B. Bellazzini (IPhT)
@ ICTS's School in Bangalore
15-19/04/2024

Positivity Constraints on EFT

(tentative) Outline:

L1 Intro and Motivations:

- Why positivity constraints? QFT-swampland/landscape
- NDA vs Symmetries vs Causality? $\rightarrow$ Pheno

L2 Positivity from Causality

- subluminality constraints on U(1)-GB, Galileons, photons
- Time delay in Gravity
- Time machines and resolvability

L3 Positivity from Analyticity + Unitarity

- microcausality $\rightarrow$ analyticity
- analyticity + unitarity $\rightarrow$ positivity : U(1)-GB & photons examples

L4 Positivity & the theory of Moments

not yet filled

⋮

L5 Gravity and eikonal

⋮

— EFT in a nutshell —

(1) $\boxed{\text{set of active/light d.o.f.} \quad + \quad \text{symmetries} \quad + \quad \text{Basic Principles QM, ...}}$

exact: Poincaré, CPT, SUSY, ...
approx: flavour ($y_i \to 0$), accidental P, B, L
scale invariance
anomalous: anomaly matching ...

QM + Poincaré:

(2) $p^2 = m^2$ mass
$m^2 = 0 \quad W_\mu = \lambda P_\mu$ helicity
$m^2 \neq 0 \quad W^2 = -m^2 \vec{J}^{\,2}$ spin

Unitarity, linearity space, RG-eqs..
$\downarrow$
Gell-mann totalitarian principle
(subject to sym. constraints)

"Light" compared to what ?

(3) $\boxed{\text{Active/Light d.o.f.} = \begin{array}{ll} m^2 \ll M^2 & \text{Low energy} \\ x \gg \ell = 1/M & \text{Long distance} \end{array}}$

(4) M = the physical cutoff where new d.o.f. become active

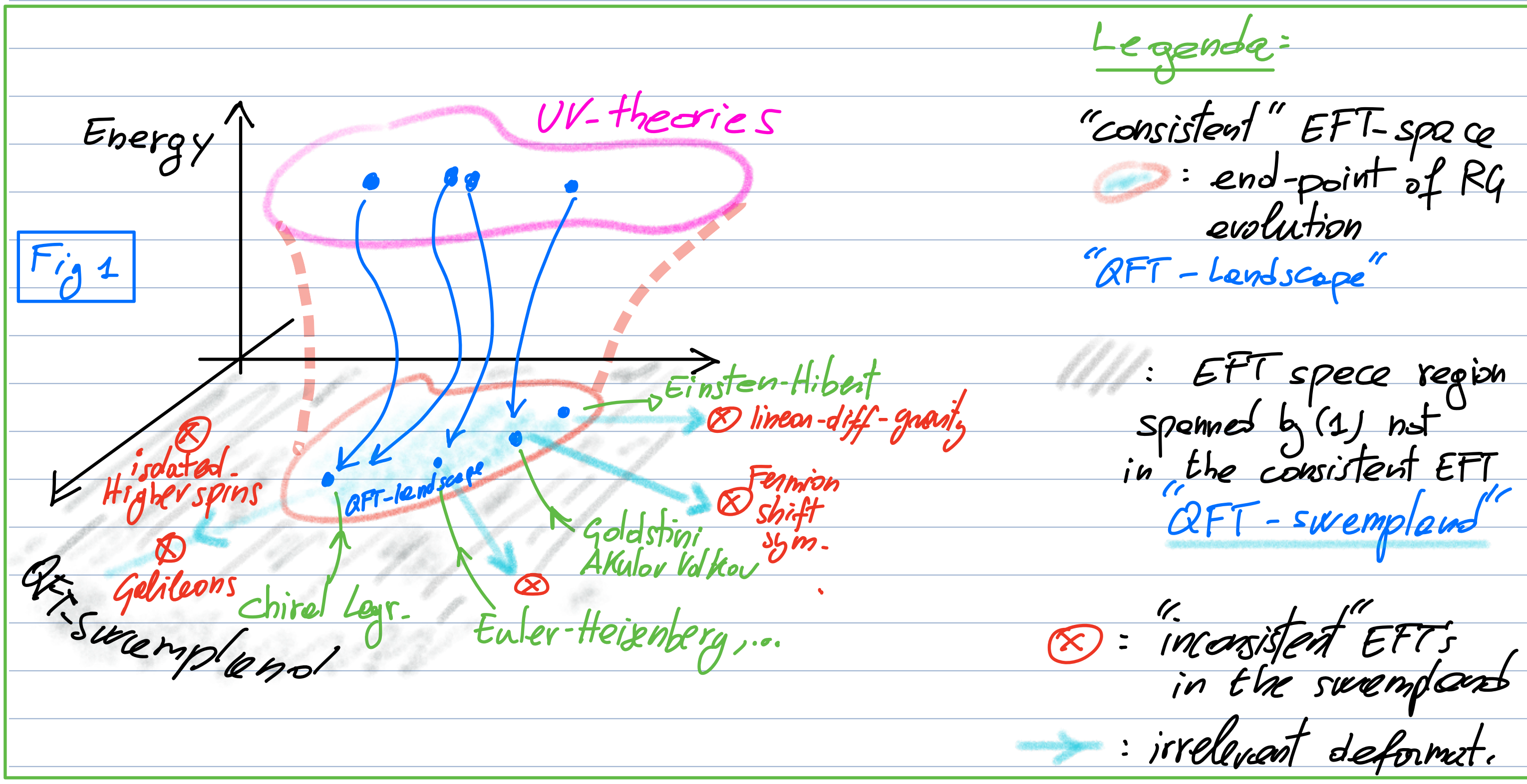
Remarks:

- M typically unknown : in practice bounded above by Λ_{strong}
  $M < \Lambda_s$ = where EFT no longer predictive/calculable
  (in strongly coupled UV-completions $M \sim \Lambda_s$)
- $m \ll E \ll M$ energy window theory predictive/calculable
- QFT = EFT with M very large (possibly $M \to \infty$)

Let's use these basic ingredients in (1) to see the space of EFT's that one can span with them

Spoiler: one generates much larger space than what is found by the result of RG-evolution from UV-theories

Fig 1



$$\mathcal{L} = \mathcal{L}_{\text{leading } \partial} + \sum_i c_i \, \mathcal{L}_i^{\text{Higher } \partial}$$ depending on c_i move into

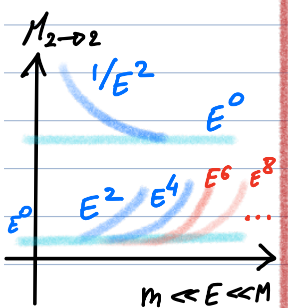
Question: What fundamental ingredients are missing?

Short & partial answer: (One is) causality / positivity $\longrightarrow$ topic of this lectures

Comment: not every "naïve" EFT, generated according to (1), can be UV-completed in a QFT. Similar spirit to QG-swampland/landscape

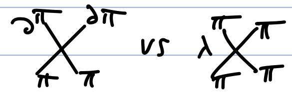
Let's generate some data point/examples (not exhaustive)

Light d.o.f.	$2\to2$ Energy scaling $m \ll E \ll M$	Lagrangian/Amplitude, UV-completions, comments, digressions,...
scalar ϕ spin-0	const	$\lambda\phi^4$
	$1/E^2$	$\mu\phi^3+\lambda\phi^4$; being relevant: $\mu^2/E^2 \gg \lambda$ vs X ; ϕ^3: dominate IR $m^2 \ll E^2 \ll \mu^2/\lambda$
	E^2 w/ flavor	non-abelian-GB $\sim (\partial\vec{\pi}\vec{\pi})^2/f^2$, $g e = e^{i\pi} e^{i\pi'} h$



Completes into QCD or linear-σ,... , before $\Lambda_s \sim 4\pi f$

irrelevant op. → dominate UV-region of EFT range:



$\lambda f^2 \ll E^2 \ll M^2 < (4\pi f)^2$ → $\lambda \ll 1$

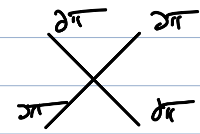
natural: λ breaks sym.

Known UV completions

unknown UV-completions

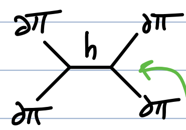
both "natural" EFT's — $E^4 > 0$

U(1)-GB: $\pi \to \pi + c$ $\qquad g_2 \frac{(\partial_\mu\pi)^4}{M^4}$:



again: dominant naturally by sym

UV-compl:



$\longrightarrow \quad g_2 = M^2/f^2 > 0$ $\qquad \frac{h}{f}(\partial\pi)^2$

supersoft, i.e. running fast with E: are they in the QFT-swamp? — E^6

Galileons: $\pi \to \pi + c + c_\mu x^\mu \qquad \partial_\mu\pi \to \partial_\mu\pi + c_\mu$

$\mathcal{L}^{Gal}$ (leading inv. up to total dr...) $\supset g\frac{(\partial\pi)^2\Box\pi}{M^3} \xrightarrow{\text{field redef.}} (\partial\partial\pi\,\partial\pi)^2 \frac{g^2}{M^6} \to \mathcal{M} \propto g^2 \frac{stu}{M^6}$

Goldstone equivalent theory for massive spin-2: $h_{\mu\nu} = \partial_\mu\partial_\nu\pi$ invariant

E^8

Generalized Galileons: $\pi \to \pi + c + c_\mu x^\mu + c_{\mu\nu} x^\mu x^\nu$

$\mathcal{L}_{\text{lead inv. up to } \partial(\ldots)} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3}{}_{\mu_4} \frac{g^2}{M^{10}} \partial_\sigma\pi(\partial_{\mu_1}\partial_{\nu_1}\partial^\sigma\pi)(\partial_{\mu_2}\partial_{\nu_2}\partial^\rho\pi)(\partial_{\mu_3}\partial_{\nu_3}\partial_\rho\pi)$

$\mathcal{M} \propto g^2 \frac{stu(s^2+t^2+u^2)}{M^{10}}$ $\qquad$ Goldstone equiv. spin-3

⋮

⋮

Comment on Approximate sym:

in an EFT one expects all higher-dim operators starting from the first — Lowest dim — that respects symmetries.
If the symmetry is approximate there is still a hierarchy of coefficients

Example: nearly galileon-invariant $\varepsilon \ll 1$

$$\mathcal{L} \sim (\partial\phi)^2 + \varepsilon\phi^4 + \varepsilon\frac{(\partial\phi)^4}{M^4} + \frac{(\partial\partial\phi)^2(\partial\phi)^2}{M^6} + \frac{(\partial\partial\phi)^4}{M^8} + \dots \tag{5}$$

$$\mathcal{M} \sim g^2\left(\varepsilon + \varepsilon\frac{E^4}{M^4} + \frac{E^6}{M^6}\left(1 + O(1)\frac{E^2}{M^2} + \dots\right)\right)$$

$\Downarrow$

Window where very irrelevant operators could dominate, naturally, within regime EFT: in the Galileon example

$$\boxed{M^2/\varepsilon < E^2 < M^2} \tag{6}$$

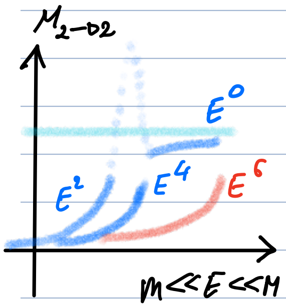
EFT - range of validity

as the symmetry become exact, $\varepsilon \to 0$, the range covers all EFT region.

Except that no UV-theory has ever be found that gives amplitudes that soft. The best "experimentally" found is starting from unsuppressed E^4.

Let's keep producing data for other spins:

Light d.o.f.	$2\to2$ Energy scaling $m \ll E \ll M$	Lagrangian/Amplitude, UV-completions, comments, digressions,...
fermion ψ spin-$1/2$	E^2	Fermi theory $(\psi^\dagger\bar\sigma^\mu\psi)^2/f^2$ $\quad \mathcal{M}(-++-) = \frac{\langle 14\rangle[23]}{f^2}$ $\propto \frac{u}{f^2}$; $m \ll E \ll M < \Lambda_s = 4\pi f$; unique op. dim-6 with chiral $U(1)$ (to suppress m) ; $\downarrow$ UV-completed by
	E^0	Gauge theory: $\mathcal{L} = \psi^\dagger i\not{D}\psi - F_{\mu\nu}^2$; $\mathcal{M}(-++-) = \frac{\langle 14\rangle[23]}{f^2}\,\underline{\mathcal{G}(s\leftrightarrow t)}$ $\to 1$ for $E \ll M$; $\to -\left(\frac{1}{s}+\frac{1}{t}\right)M^2$; ($\mathcal{G}$ analytic for $E^2 < M^2$ only)
	$E^4 > 0$	Goldstino: (non-abelian spinor shift sym.) ; $\mathcal{L}_g = \frac{g}{M^4}\psi^\dagger\bar\sigma^\mu\partial^\nu\psi\,\psi^\dagger\bar\sigma_\nu\partial_\mu\psi + \dots$ (unique op. with χ-sym & 2-deriv.) ; it respects $\psi \to \psi + \xi - i\partial_\mu\psi(\psi^\dagger\bar\sigma^\mu\xi - \xi^\dagger\bar\sigma^\mu\psi) + \dots$; Goldstino of susy-breaking: $U(\psi) = e^{i(\psi Q + \psi^\dagger Q^\dagger + x_\mu P^\mu)}$; $U(\psi) \xrightarrow{\text{susy}} g\,U\,h^{-1} = U'$ $\quad g = e^{i(\xi Q + \xi^\dagger Q^\dagger)}e^{ix_\mu P^\mu}$ (supercharges) ; $U^{-1}dU = dx^\mu E_\mu^a(P_a + \nabla_a\chi\,Q + \nabla_a\chi^\dagger Q^\dagger)$; $\int\mathcal{L}_{\text{Akulov-Volk}} : \int -F^2 \det E_\mu^a\, d^4x$ (invariant) $= \dots = \mathcal{L}_g\big|_{g=\frac{-1}{M^4F^2}>0}$; $E_\mu^a = \delta_\mu^a - i\chi^\dagger\bar\sigma^a\partial_\mu\chi + \text{h.c.}$
	E^6	Fermion shift-symmetric: $\psi \to \psi + \xi$; $\mathcal{L}_{\text{leading}} = (\partial\psi^\dagger\partial\psi^\dagger)(\partial\psi\partial\psi)\frac{g^2}{M^6}$ $\downarrow$
	$\vdots$ $E^{n>6}$	$\to$ unknown UV-completions



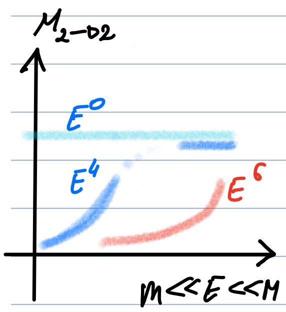
Known UV completions (blue)

unknown UV-completions (red)

TABLE 3

Light d.o.f.	$2\to 2$ Energy scaling $m \ll E \ll M$	Lagrangian/Amplitude, UV-completions, comments, digressions,...

Vectors A_μ, spin-1



Known UV completions

unknown UV-completions

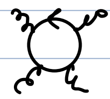
$E^4 > 0$

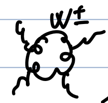
Euler-Heisenberg massless $A_\mu \to A_\mu + \partial_\mu \Omega$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{g_1}{M^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 - \frac{g_2}{M^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2$$

$$M(-++-) \propto \frac{\langle 14\rangle^2[23]^2}{M^4}(g_1+g_2)$$

$$M(----) \propto \frac{(s^2+t^2+u^2)}{M^4}(g_1-g_2)$$

UV-completion: QED-like  $\longrightarrow \begin{cases} g_1 = \left(\frac{g^2}{16\pi^2}\right)\frac{g^2}{90} > 0 \\ g_2 = \left(\frac{g^2}{16\pi^2}\right)\frac{7g^2}{360} > 0 \end{cases}$

(or spont. broken $SU(2)\to U(1)$: )

E^0 + E^2

YM $\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{g}\frac{f^{abc}}{M^2}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}$

$$M(-_a +_b +_a -_b) \sim g^2\langle 14\rangle^2[23]^2 (f_{abe})^2\frac{1}{st}$$

$$M(-+--) \sim g\bar{g}\frac{E^2}{M^2}$$

E^6

E^8

$\downarrow$

"Vector Galileons" $A_\mu \to A_\mu + c_{\mu\nu}x^\nu$ $\quad c_{\mu\nu} = -c_{\nu\mu}$

$$F_{\mu\nu} \to F_{\mu\nu} + 2c_{\mu\nu}$$

$$\mathcal{L}_{D=3} = -\frac{1}{g_3}\frac{F_{\mu\nu}^2}{2} + \bar{g}\left(F_{\mu\nu}^2\right)\frac{\varepsilon^{\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma}}{M^3} \qquad [g_3]=[M] \quad [A]=[M]$$

invariant up to tot. deriv.

dual to cubic galileon via $\varepsilon^{\mu\nu\rho}F_{\nu\rho} = \partial^\mu\phi$

$$\mathcal{L}_{D=4} = -F_{\mu\nu}^2 + g_4\frac{(\partial F)^4}{M^8}$$

$\vdots \ \downarrow$

Higher E^n by $A_\mu \to A_\mu + c_{\mu\nu_1\ldots\nu_n}x^{\nu_1\ldots\nu_n}$ $\quad c_{\mu\nu_1\ldots} = -c_{\nu_1\mu\ldots}$

What about Gravity?

$\boxed{\text{Tensor } h_{\mu\nu} \text{ spin}-2}$

(7) $$\mathcal{L}_{grav.} = -\sqrt{-g}\,\frac{M_{PL}^2}{2}\left(R + c_3\frac{Rie^3}{M^4} + c_4\frac{Rie^4}{M^6} + \dots\right)$$ Known UV completion: strings

in D=4 G.B. ($Rie^2+\dots$) missing

leading elastic amp. E^2/M_{PL}^2, specifically

(8) $$\underline{\mathcal{M}(-++-)} \propto \frac{\langle 14\rangle^4[23]^4}{stu}\frac{1}{M_{PL}}\left(1 + \ast\frac{stu}{M^6}c_4 + \ast\frac{s^2u^2}{M^8}c_3^2 + \dots\right)$$

$$= \frac{E^2}{M_{PL}^2}\left(1 + c_4\left(\frac{E}{M}\right)^6 + c_3\left(\frac{E}{M}\right)^8 + \dots\right)$$

(9) $$\underline{\mathcal{M}(-+++)} \propto \frac{(\langle 14\rangle[42][43])^4}{stu}\frac{c_3}{M_{PL}^2M^4}\left(1+\dots\right) = \frac{E^2}{M_{PL}^2}\left(c_3\left(\frac{E}{M}\right)^4 + \dots\right)$$

(10) $$\underline{\mathcal{M}(----)} \propto \left(\frac{\langle 12\rangle\langle 34\rangle}{[12]\,[34]}\right)^2\frac{1}{M_{PL}^2}\left(\frac{stu}{M^4}c_3 + \frac{(s^2+t^2+u^2)^2}{M^6}c_4 + \dots\right)$$

$$= \frac{E^2}{M_{PL}^2}\left(c_3\left(\frac{E}{M}\right)^4 + c_4\left(\frac{E}{M}\right)^6 + \dots\right)$$

As long as $\boxed{M \ll 4\pi M_{PL}}$ — weakly coupled UV completion — one may naively expect large c_n:

(11) $$c_n \sim \left(M_{PL}/M\right)^2 \gg 1 \quad ?$$

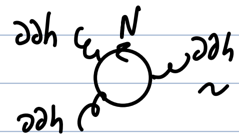
since $\left(E/M_{PL}\right)^2$-suppression in (9–11).

In fact, no string theory nor QFT example that generates $c_n \gg 1$:

(12) $$\underline{c_n \lesssim O(1)}$$ or in the swampland (by causality/positivity)

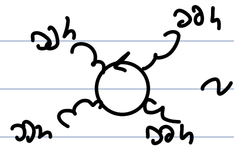
Power Counting: could have gone one way or the other, e.g.:

- **bread & butter**: c_3, c_4: N matter at $E = M$ min. coup.

(13) 

$\sim \frac{N}{16\pi^2} \frac{1}{M_{Pl}^3} \int d^4K \frac{K^3}{K^3} \frac{1}{(K^2)^3} \longrightarrow \frac{Rie^3}{M^2} \frac{N}{16\pi^2}$

vertex; prop.; ext. mom. pull out

(14) 

$\sim \frac{N}{16\pi^2} \frac{1}{M_{Pl}^4} \int d^4K \frac{K^4}{K^4} \frac{1}{(K^2)^4} \longrightarrow \frac{Rie^4}{M^4} \frac{N}{16\pi^2}$

$$\boxed{c_n = \frac{N}{16\pi^2}\left(\frac{M}{M_{Pl}}\right)^2 \lesssim 1}$$

at very best

- **Exotic "linear-diff" power counting** (for "composite" $h_{\mu\nu}$)

  linearized-$R_{\mu\nu\rho\sigma}$ invariant under $h_{\mu\nu} \to h_{\mu\nu} - \partial_{(\mu}\xi_{\nu)}$ linear diffs

  $\Rightarrow Rie^3$ unsuppressed w.r.t. R. $\Rightarrow$ leading $(+++)$ amp.

  The minimal coupling $(++-)$ generated by breaking to full diff. invariance $h_{\mu\nu} \to h_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)}$ so that R Einstein-Hilbert is generated.

Aside Comment: the curious case of "Remedios"

This is the gauge theory analog of composite $h_{\mu\nu}$ above, providing a power counting for composite transverse gauge bosons:

(15) $\underset{\text{gauge}}{U(1)^3} \rtimes \underset{\text{global}}{SO(3)}$: $\mathcal{L} = -\frac{1}{4}F^{a\,2}_{\mu\nu} + \bar{g}\frac{\varepsilon_{abc}}{M^2} F^a_{\mu\nu} F^b_{\nu\rho} F^c_{\rho\mu}$ $F^a_{\mu\nu} \equiv \partial_{[\mu} A^a_{\nu]}$

so(3)-flavor; abelian

$\mathcal{M}(---)$, $\mathcal{M}(+++)$ unsuppressed; minimal coupl. $(--+)$ vanishing.

Adding $(--+)/(++-)$ corresponds to gauge the $SU(2)$: $F^a_{\mu\nu} \to \mathcal{F}^a_{\mu\nu} = [D_\mu, D_\nu]^a$ with $g \ll \bar{g}$. Contrary to gravity, remedios are ok with positivity.

— Massive Spin EFT —

Clearly, the story so far covers also massive spinning particles: since we are interested in the high-energy limit, one can focus on the various "eaten GBs"

(16) massive vectors: $3\,\text{d.o.f} = A_\mu^a \longrightarrow \partial_\mu \pi^a + A_\mu^{a\perp} = 1 + 2$

massive tensors: $5\,\text{d.o.f} = h_{\mu\nu} \longrightarrow \underset{1}{\partial_\mu\partial_\nu \pi} + \underset{2}{\partial_{(\mu} A_{\nu)}} + \underset{2}{h_{\mu\nu}^\perp}$

$\vdots$

It's clear that $\pi \to \pi + c_\mu x^\mu$ for massive spin-2 states, so that constraining Galileons constraints massive spin-2 EFT, too.

Example: Stuckelberg for non-abelian massive vectors

(17) $$\frac{m^2}{2} A_\mu^{a\,2} \xrightarrow[\substack{\text{un-eat GB's} \\ \text{(stuckelberg trick)}}]{\text{restore gauge}} \frac{f^2}{2}\Big(g A_\mu^a - i\underbrace{U^{-1} dU^a}_{d_\mu^a}\Big)^2 \longrightarrow \frac{(\partial\pi^a)^2}{2} + \frac{1}{f^2}\,\partial\pi\partial\pi\,\pi\pi + \dots$$

$f \equiv m/g$

Maurer-Cartan 1-form: $d_\mu^a = \partial_\mu \pi^a + \frac{1}{2} f^{abc}\pi^b \partial_\mu \pi^c + \dots$

Example: Fierz-Pauli tuning

(18) $$\frac{m^2}{2}\left[a\, h_{\mu\nu}^2 + b\, h^2\right] \xrightarrow[\text{restore diffs}]{} \frac{m^2}{2}\left[a\,(h_{\mu\nu} - \partial_\mu\partial_\nu\pi + \dots)^2 + b\,(h - \Box\pi)^2\right]$$

no-ghost: $a\,(\partial_\mu\partial_\nu\pi)^2 + b\,(\Box\pi)^2 = 0$ (up to t.d.) $\Longrightarrow$ $a = -b$

Resolving kinetic mixing, $\propto m^2$, generate π K.T. for $\pi \propto m^2$
$\Longrightarrow$ Galileon interactions scale as $\frac{(\partial\pi)^2\Box\pi}{m^2 M_{Pl}}$ (at best)

— Pheno? —

While understanding the space of EFT is by itself interesting, it has also a nice Pheno application, e.g. at collider.
Contrast for example these two paradigms for

Composite Fermions

Chiral composite

$g_*^{(6)} \gtrsim O(1)$, $M \gg v_{EV}$

$m_\psi \ll M$ by chiral sym

$$\mathcal{L}_{\chi\text{-sym}} = \frac{g_*^{(6)\,2}}{M^2} (\psi^\dagger \psi^\dagger)(\psi\psi)$$

(in partial compositeness picture generated by mixing $\psi\, \mathcal{O}_{strong}$)

Supersoft composite

$g_*^{(10)} \gtrsim O(1)$, $M \gg v_{EV}$

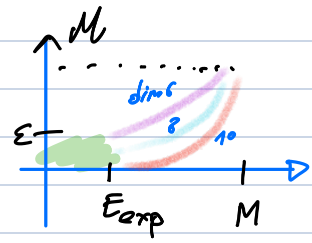
$m_\psi \ll M$ by $\psi \to \psi + \xi$

$$\mathcal{L}_{\text{shift sym}} = \frac{(g_*^{(10)})^2}{M^6} (\partial\psi^\dagger \partial\psi^\dagger)(\partial\psi\partial\psi)$$

(In partial compositeness picture generated by mixing $\partial\psi\, \mathcal{O}^\mu_{strong}$)

dim-6 vs dim-10

Experimentally big difference:



• Fixed $M^{(6)} = M^{(10)} = M$ different $g_*^{(6)} \neq g_*^{(10)}$ ⟵ (either M closer or g_* larger!)

(19) $$(g_*^{(6)})^2 \left(\frac{E_{exp}}{M}\right)^2 < \varepsilon_{exp} \xrightarrow{\text{bound rescaled}} (g_*^{(10)})^2 \left(\frac{E_{exp}}{M}\right)^6 < \left(\frac{g_*^{(6)} E_{exp}}{M}\right)^2\Bigg|_{exp} \to g_*^{(10)} < g_*^{(6)}\Big|_{exp} \left(\frac{M}{E_{exp}}\right)^3$$

say $E_{exp} \simeq M_{jj}\text{-cut} \sim$ TeV $\quad M \sim 2$ TeV $\Rightarrow$ $\mathcal{O}(10)$-large coupling!

($E_{exp} < M$ by EFT assumption)

Positivity will rule out theoretically dim-10, but not Goldstini-compositeness dim-8

Main Lessons:

- E^4-coefficients positives
- $E^{n\geq 6}$ never leading

) despite symmetries would allow them!

$\Downarrow$

There is a new, missing, ingredient to build EFT that live in the QFT-landscape (or string-landscape if gravity)