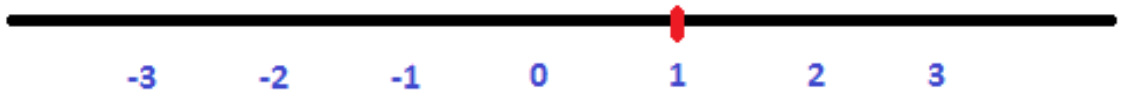

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Simple random walk

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1. Start at 0.

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3. If head appears, then take a step $+1$ and if tail appears, then take a step -1.

Simple random walk

1. Start at 0.
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3. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
4. Toss a fair coin in an *independent fashion*.

Simple random walk

1. Start at 0.
2. Toss a fair coin.
3. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
4. Toss a fair coin in an *independent fashion*.
5. If head appears, then take a step $+1$ and if tail appears, then take a step -1.

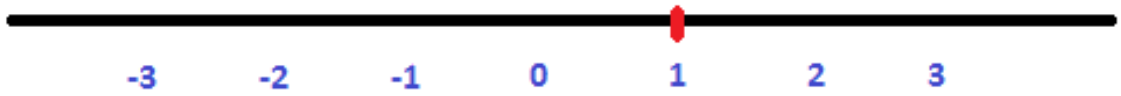
Simple random walk

1. Start at 0.
2. Toss a fair coin.
3. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
4. Toss a fair coin in an *independent fashion*.
5. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
6. Repeat 4 and 5 again and again.

QUESTIONS?
COMMENTS?
CONCERNS?

Simple random walk movie

Simple random walk movie



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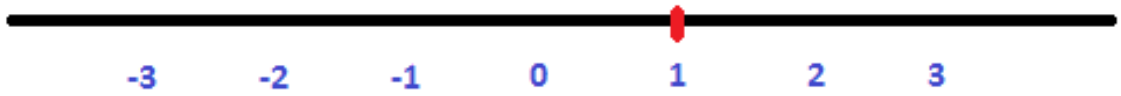

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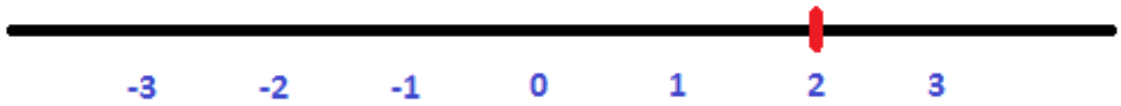

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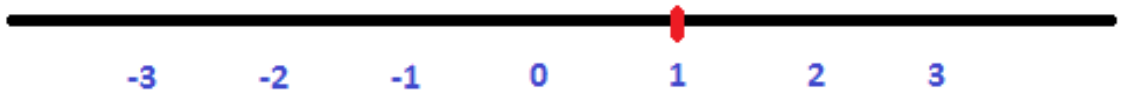

Simple random walk movie

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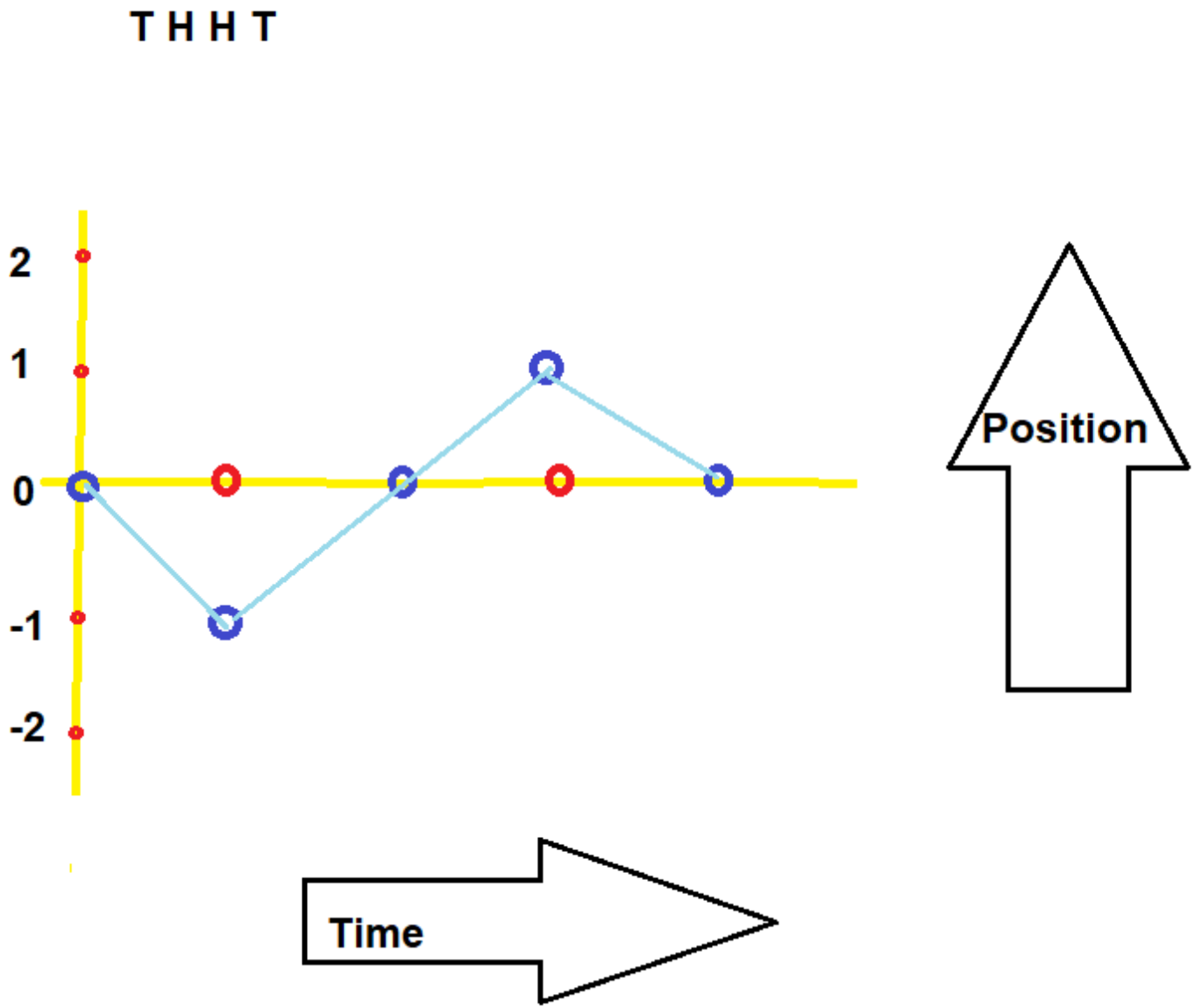
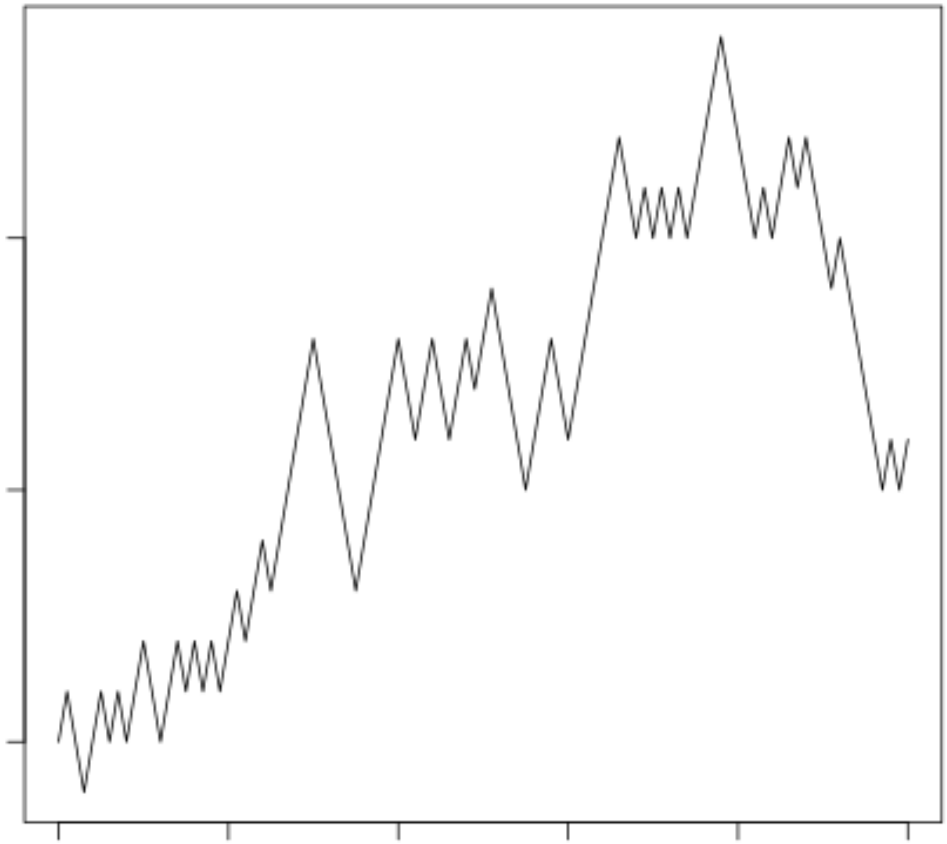
How to graphically represent a random walk?



How does a random walk grow?



Conclusion: We should scale the time by n and the position by $\sqrt{n}$ for a large value of n.

The scaled picture

We should scale the time by n and the position by $\sqrt{n}$ for a large value of n.

Question: What do we mean by this?

Suppose S_n is your position after n tosses. Since you start at 0, take $S_0 = 0$. It is easy to see that S_1 is either $+1$ or -1 depending on whether the first too yields a H or a T (resp.), and so on.

Earlier, we were representing a random walk graphically by joinging the points:

$$(0, 0), (1, S_1), (2, S_2), (3, S_3), \ldots, (n, S_n).$$

The scaled picture

We should scale the time by n and the position by $\sqrt{n}$ for a large value of n.

Suppose S_n is your position after n tosses. Since you start at 0, take $S_0 = 0$. It is easy to see that S_1 is either $+1$ or -1 depending on whether the first too yields a H or a T, and so on.

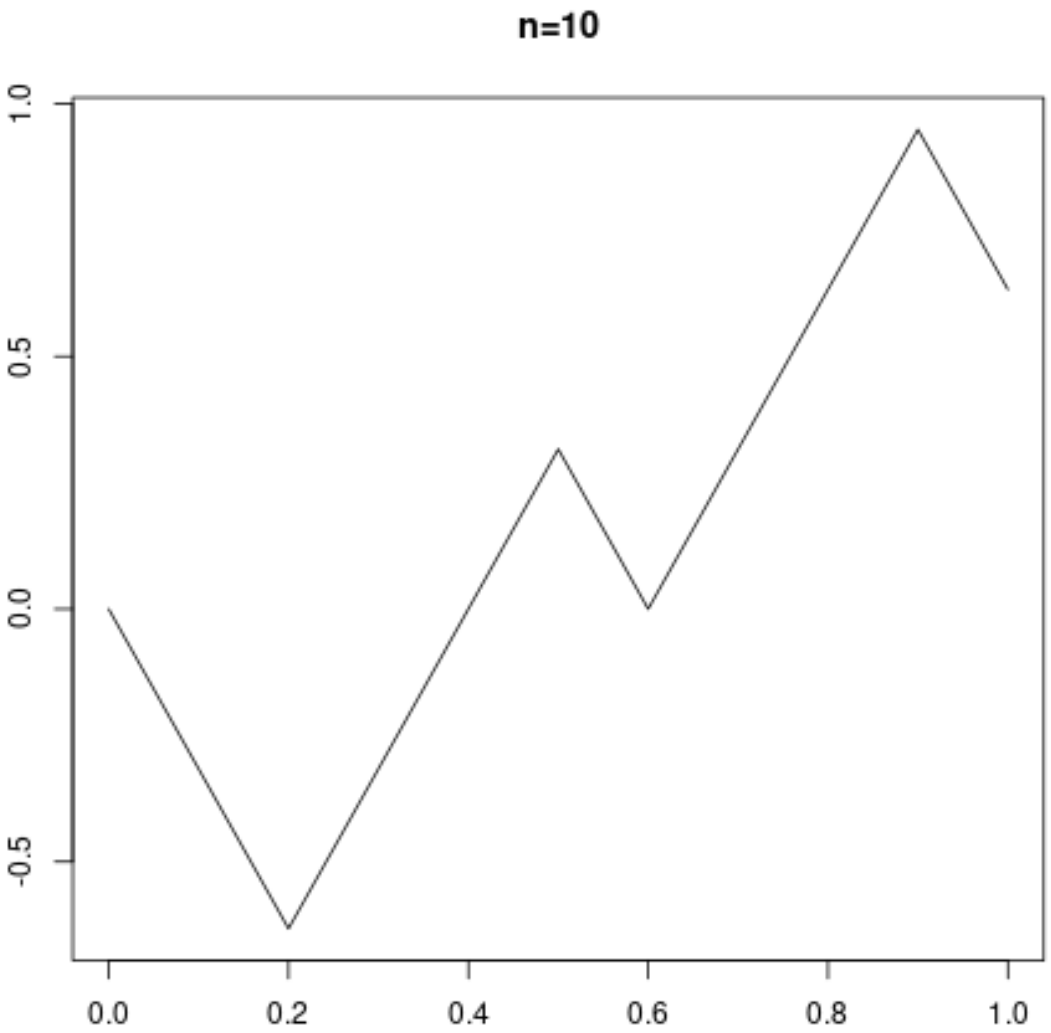
Earlier, we were representing a random walk graphically by joinging the points:

$$(0,0),(1,S_1),(2,S_2),(3,S_3),\ldots,(n,S_n).$$

Now we shall join the points

$$(0,0),\left(\frac{1}{n},\frac{S_1}{\sqrt{n}}\right),\left(\frac{2}{n},\frac{S_2}{\sqrt{n}}\right),\left(\frac{3}{n},\frac{S_3}{\sqrt{n}}\right),\ldots,\left(\frac{n}{n},\frac{S_n}{\sqrt{n}}\right).$$

The scaled picture



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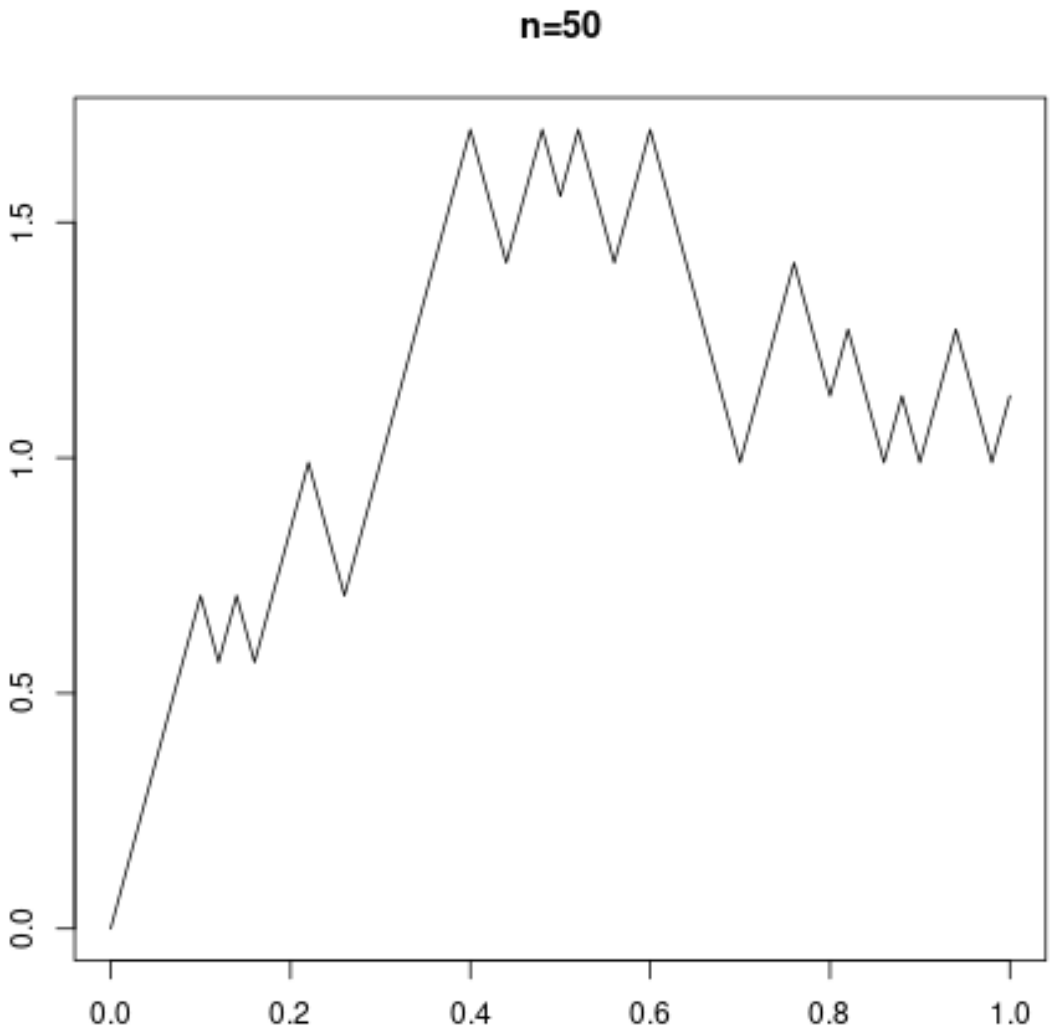

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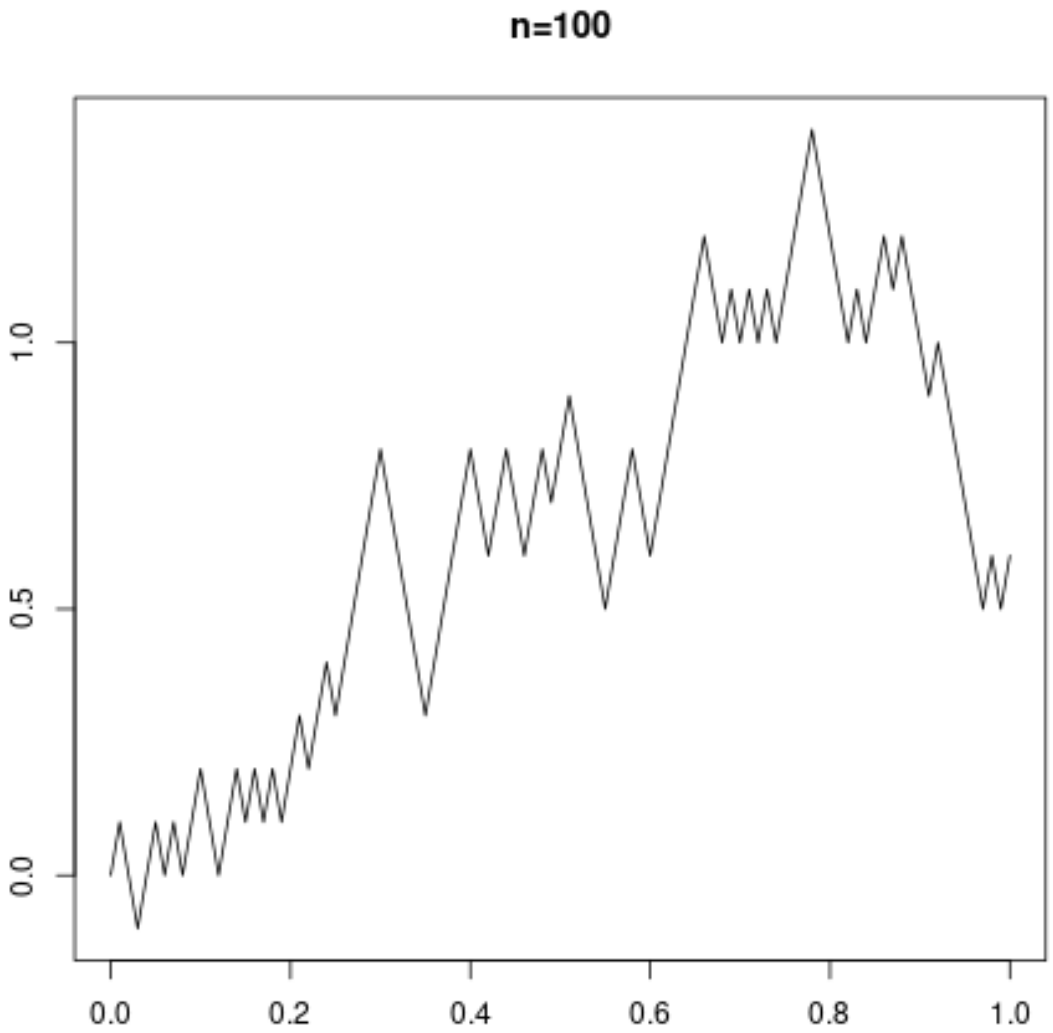

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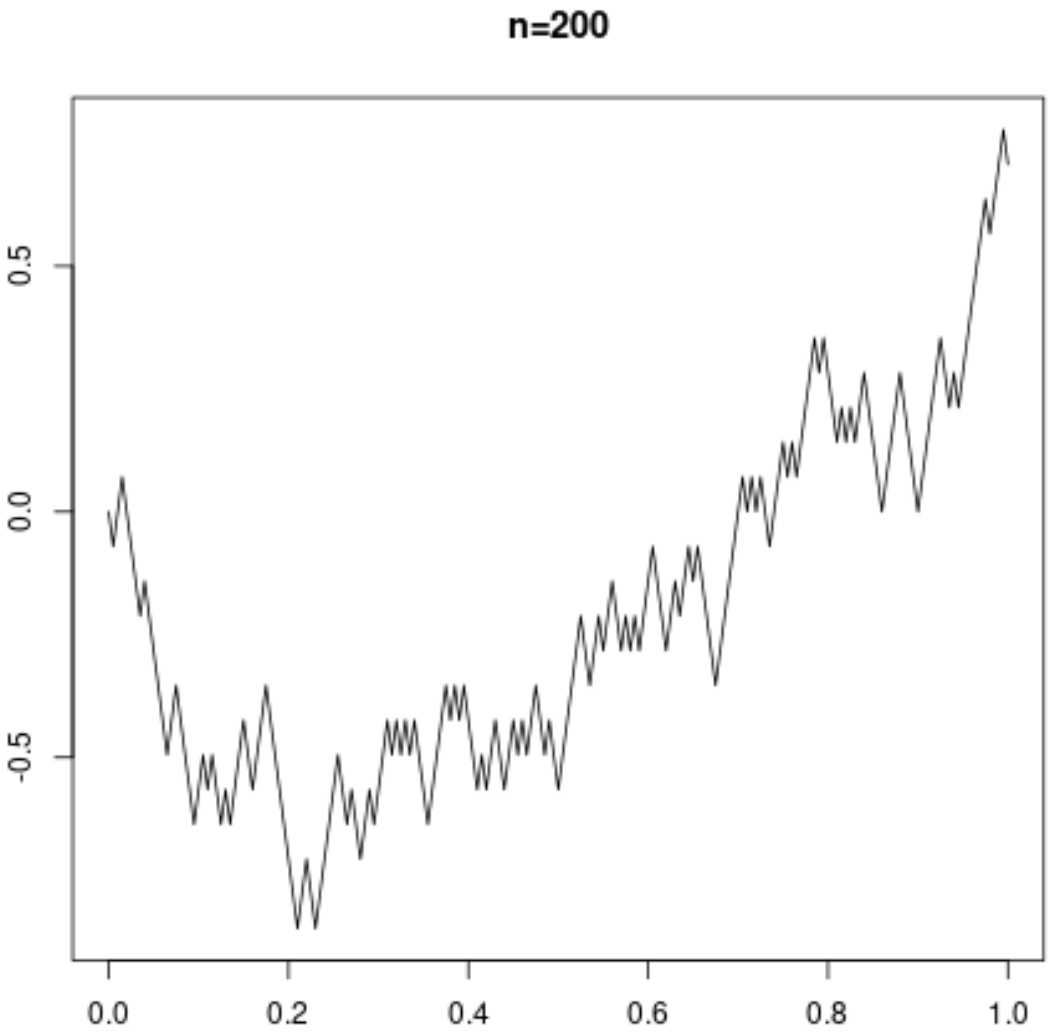

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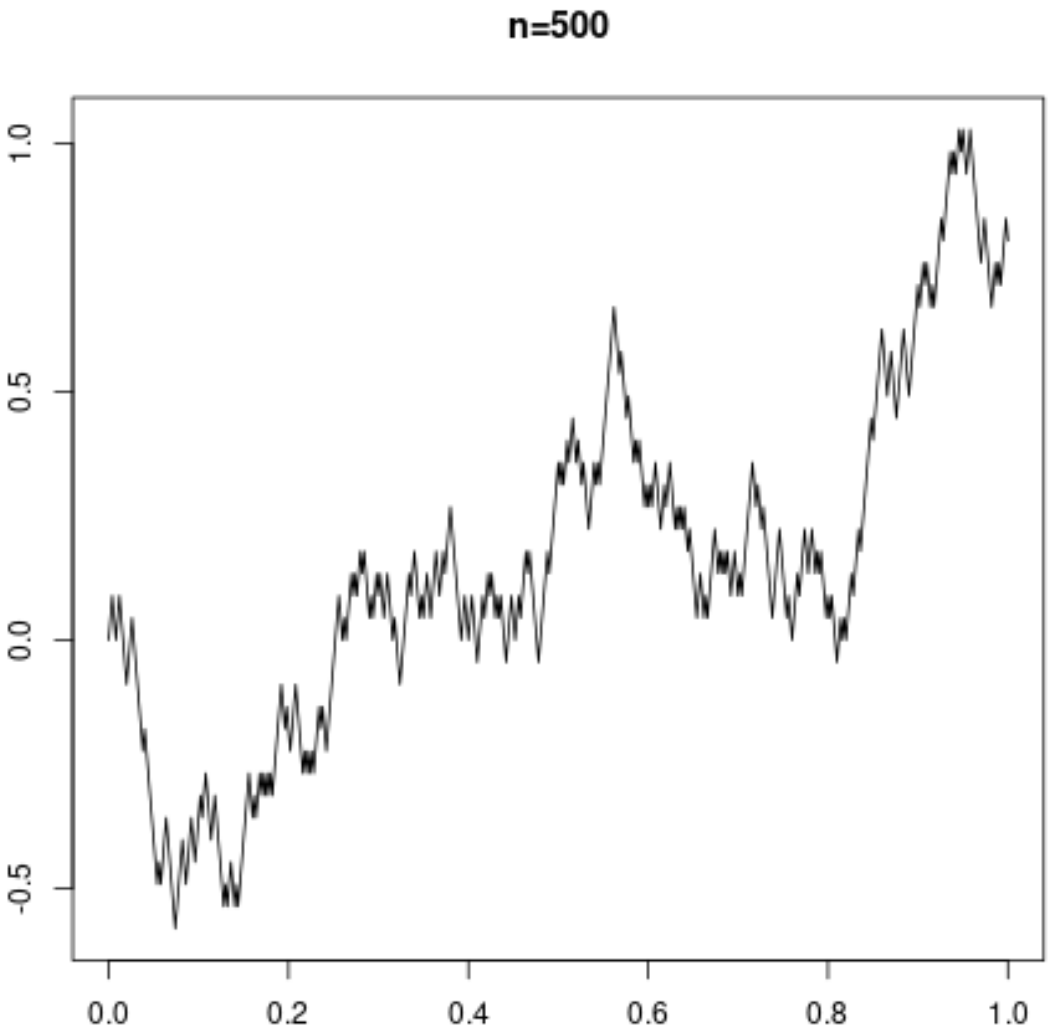

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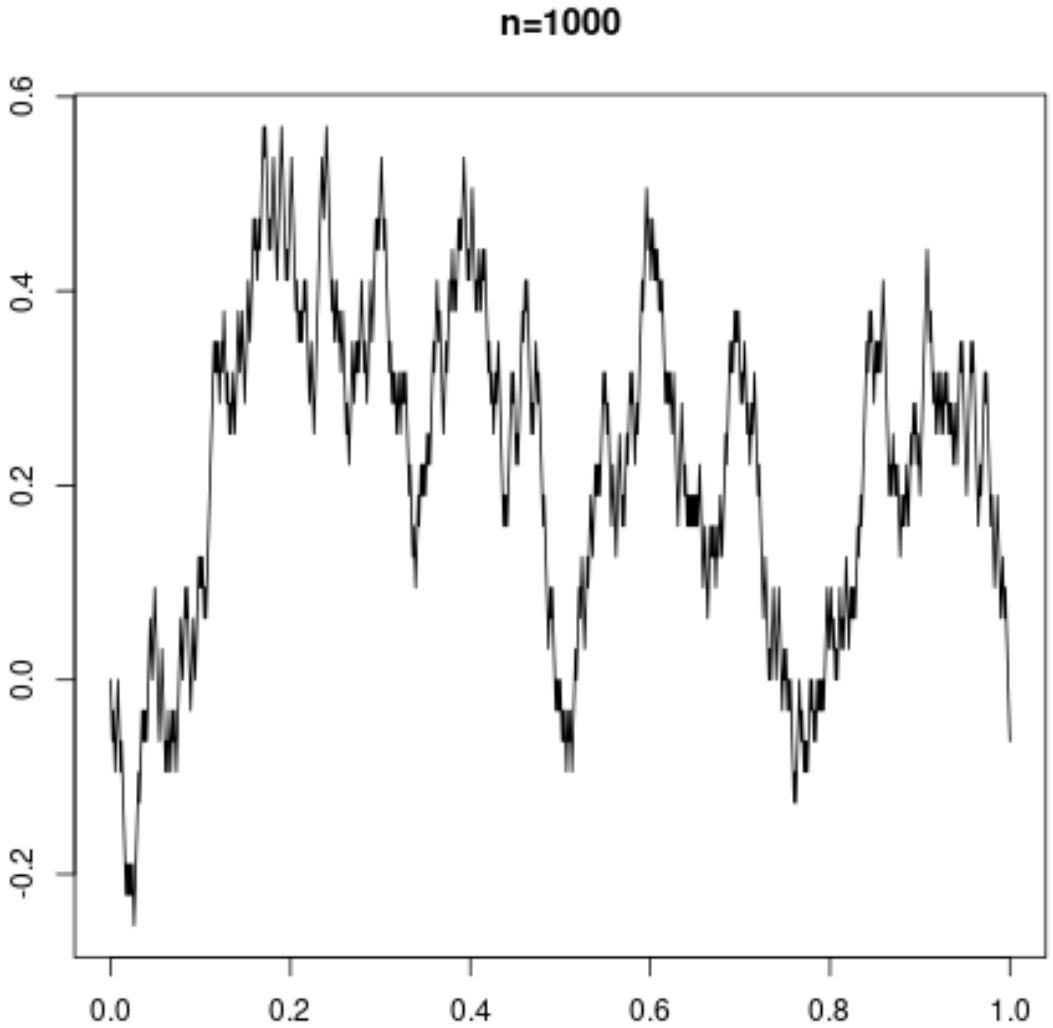

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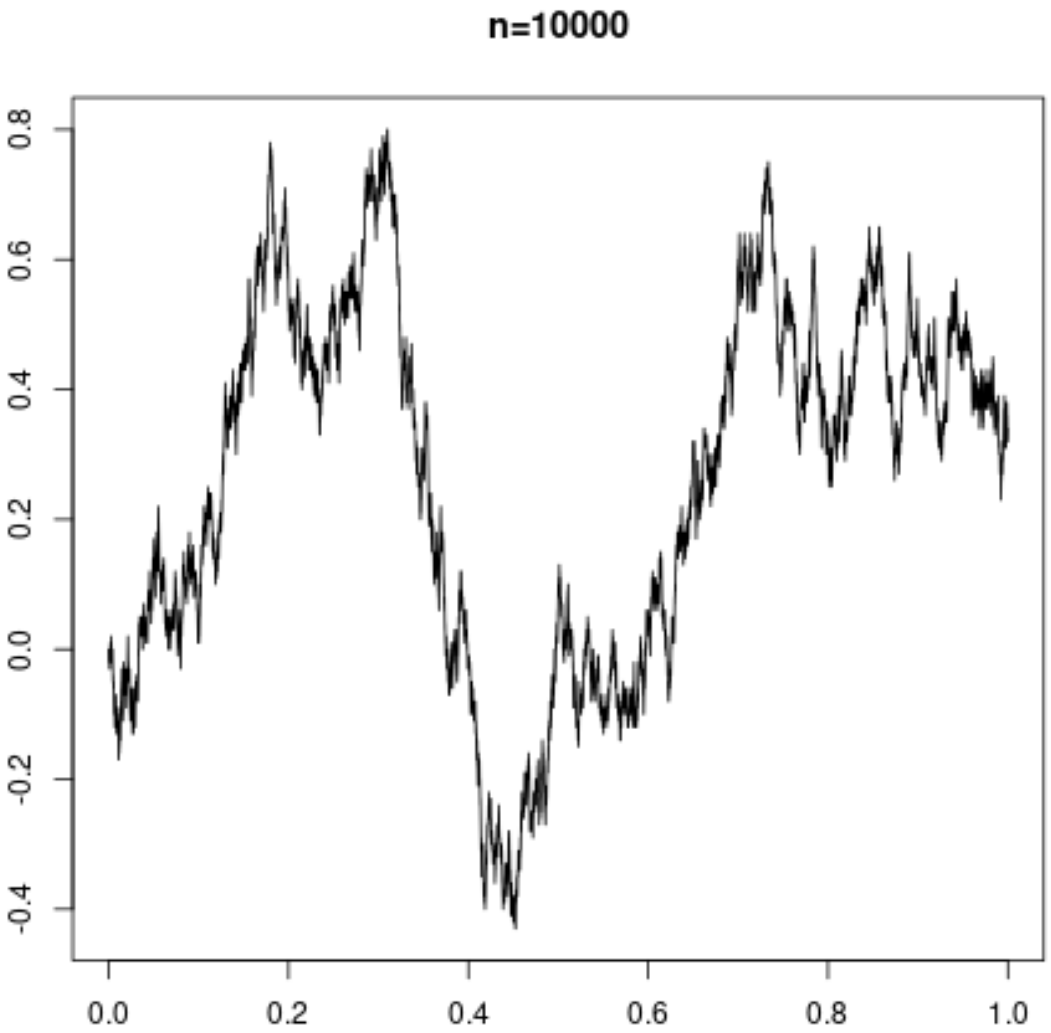

From random walk to brownian motion

As we can clealy see, the picture of random walk is becoming more and more jittery as n becomes larger and larger.

This was formally proved by Norbert Wiener. He showed that the scaled picture actually becomes Brownian motion when n is very very large.

We can actually use Wiener's theorem to simulate Brownian motion - first simulate a simple random walk, and then scale the time and position appropriately.

Final remarks

- Simulation is basically immitation of a real-world system or phenomenon.
- We use models (e.g., Brownian motion) that represents key characteristics or behaviour of the underlying process or system.
- A simulation essentially mimics the dynamics and evolution of the model over time. Most often, this is carried out with the help of computers.

Thank You Very Much!