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Positivity Constraints on EFT

We have seen in L1 and L2 that subluminality/causality provides interesting positivity bounds on EFT coefficients. It's shown however also some of its limitations:

- no systematics in building bkg's where to look for superluminality
- "easy" for lowest dim. operators, what about higher-dim operators?

In this L3 we recast these positivity bounds in terms of scattering amplitudes, which allow to draw (often, not always) more general conclusions.

— Subluminality ⟹ Microcausality —

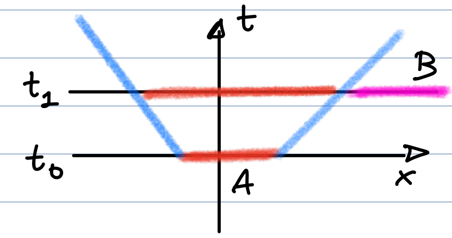
Let's first consider a classical field $\phi(t,\vec{x})$ specified at some time slice $t=t_0$ along with its time derivative

$$(1) \qquad t = t_0 \qquad \phi(t_0,\vec{x}) = \phi_0(\vec{x}) \qquad \dot{\phi}(t_0,\vec{x}) = \dot{\phi}_0(\vec{x})$$

Solving its e.o.m. we get $\phi(t,\vec{x})$ and $\dot{\phi}(t,\vec{x})$ at late times as functionals of the initial conditions $\phi_0(\vec{x})$, $\dot{\phi}_0(\vec{x})$

$$(2) \qquad \phi(t,\vec{x}) = \phi[\phi_0,\dot{\phi}_0](t,\vec{x}) \qquad \dot{\phi}(t,\vec{x}) = \dot{\phi}[\phi_0,\dot{\phi}_0](t,\vec{x})$$

Subluminality means that varying the initial conditions in some region A it will not affect the solution in B, if A x B are spacelike



(3) $$\frac{\delta\phi(t,x)}{\delta\phi(t_0,y)}\bigg|_{\substack{(t_0,\vec y)\in A\\ (t,\vec x)\in B}} = 0 = \frac{\delta\phi(t,x)}{\delta\dot\phi(t_0,y)}$$

This can be written in terms of Poisson breckets

second $\qquad\qquad\qquad\qquad$ first

(4) $$\{\phi(t,\vec x),\ \phi(t_0,\vec y)\}\Big|_{\text{spacelike}} = 0 \quad \text{and} \quad \{\phi(t,\vec x),\ \dot\phi(t_0,\vec y)\}\Big|_{\text{spacelike}} = 0$$

$\Downarrow$ canonical quantization

(5) $$\boxed{[\phi(t,\vec x),\ \phi(t_0,\vec y)]\Big|_{\text{spacelike}} = 0}$$ Micro-causality

— Micro-causality $\xrightarrow{LSZ}$ Analyticity —

Elevating (5) to axiom for any local indipendent op., The 2-to-2 scattering amplitudes can be written via LSZ formula in terms of the retarded commutators which thus vanish at spacelike separation (as well as in the past):

(6) $${}^{out}\langle 3\,4|1\,2\rangle^{in} - {}^{in}\langle 3\,4|1\,2\rangle^{in} = i\,\delta^4(K_1+K_2-K_3-K_4)\,M(12\to 34)$$

(7) $$\boxed{\underbrace{M(12\to 34)}_{\text{LSZ 2\&4}} \propto \int d^4x\, e^{i(K_2+K_4)\frac{x}{2}}\,\theta(x^0)\,\langle 3|\,[\underbrace{J(x)}_{(\Box+m^2)\phi},\ \underbrace{J^\dagger(0)}_{(\Box+m^2)\phi^\dagger}]\,|1\rangle}$$

Aside comment:

The LSZ (7) differs by LSZ with T-ordered operators by terms proportional to $(3 \leftarrow \text{anything})$ and $(1 \rightarrow \text{anything})$, which vanish by stability of 3 and 1. This follows by the identity

$$(8) \qquad T\,J(x)\,J^\dagger(0) = \theta(x^0)\,[J(x), J^\dagger(0)]_\mp \pm J^\dagger(0)\,J(x)$$

and by inserting in the last term a complete set of states. See e.g. Weinberg chap. 10 for a simple proof. Or LSZ-I & -II paper

$\left(LSZ_{2+4} \langle 3|\phi^\dagger(0)|n\rangle\langle n|\phi(x)|1\rangle \propto \sum_n M(2+n\to 3)\, M(1\to n+4) = 0 \right)$

The interesting point of (7) is that allows analytic continuation to complex momenta, which are key to positivity.

Forward elastic scattering

Let's discuss the simplest and paradigmatic example of forward elastic scattering:

$$(8) \qquad M(12\to 12) = \int d^4x\, e^{ik_2 x}\, \theta(x^0)\,\langle 1|[J(x), J^\dagger(0)]|1\rangle \qquad \begin{matrix} k_1 \leftrightarrow k_3 \\ k_2 \leftrightarrow k_4 \end{matrix}$$

without lost of generality let's take $\vec{k}_i$ along the $\hat{x}$ direction, and move to light-cone coordinates

$$(9) \quad \begin{cases} u = x^0 - x^1 \qquad v = x^0 + x^1 \\ ds^2 = du\,dv - d\vec{x}_\perp^{\,2} \end{cases} \qquad g_{\mu\nu} = \left(\begin{array}{cc|c} 0 & 1/2 & \\ 1/2 & 0 & 0 \\ \hline & 0 & -\mathbb{1} \end{array} \right) \qquad g^{\mu\nu} = \left(\begin{array}{cc|c} 0 & 2 & \\ 2 & 0 & 0 \\ \hline & 0 & -\mathbb{1} \end{array} \right)$$

(10) $$\mathcal{M}(12 \to 12) = \int du\, dv\, e^{i\frac{(k^u v + k^v u)}{2}} \left[\int d^2 x_\perp\, \theta(x^0) \langle 1 | [J(x), J^\dagger(0)] | 1 \rangle \right]$$

where we used the coordinate choice $\vec{k}_\perp = 0$. $\xrightarrow{1}\ \xleftarrow{2}$ ($k = k_2$ here)

Paradigmatic Example of Analytic Extension from causality

(11) $$\hat{f}(k) = \int dx\, e^{ikx}\, \theta(x)\, f(x) \qquad k \in \mathbb{R}$$

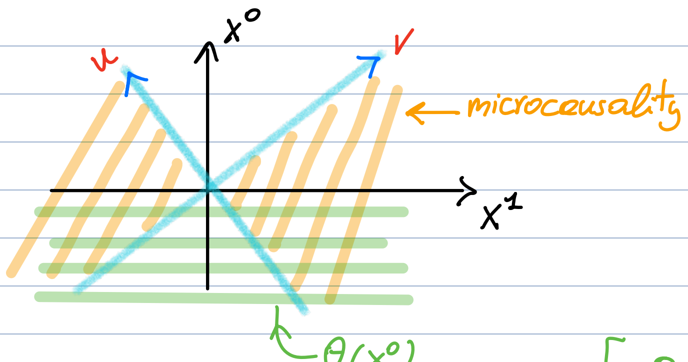
$$\Downarrow$$

(12) $$\hat{f}(p = k + iq) \equiv \int dx\, e^{ipx}\, \theta(x)\, f(x) = \int dx\, e^{ikx \underbrace{- qx}_{\text{exp. dumping factor}}}\, \theta(x)\, f(x)$$

$q \geq 0$

ANALYTIC Extension
upper plane

The (10) is just the 2D-version of this: the integrand inside [...] vanishes by microcausality + $\theta(x^0)$ at

(13) 

Support of integrand in (10)
$u > 0$ & $v \geq 0$

[$0 < x^2 = uv - \vec{x}_\perp^{\,2} \Rightarrow uv > \vec{x}_\perp^{\,2}$ with $\vec{x}_\perp$ integrated over in (10)]

(14) $$\boxed{\mathcal{M}(12 \to 12) = N \cdot \int_0^\infty du \int_0^\infty dv\, e^{i\frac{(k^u v + k^v u)}{2}} \left[\int d^2 x_\perp\, \theta(x^0) \langle 1 | [J(x), J^\dagger(0)] | 1 \rangle \right]}$$

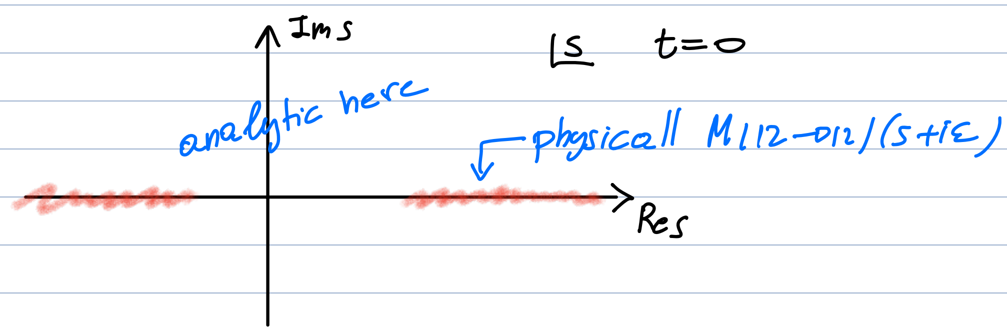
defines Analytic Extension to complex $k^{u,v}$ with

(15) $\mathrm{Im}\, K^{u,v} \geq 0$ upper planes

What does that mean for analyticity in the Mandelstam variables? Since $M(12 \to 12)$ is a scalar, it implies analyticity in s (at $t=0$). For instance, let's take at least one of them massive for concreteness, say 1, $K_1^2 = m_1^2 \Rightarrow$ in its c.o.m. $2K_1 \cdot K_2 = 2m K_2^0 = m(K_2^u + K_2^v)$. This is actually general:

(16) $\boxed{M(12 \to 12) \text{ analytic in Mandelstam-}s \text{ for } \mathrm{Im}\, s \geq 0,\ t=0}$

The $M(12 \to 12)$ is the boundary value $\mathrm{Im}\, s \to 0^+$ for $\mathrm{Re}\, s > 0$



Remark: if 2 is massless $K_2^2 = K_2^u K_2^v = 0 \to K_2^v = 0$, the analytic extension in K_2^u is perfectly consistent with the on-shell condition $m_2^2 = 0$. On the other hand, for $m_2 \neq 0$ one needs to work harder to extend the primitive domain of analyticity (15) & ( ) to include the mass-shell. this is possible, but not covered in this lectures.

From the "time-symmetric" version of (8), namely

$$(17) \qquad T\,J(x)\,J^\dagger(0) = -\,\theta(-x^0)\,[J(x), J^\dagger(0)]_\mp \pm J(x)\,J^\dagger(0)$$

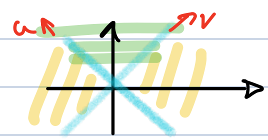
it follows (again observing the last term in (17) drops by stability)

the Advanced-commutator version of LSZ reduction formula:

$$(18) \quad \mathcal{M}(1\bar{2} \to 3\bar{4}) \underset{\text{LSZ } \bar{2}\,\&\,\bar{4}}{\propto} -\int d^4x\, e^{-i(k_2+k_4)\frac{x}{2}}\,\theta(-x^0)\,\langle 3|\,[J(x), J^\dagger(0)]\,|1\rangle$$

for the crossed process $1\bar{2} \to 3\bar{4}$ (equivalently $\bar{1}2 \to \bar{3}4$) with antiparticles (this is required by requiring dropping $\pm J(x)J^\dagger(0)$ in (17)).
Mutatis mutandis, up to $x \to -x$ this is analytic in upper $k_2^{u,v}$ plane (of course, whether we call it $\bar{2}$ or 2 doesn't matter!)
The interesting point, however, is to see it as function of $k^\mu = -(k_2+k_4)^\mu/2 \underset{\text{forward}}{=} -k_2^\mu$

$$(19) \quad \mathcal{M}(1\bar{2} \to 1\bar{2})(k=-k_2) = -N\int d^4x\, e^{+ik\cdot x}\,\theta(-x^0)\,\langle 1|\,[J(x), J^\dagger(0)]\,|1\rangle$$

analytic in lower $k^{u,v}$-plane 

This is useful because the difference with $\mathcal{M}(12\to 12)(k)$ at the common boundary, real k^μ, is just the F.T. $[\,,\,]$:

$$(20) \quad \mathcal{M}(12 \to 12) - \mathcal{M}(1\bar{2} \to 1\bar{2}) = N\int d^4x\, e^{ik\cdot x}\,\langle 1|\,[J(x), J^\dagger(0)]\,|1\rangle$$

and the r.h.s vanishes for real values of K (or s) "below treshold", when no intermediate state contributes:

$$(21)\quad \int d^4x\, e^{iKx} \langle 1| J(x) \overbrace{}^{|n\rangle\langle n|} J^\dagger(0)|1\rangle = \sum\!\!\!\!\!\!\int_n \int d^4x\, e^{+i(K+P_1-P_n)x} |\langle 1|J(0)|n\rangle|^2$$

$$= \sum\!\!\!\!\!\!\int_n (2\pi)^4 \delta^4(K+P_1-P_n) |\langle 1|J(0)|n\rangle|^2$$

$$(22)\quad \int d^4x\, e^{iKx} \langle 1| J^\dagger(0) J(x)|1\rangle = \ldots = \sum\!\!\!\!\!\!\int_n (2\pi)^4 \delta^4(K+P_1-P_n) |\langle n|J(0)|1\rangle|^2$$

(with $\sum\!\!\!\!\!\!\int \propto \int \prod_n (2\pi)\delta(p_n^2-m_n^2)\,\theta(p_n^0)\ldots$)

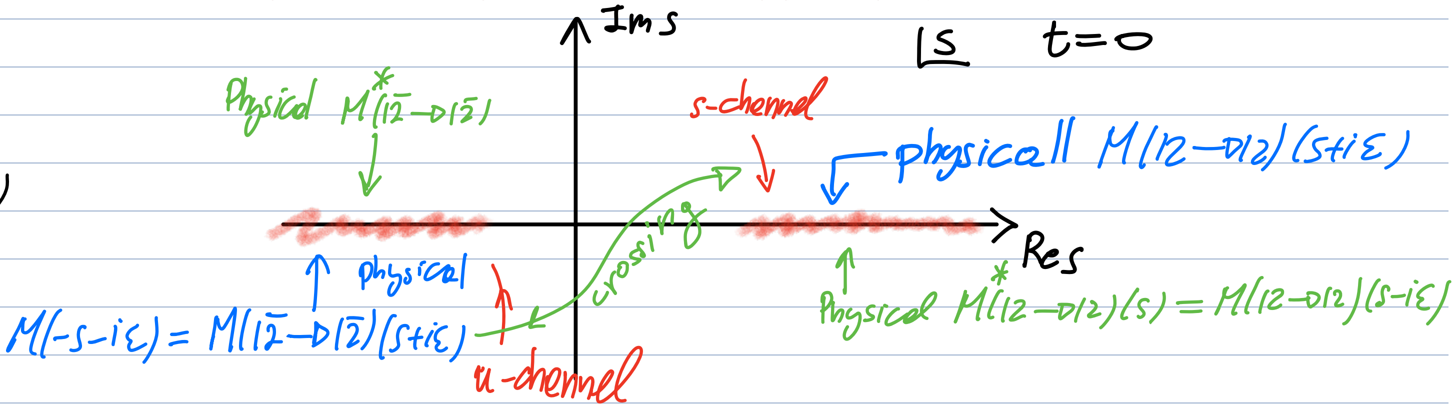
so that if one takes e.g. $K^0+P_1^0=\sqrt{s} < m_n^0 \leq p_n^0$ indeed the r.h.s. of (20) has no support.

We have thus two analytic functions that agree on a common boundary on the real axis below threshold: by the Morera's theorem they define a unique function analytic in both upper and lower s-plane

$$(23)\quad \boxed{M(s) = \begin{cases} M(12\to12)(s) & \operatorname{Im} s > 0 \\ M(1\bar{2}\to1\bar{2})(\underbrace{-s}_{\text{"}u\text{"}}) & \operatorname{Im} s < 0 \end{cases}}$$

Analytic except where $[\,,\,]$ has support on Re s

(24)



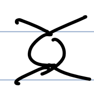
Remarks

- The amplitude satisfies also a reality condition

$$(25) \qquad \mathcal{M}^*(s^*) = \mathcal{M}(s) \qquad \text{hermitian analyticity}$$

This follows directly from definitions, or equivalently from Schwartz reflection principle + (24) being satisfied on the real axis below threshold.
(For particles with spin: $\mathcal{M}(1^{\lambda_1}2^{\lambda_2} \to 3^{\lambda_3}4^{\lambda_4})^*(s-i\varepsilon, t) = \mathcal{M}(3^{\lambda_3}4^{\lambda_4} \to 1^{\lambda_1}2^{\lambda_2})(s+i\varepsilon, t)$)

- The statements (23 + 24) are basically crossing symmetry in the special kinematics $t=0$
In the absence of mass gap, when the branch cuts close and separate the planes, crossing symmetry is taken as assumption

- Analiticity in s holds also for negative values (physical) of $t<0$, as long as its not too negative (e.g. $\pi\pi \to \pi\pi$ is s-analytic for $O(100)m_\pi^2 < t < 0$)
The proofs are rather cumbersome. For scattering the lightest state in the theory is conjectured Maximal Analyticity where $0 < -t < |s|$ (or M^2 in EFT).
Extension to $0<t$ can reach $t \le 4m_\pi^2$ for instance.
After that  does not allow to go further.

Analyticity $\xrightarrow{\text{Unitarity + locality}}$ Positivity

Let's add now 2 more assumptions/facts:

(26) Decay-rate of amplitudes: $\lim_{|s|\to\infty} \frac{\mathcal{M}(12\to12)(s)}{s^2} = 0$

(weak form of locality)

(27) Unitarity: $S^\dagger S = S S^\dagger = \mathbb{1}$

- The (26) is actually a theorem in axiomatic QFT for gapped theories known as the Froissart-Martin bound

(28) $\mathcal{M} \sim s \log^2 s$ at large s. gapped theory.

Gravity actually marginally violates (26), but the following still holds

(29) $\frac{\mathcal{M}(s, t<0)}{s^2} \xrightarrow[s\to\infty]{} 0$ in Gravity (Zhiboedov-Häring)

(the (29) is proven by observing that for t-fixed $|t|/|s| \to 0$ so that one enters always the eikonal gravitational regime (see also 2211.00085), where the resummation of leading ladder diagrams confirms (29): see 2202.08280 for detailed discussion)

- The (27) implies the optical theorem, which in its $t\to0$ is

$$(30)\quad \boxed{\frac{\operatorname{Disc}\mathcal{M}(12\to12)(s)}{i} \equiv \frac{\mathcal{M}(12\to12)(s+i\varepsilon) - \mathcal{M}(12\to12)(s-i\varepsilon)}{i} \underset{(25+27)}{=} \langle 12|\mathcal{M}^\dagger \mathcal{M}|12\rangle \geq 0}$$

— Positivity in $\pi\pi \to \pi\pi$ —

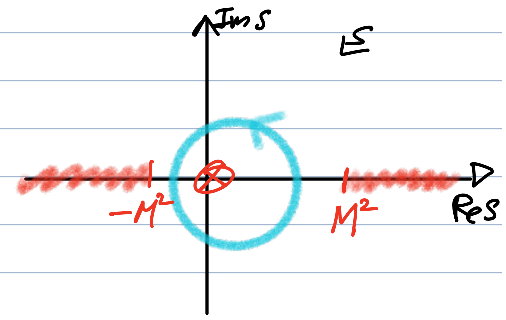
Let's consider single-GB's EFT with vanishing or negligible mass (keeping: analyticity + crossing + unitarity + energy-rate/locality)

(31) $$\mathcal{M}(12\to 34) \underset{E\ll M}{=} \frac{c_2}{2M^4}(s^2+t^2+u^2) + \frac{c_3}{M^6}\, stu + \frac{c_4}{4M^8}(s^2+t^2+u^2)^2 + \dots$$

(32) $$\mathcal{M}(12\to 12)(s,t\to 0) \underset{E\ll M}{=} c_2\, s^2 + c_4\, s^4 + \dots$$ ← only even powers

Thanks to analyticity, one can extract c_{2n} coefficients via Cauchy theorem

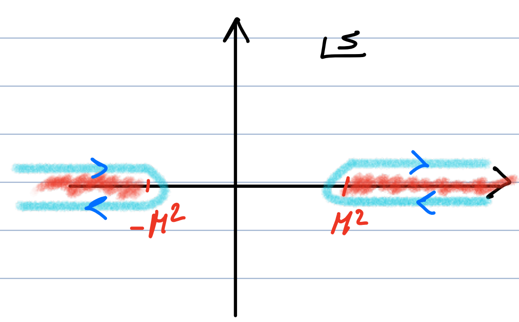
(33) $$\boxed{\frac{c_{2n}}{M^{4n}} = \frac{1}{2\pi i}\oint \frac{\mathcal{M}(s,t=0)}{s^{2n}}\,\frac{ds}{s}}$$



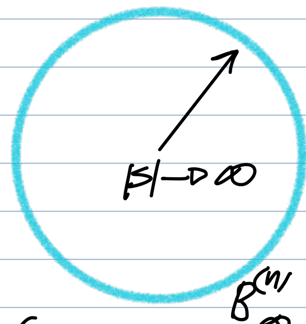
if theory gapped, or IR branch-cuts below M negligibles (that is ignoring IR running from $E=M$ to the scale of the EFT)

The (33) is cool because we can deform the contour to wrap around the Disc's on the real axis:

(34)



dropping for $n\geq 1$ thanks to (26).



(35) $$\frac{C_{2n}}{M^{4n}} = \frac{1}{\pi}\left(\int_{M^2}^{\infty}\frac{ds}{s} + \int_{-\infty}^{M^2}\frac{ds}{s}\right)\frac{M(s+i\varepsilon)-M(s-i\varepsilon)}{2i} + \underset{\displaystyle 0}{\underset{\downarrow}{B_\infty^{(n\geq 1)}}}$$

$$\underset{\text{in } 2^\circ \text{ int.}}{\overset{s\to -u}{=}} \frac{1}{\pi}\int_{M^2}^{\infty}\frac{ds}{s}\,\frac{M(s+i\varepsilon)-M(s-i\varepsilon)}{2i} + \int_{M^2}^{\infty}\frac{du}{u}\,\underbrace{\frac{M(-u-i\varepsilon)-M(-u+i\varepsilon)}{2i}}$$

$M_{1\bar{2}\to 1\bar{2}}(u+i\varepsilon) - M_{1\bar{2}\to 1\bar{2}}(u-i\varepsilon)$ by crossing: in our case moreover $M_{1\bar{2}\to 1\bar{2}} = M_{12\to 12}$

$$\underset{\text{crossing + rename } u\to s}{=} \frac{2}{\pi}\int_{M^2}^{\infty}\frac{ds}{s}\,\frac{M(s+i\varepsilon)-M(s-i\varepsilon)}{2i}$$

$$= \frac{2}{\pi}\int_{M^2}^{\infty}\frac{ds}{s}\,\frac{\mathrm{Disc}\,M(s)}{2i} \underset{\text{unitarity (33)}}{=} \frac{1}{\pi}\int_{M^2}^{\infty}\frac{ds}{s}\,\langle 12|M^\dagger M|12\rangle \geq 0$$

(36) $$\boxed{C_{2n} \geq 0}$$ positivity of s^{2n}-coefficients

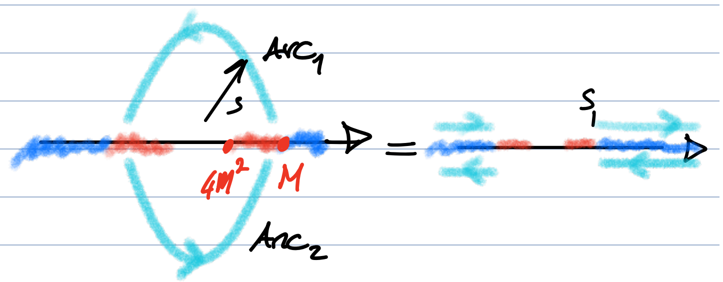
Remarks:

- The $=0$ sign is reached only in free theory since $\sum_n |\langle 12|M^\dagger|n\rangle|^2 = 0$ implies $\langle 12|M^\dagger|n\rangle = 0 = |\text{anything}\rangle \to |12\rangle = 0$

- In the presence of IR loops that one wants to keep into account it's actually better to define "arcs$_n$"$(s) = a_n(s)$

"Arcs"

(37) $$\boxed{a_n(s) \equiv \frac{1}{2\pi i}\int_{\substack{\text{Arcs}_{1+2}\\ \text{of radius } s}} \frac{M(s')}{(s')^n}\frac{ds'}{s'} \geq 0}$$



which by analyticity are related to the coefficients $c_{2n}^{(m^2_{IR})}$ defined by expanding $\mathcal{M}(12\to12)$ below threshold (where is analytic) as

$$(38) \quad a_{2n}(s) = \frac{c_{2n}^{(m^2_{IR})}}{M^{4n}} - \#^2 \int_{m^2_{IR}\ (\text{e.g. } 4m^2)}^{s} \langle 12 | M^\dagger M | 12 \rangle \to \frac{c_{2n}^{(m^2_{IR})}}{M^{4n}} > a_{2n}$$

More generally, $a_n(s)$ is calculable in EFT in terms of Wilson coefficients matched at some μ, so the (37) will represent some positivity conditions on combinations of Wilson coefficients.

Notice, however that $c_2 > 0$ can't be undone in any way (keeping EFT fixed, without adding new light d.o.f's) since

$$(39) \quad \mathcal{M}(12\to12) = \frac{c_2}{M^4} s^2 + \Big(\underset{\uparrow\, -7/10}{\#} \frac{c_2^2}{16\pi^2} \log s/\mu^2 + c_4(\mu)\Big) \frac{s^4}{M^8} + \dots$$

$$(40) \quad a_2(s) = \frac{1}{2\pi i} \oint \frac{ds'}{s'} \frac{\mathcal{M}(12\to12)}{s'^2} \sim \frac{c_2}{M^4} - \frac{\#}{\pi} \int_{m^2_{IR}}^{s} \frac{\text{Disc}}{i} \frac{1}{s'^2} \frac{ds'}{s'}$$

$\to \frac{s^2 c_2^2}{M^8 16\pi^2}$

$\frac{c_2^2 \cdot \cancel{\pi}}{16\pi^2} \frac{s^2}{M^8} \frac{1}{\cancel{\pi}}$

$$= \frac{c_2}{M^4}\Big(1 - \# \underbrace{\Big(\frac{c_2}{16\pi^2}\Big) \frac{s^2}{M^4}}_{\ll 1} + \dots\Big)$$

$\underline{c_2 > 0}$ can't reverse the sign in EFT

The story is not as simple for the higher Wilson coefficients, e.g.

$$(41) \quad M^8 a_4(s) = \underbrace{c_4 + \# \frac{c_2^2}{16\pi^2} \log s}_{\text{running } c_4(s) \geq 0} + \dots \quad \text{but: } M^{12} a_6(s) = \underbrace{\frac{c_2^2 M^4}{s^2\, 16\pi^2}}_{\text{dominated by IR!}} + c_6 + \dots$$

($M^{2n} a_n = c_n + \frac{c_2^2}{16\pi^2} \frac{M^{2n-4}}{s^{n-2}} + \dots \xrightarrow[\text{tree} \gg \text{loop}]{} \frac{c_2^2}{16\pi^2} \ll \left(\frac{s}{M^2}\right)^{n-2} \longrightarrow$ eventually violated
no matter how small the coupling was, since $s/M^2 \ll 1$)

— Positivity in Euler-Heisenberg —

Let's study another case with spin: the theory of $U(1)$ gauge bosons below the mass of lightest charged state:

$$(42) \quad \mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a}{M^4} \left(F_{\mu\nu} F^{\mu\nu}\right)^2 - \frac{b}{M^4} \left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right)^2 + \dots$$

$$(43) \quad \begin{cases} M(1^- 2^+ 3^+ 4^-) = \frac{\langle 14 \rangle^2 [23]^2}{M^4} F_{-+}(s,t) \\ F_{-+}(s,t) = F_{-+}(t,s) \underset{IR}{=} a + b + \dots \end{cases}$$

helicity preserving $1^- 2^+ \to 3^- 4^+$

$s \leftrightarrow t$ symmetric

$$(44) \quad \begin{cases} M(1^- 2^- 3^- 4^-) = \left(\frac{\langle 12 \rangle \langle 34 \rangle}{[12]\,[34]}\right) \frac{(s^2 + t^2 + u^2)}{M^4} F_{--}(s,t,u) \\ F_{--}(s,t,u) \underset{IR}{=} a - b + \dots \end{cases}$$

fully s-t-u symmetric

Once the little-group structure is removed, the F_{-+} and F_{--} have exactly the same analyticity properties as in the scalar theory. So it's actually immediate to get $a + b > 0$ since

$$(45) \quad \frac{1}{2\pi i} \oint \frac{M(1^- 2^+ 3^+ 4^-)\, M^4}{\langle 14 \rangle^2 [23]^2} \frac{ds}{s} \Bigg|_{t=0} = a + b > 0$$

$$\parallel$$

$$\frac{2}{\pi} \int_{M^2}^{\infty} \frac{ds}{s} \frac{1}{s^2} \frac{\text{Disc}\, M(1^- 2^+ 3^+ 4^-)}{2i} > 0$$

In fact, one can do better since we can scatter any state we like, e.g. linearly polarized states

(46) $$|\uparrow\uparrow\rangle \longrightarrow |\uparrow\uparrow\rangle \quad \text{or} \quad |\uparrow\downarrow\rangle \longrightarrow |\uparrow\downarrow\rangle$$

or any linear combination in between. Since these are still elastic the r.h.s. of the dispersion relation is still delivering a positivity:

(47) $$\langle\uparrow\uparrow| M^{\dagger}M |\uparrow\uparrow\rangle \geq 0$$
$$\langle\uparrow\downarrow| M^{\dagger}M |\uparrow\downarrow\rangle \geq 0$$

after all as matrix! $\boxed{M^{\dagger}M \geq 0}$

What's interesting about this is that it will probe also the inelastic-helicity configuration

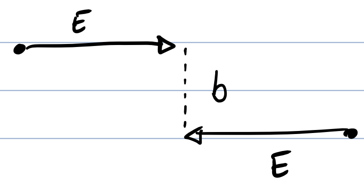
(48) $$|\uparrow\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \qquad |\downarrow\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}i}$$

(which are a nice basis under crossing since $|\uparrow\rangle \leftrightarrow |\uparrow\rangle$, $|\downarrow\rangle \leftrightarrow |\downarrow\rangle$ at $t=0$ when $3 \to 1$, $4 \to 2$)

(49) $$4 \cdot \langle\uparrow\uparrow| M |\uparrow\uparrow\rangle\Big|_{t=0} = M(1^+2^+1^+2^+) + M(1^+2^+1^+2^-) + \dots$$

(50) $$\begin{cases} \langle\uparrow\uparrow| M |\uparrow\uparrow\rangle\Big|_{\substack{t=0\\ IR}} = 16\,\dfrac{a}{M^4} \\ \langle\uparrow\downarrow| M |\uparrow\downarrow\rangle\Big|_{\substack{t=0\\ IR}} = 16\,\dfrac{b}{M^4} \end{cases} \Longrightarrow \boxed{\begin{matrix} a \geq 0 \\ b \geq 0 \end{matrix}}$$

- Heuristic Derivation   Froissart Bound



coupling: $g = \begin{cases} \text{strength:} & \lambda E^{n} \\ \text{locality:} & e^{-mb} \end{cases}$ m: mass of mediator

Relevant range of b: $e^{-m b^{max}} E^{n} \sim O(1) \Rightarrow \boxed{b^{max}(E) = \frac{n}{m} \log E}$

$\Rightarrow \boxed{\sigma \sim (b^{max})^2 \sim \frac{1}{m^2} \log^2 E}$

main lesson: locality is crucial ingredient $\left\{ \begin{array}{l} \text{coupling growth } E\text{: polynom.} \\ \text{decoupling distance: expon.} \end{array} \right.$

$\Rightarrow$ range interact. $b \sim \log E$

— S-matrix Heuristic Derivation —

trade: $b \longleftrightarrow \ell = E \cdot b$ angular momentum $\rightarrow$ Partial Waves

$$\boxed{M(s,t) = 16\pi \sum_\ell (2\ell+1)\, a_\ell(s)\, P_\ell(\underset{1+\frac{2t}{s}}{\underset{\shortparallel}{\cos\theta}})}$$

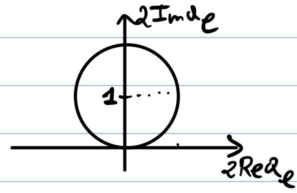
unitarity:

$$s\,\sigma_{TOT}(s) \underset{\text{opt. th.}}{\propto} \mathrm{Im}\, M(s, t=0) = \sum_\ell (2\ell+1)\, \mathrm{Im}\, a_\ell(s)$$

$$|s_\ell|^2 = |e^{2i\delta_\ell}|^2 = |1+2i\,a_\ell|^2 \leq 1 \qquad a_\ell \equiv \frac{e^{2i\delta_\ell(s)} - 1}{2i}$$

$$0 \leq |a_\ell|^2 \leq \mathrm{Im}\, a_\ell \leq 1$$

$$\left\lfloor (1-2\,\mathrm{Im}\,a_\ell)^2 + (2\,\mathrm{Re}\,a_\ell)^2 \leqslant 1 \right.$$



unitarity $0 \leqslant \mathrm{Im}\,a_\ell \leqslant 1$ not enough $\mathrm{Im}\,M(s,t=0) \leqslant 16\pi \sum_\ell (2\ell+1) = \infty$

needed: $\boxed{\text{decoupling large } b \longleftrightarrow \text{decoupling large-}\ell}$

$$\frac{\mathrm{Im}\,M(s,t=0)}{16\pi} \leqslant \underbrace{\sum_\ell^{\ell_{max}} (2\ell+1)}_{\ell_{max}^2} + \sum_{\ell > \ell_{max}}^{\infty} (2\ell+1)\,\mathrm{Im}\,a_\ell(s)$$

need to estimate $\mathrm{Im}\,a_\ell(s)$ at large ℓ

$\boxed{* \quad M(s,t) \text{ is analytic in } s \ \& \ t, \text{ even for } 0 \leqslant t \leqslant \mu_{IR}^2}$

closest threshold (e.g. $4m^2$)

$\boxed{** \quad \text{Polynomial Boundedness: } |M(s\to\infty, t)| < \text{const}\cdot s^N \quad \text{some } N}$

From $*$: $\mathrm{Im}\,M(s,t) = 16\pi \sum_\ell (2\ell+1)\,\mathrm{Im}\,a_\ell\, P_\ell\left(1+\frac{2t}{s}\right)$ with $0 \leqslant t \leqslant \mu_{IR}^2$

but now $P_\ell(\underbrace{1+2t/s}_{>1}) \sim \frac{e^{+2\ell\sqrt{t/s}}}{\sqrt{\ell}}$ @ large ℓ

From $**$: $\mathrm{Im}\,a_\ell(s)$ needs to decay exponentially: best for $t = \mu_{IR}^2$

$$\frac{\mathrm{Im}\,M(s,t=0)}{16\pi} \leqslant \underbrace{\sum_\ell^{\ell_{max}} (2\ell+1)}_{\ell_{max}^2} + \underbrace{\sum_{\ell > \ell_{max}}^{\infty} (2\ell+1)\,\mathrm{Im}\,a_\ell(s)}$$

can drop this from ℓ_{max}: $e^{2\ell_{max}\sqrt{\frac{\mu_{IR}}{s}}} \sim s^N$

i.e. $\boxed{\ell_{max}(s) = \frac{N}{2\mu_{IR}} \sqrt{s} \log s}$

Froissart – Martin Bound

$$\begin{cases} \sigma_{TOT}(s \to \infty) \leq \frac{\text{const}}{\mu_{IR}^2} \log^2 s \\ |\mathcal{M}(s \to \infty, t=0)| \leq \frac{\text{const}}{\mu_{IR}^2} s \cdot \log^2 s \end{cases}$$

$(|\mathrm{Re}\,M| < |M| \ldots)$

$\mathcal{M}$ is polynom. bounded by s^N with $N=2$

Summary:

$$\left[\begin{array}{ll} \text{unitarity} & (|S_\ell|^2 \leq 1) \\ \text{causality} & (\mathcal{M} \text{ analytic \& gapped}) \\ \text{locality} & (\leq s^N) \end{array}\right] \Longrightarrow \underline{\sigma_{TOT}(s \to \infty) < \log^2 s}$$

Axiomatic Lorentzian QFTs satisfy these $\to$

$\sum_\ell (2\ell+1)\, a_\ell\, P_\ell$