The local cohomology theorem for group cohomology

The philosophy of this series of lectures is that we spot pretty patterns & then develop theory to explain the patterns. Eventually we hope to give a structural understanding of the phenomena. In our case the pattern is in the cohomology ring of a finite group

Poincaré duality

The very first pattern is that the cohomology (as a graded vector space) is palindromic.

If M^m is a (connected k-orientable) m-manifold we consider

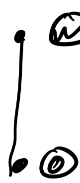
$A = H^*M$ has $H^i(M) \cong H^{m-i}(M)$

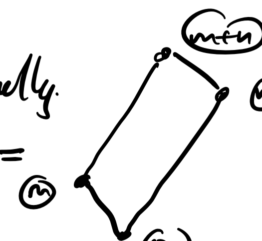
or, better still $H^i(M) = H^{m-i}(M)^\vee$, with the duality given by cup product into the top degree.

$$A^\vee \cong \Sigma_{-m} A$$

$$A^\vee = \mathrm{Hom}(H^*M, k) = H_*M \cong \Sigma_m H^*M$$

Example 1 (a bit too simple to show the pattern clearly)

If $X = S^m$ then $H^*S^m =$ 

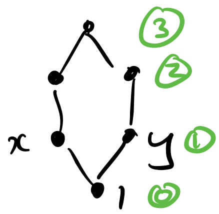
More generally $H^*S^m \times S^n =$ 

Poincaré series $f(t) = 1 + t^m$, $f(\frac{1}{t}) = 1 + \frac{1}{t^m} = \frac{1}{t^m} f(t)$.

$f(t) = (1+t^m)(1+t^n)$
$f(1/t) = (1+\frac{1}{t^m})(1+\frac{1}{t^n}) = \frac{1}{t^{m+n}} f(t)$

Example 2: If $X = Sp(1) \cong S^3$, take $M = X/G$

$G = Q_8$ char $k = 2$

then $H^*(M) =$ 

$= k[x,y]/(x^2+xy+y^2,\ x^3, y^3)$

$f(t) = 1 + 2t + 2t^2 + t^3$

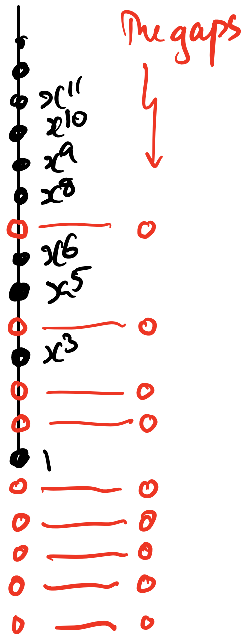
$f(\frac{1}{t}) = 1 + \frac{2}{t} + \frac{2}{t^2} + \frac{1}{t^3} = \frac{1}{t^3} f(t)$

Clearly M is a 3-mfld so $H^0 = H^3 \cong k$.
For H^1 we see $H^1(M) = \mathrm{Hom}(\pi_1 M, k) = \mathrm{Hom}(Q_8, k) = k^2$.

But this clearly only applies to finite dimensional spaces

<u>Example</u>: Consider $k[x]$, again a bit too simple, so we will return to it.

Instead consider $R = k[x^3, x^5] \subseteq k[x]$



Note that the original black dots, when turned upside down & shifted by 7 coincide with the gaps

$$f(t) = 1 + t^3 + t^5 + t^6 + \frac{t^8}{1-t} = \frac{1 - t + t^3 - t^4 + t^5 - t^7 + t^8}{1-t}$$

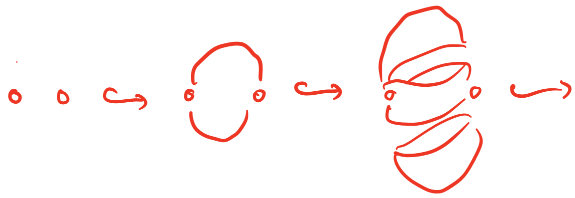
$$f(1/t) = \frac{1 - \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4} + \frac{1}{t^5} - \frac{1}{t^7} + \frac{1}{t^8}}{1 - 1/t} = \frac{1}{t^7}\,\frac{t^8 - t^7 + t^5 - t^4 + t^3 - t + 1}{1-t}$$

$$= -t^{-7} f(t)$$

But this can only work for cohomology of finite dimension
I'm going to talk about finite groups G.

& $H^*BG = \mathrm{Ext}^*_{kG}(k,k) \leftarrow$ By definition $BG = EG/G$ where EG is G-free, contractible & of the homotopy type of a GCW complex.

Give the example of the infinite sphere & $\mathbb{R}P^\infty =$



$EG^{(0)} \hookrightarrow EG^{(1)} \hookrightarrow EG^{(2)} \hookrightarrow$

Hence $H_*(EG) \simeq H_*(pt) = k$

But

$$H_*(EG) = H_*\big(\underset{\overset{\|}{P_0}}{C_0(EG)} \leftarrow \underset{\overset{\|}{P_1}}{C_1(EG)} \leftarrow \cdots\big)$$

& since EG is free $EG^{(n)}/EG^{(n-1)} \simeq \bigvee \Sigma^n G_+$

& $C_n(EG) = \bigoplus kG$ is free

Hence $*$ is a free resolution

Also since EG is free

$$H^*BG = H^*(\mathrm{Hom}_{kG}(C_*EG, k))$$
$$= H^*(\mathrm{Hom}_{kG}(P_*, k))$$
$$= \mathrm{Ext}^*_{kG}(k,k)$$

<u>Example 1</u>: $G = C_2 = \langle g \rangle$

<u>(a) Algebra</u>: $0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}G \xleftarrow[①]{1-g} \mathbb{Z}G \xleftarrow{1+g} \mathbb{Z}G \xleftarrow{1-g} \mathbb{Z}G \leftarrow \cdots$

$$\mathrm{Ext}^*_{kG}(k,k) = H^*\big(k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0}\big)$$

On C_2 & acts freely on $S(\mathbb{R}) \subseteq S(2\mathbb{R}) \subseteq \cdots \subseteq S(\infty\mathbb{R}) = EC_2$
$\uparrow$ 0-sphere $\quad$ $\uparrow$ 1-sphere.

& hence $BG = \mathbb{RP}^\infty$

Either way $H^* BC_2 = \mathrm{Ext}^*_{kC_2}(k,k) = k[\overset{(1)}{t}]$

Example 2: $G = Q_8$

G acts freely on $S(\mathbb{H}) \subseteq S(2\mathbb{H}) \subseteq \cdots$
& hence

$$H^* BQ_8 = H^*(Sp(1)/Q_8)[\overset{(4)}{z}]$$

Example 3: If G acts freely on $S(V)$
H —— $S(W)$

then $G \times H$ acts freely on $S(V) \times S(W) \subseteq S(2V) \times S(2W) \subseteq \cdots$

& $B(G \times H) = BG \times BH$.

Example 4: We will see later that if G is nilpotent then G acts freely on $S(V_1) \times \cdots \times S(V_n)$ for some rep^ns V_i & hence

$$EG = S(\infty V_1) \times \cdots \times S(\infty V_n).$$

<u>Theorem</u> (Benson-Carlson) If H^*BG is CM

for $f(t) = \sum_i \dim_k H^i(BG;k)\, t^i$

then $f\left(\frac{1}{t}\right) = (-t)^r f(t)$.

If H^*BG is almost CM (in the sense that depth $= r-1$)

then $f\left(\frac{1}{t}\right) - (-t)^r f(t) = (-1)^{r-1}(1+t)\, g(t)$

& $g\left(\frac{1}{t}\right) = (-t)^{-(r-1)} g(t)$ (note the sign of the power of t)

<u>Example</u>: $G = C_2$ has $H^*BG = k[\overset{1}{x}]$ if k is of characteristic 2

& $f(t) = \frac{1}{1-t}$ & certainly satisfies $f\left(\frac{1}{t}\right) = (-t) f(t)$.
The cohom. ring is CM

<u>Example</u>: $G = Q_8$ has $H^*BQ_8 = H^*(S(\mathbb{H})/Q_8)[z] = \dfrac{k[\overset{1}{x}, \overset{1}{y}, \overset{4}{z}]}{x^2+xy+y^2,\ x^3, y^3}$

with $f(t) = \dfrac{1+2t+2t^2+t^3}{1-t^4}$. Again this CM

& $f\left(\frac{1}{t}\right) = \dfrac{1 + 2/t + 2/t^2 + \frac{1}{t^3}}{1 - 1/t^4} = (-t) f(t)$.

<u>Example</u>: If $G = H \times K$ then $H^*BG = H^*BH \otimes H^*BK$

& $f_G(t) = f_H(t)\, f_K(t)$, so if it holds for H, K it holds for G.

<u>Example</u>: If $G = D_8$, $H^*(BD_8) = \dfrac{k[\overset{1}{x}, \overset{1}{y}]}{(xy)}[\overset{2}{z}]$

& $f(t) = \frac{1}{(1-t)^2}$. This is now CM of dim^n 2 & again

we verify $f\left(\frac{1}{t}\right) = (-t)^2 f(t)$.

<u>Example</u>: If $G = SD_{16}$, one may check this is almost CM

& $f(t) = \dfrac{1}{(1-t)^2(1+t^2)}$. One may note this satisfies $f\left(\frac{1}{t}\right) = \dfrac{1}{(1-1/t)^2(1+1/t^2)}$
$= t^4 f(t)$ as def shift 2

But this is probably a coincidence. One checks $f(\frac{1}{t})-(-t)^2 f(t) = \frac{t^2(t+1)}{(1-t)(1+t^2)}$ so $g(t)=\frac{t^2}{(1-t)(1+t^2)}$

[Lecture 2 began here]

which satisfies $g(1/t) = (-t)^{-1} g(t)$ as required

Local cohomology

The algebraic structure that lets us proceed is local cohomology. We will work with graded commutative rings R (i.e. $yx=(-1)^{|x||y|}xy$)

To start with we suppose $x \in R$ & define the stable Koszul complex

$$K_\infty(x) = (\overset{0}{R} \longrightarrow \overset{-1}{R[\tfrac{1}{x}]})$$

(note for later that this may be viewed as the fibre of a map of chain complexes in degree 0 namely $R \longrightarrow R[\frac{1}{x}]$)

Now suppose $I = (x_1, \dots, x_n)$ & define

$$K_\infty(\underset{\sim}{x}) = K_\infty(x_1) \otimes_R \cdots \otimes K_\infty(x_n)$$

(a chain complex nonzero in degrees $0, -1, \dots, -n$).

In a moment we will show this depends only on I, so we may define the I-local cohomology by

Def^n: $H^*_I(R;M) = H^*(K^\bullet_\infty(\underset{\sim}{x}) \otimes_R M$

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Lemma 1 (a) $H^0_I(R;M) = \Gamma_I M$

(b) $H^*_I(R;M)$ is I-power torsion

Pf: (a) Clear $H^0 \to R \to \bigoplus_i R[\frac{1}{x_i}]$

(b) Certainly $(R \to R[\frac{1}{x}])[\frac{1}{x}] = R[\frac{1}{x}] \to R[\frac{1}{x}]$
so inverting x_i kills.
If $y = \sum_i \lambda_i x_i$ then y^{ns} is divisible by x_i^s for one s

Corollary: $K_\infty(I)$ depends only on $\sqrt{I}$.
Proof: If $y \in \sqrt{I}$ then $y = \sum_i \lambda_i x_i$
& so $H_*(K_\infty(I) \otimes R[\frac{1}{y}]) = 0$
Hence $K_\infty(I) \xrightarrow{\simeq} K_\infty(I, y)$ //.

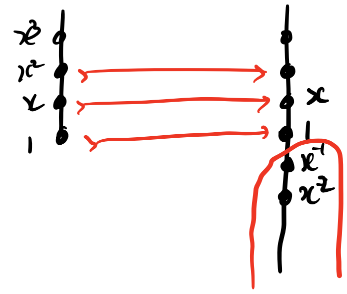
Lemma: The first non-vanishing local cohomology group is the I-depth of M

Lemma: If R is Noetherian
then $H_I^*(R; M) = R\Gamma_I(M)$
where $\Gamma_I M = \{m \in M \mid I^N m = 0 \text{ for } N \gg 0\}$
Hence also $H_I^i(R; M) = 0$ for $i > \text{Krulldim}(R)$

Pf: Γ_I is left exact, $H_I^0(M) = \Gamma_I(M)$ & $H_I^*(M) = 0$ if M is injective //

Example: If $R = k[\overset{d}{x}]$

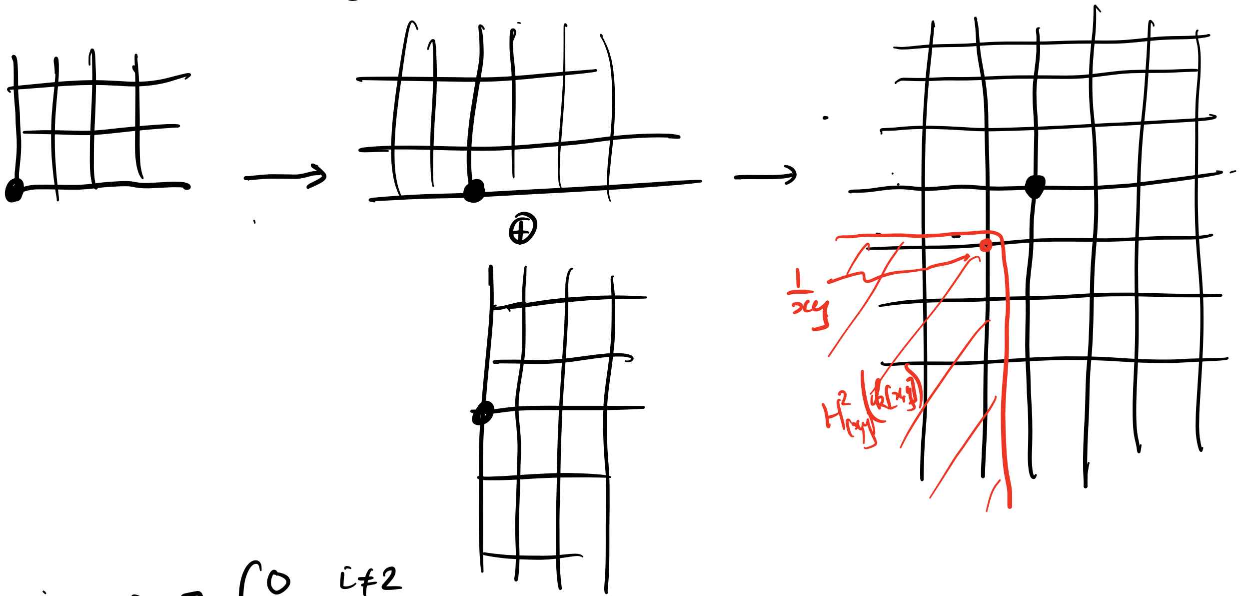
then $H^i_{(x)}(k[x]) = H(k[x] \longrightarrow k[x,x^{-1}]) = \begin{cases} 0 & i=0 \\ \Sigma^{-d} k[x]^\vee & i=1 \end{cases}$



[Look back at the $R=k[x^3,x^5]$ example & see we essentially calculated

$$H^i_{(x^3)}(R) \simeq \begin{cases} 0 & i \neq 1 \\ \Sigma^7 R^\vee & i=1 \end{cases}$$

Example 2: If $R = k[\overset{d}{x}, \overset{e}{y}]$
$I = (x,y)$



$$H^i_{(x,y)} k[x,y] = \begin{cases} 0 & i \neq 2 \\ \Sigma^{-d-e} k[x,y]^\vee & i=2 \end{cases}$$

Conclusion: For the polynomial ring $P = k[\overset{d_1}{x_1}, \overset{d_2}{x_2}, \dots, \overset{d_r}{x_r}]$
$I = (x_1, x_2, \dots, x_r)$

we have $H^i_{(x)}(P) = 0$ for $i \neq r$ & $\boxed{H^r_{(x)}(P)^\vee = \Sigma^d P \quad \text{where } d = \sum d_i}$

The Hilbert series is

$$f(t) = \frac{1}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_r})} \text{ \& then } f(\tfrac{1}{t}) = \frac{1}{(1-1/t^{d_1})\cdots(1-1/t^{d_r})} = \frac{t^d}{(t^{d_1}-1)\cdots(t^{d_r}-1)}$$

$$= t^d (-1)^r f(t)$$

Gorenstein duality

Def^n (a) We say R is <u>Cohen-Macaulay</u> if depth = dimension.

(b) R is <u>Gorenstein</u> if R is of finite injective dimension. In this case $\mathrm{Ext}^*_R(k,R) = \mathrm{Ext}^r_R(k,R) \cong \Sigma^{a+r} k$

(c) We say R has <u>Gorenstein duality</u> of shift a if

$$H^r_{\mathfrak{m}}(R) \cong \Sigma^{a+r} R^\vee.$$

Example: $k[x_1, \dots, x_r]$ is Gorenstein of shift $d+r$, $d = \Sigma d_i$

For example if $r=1$ $\quad 0 \to \Sigma^d k[x] \to k[x] \to k \to 0$

$$0 \leftarrow \underset{\Sigma^{-d}k}{\mathrm{Ext}^1(k,R)} \leftarrow \Sigma^{-d} k[x] \leftarrow k[x] \leftarrow \underset{0}{\mathrm{Hom}(k,R)} \leftarrow 0$$

(Gor of shift $-1-d$)

$$0 \to \underset{0}{H^0_{(x)}(k[x])} \to k[x] \to k[x, x^{-1}] \to \underset{\Sigma^{-d} k[x]^\vee}{H^1_{(x)}(k[x])} \to 0$$

(Gor D of shift $-1-d$)

Theorem: For every finite group G there is a SS $\quad H^*_{\mathfrak{m}}(H^*BG) \Rightarrow H_*BG$

If H^*BG is CM then (a) H^*BG is Gorenstein of shift 0

& (b) $f(\frac{1}{t}) = (-t)^r f(t)$.

Remark: Parts (a) & (b) follow from the SS.

Indeed if CM then SS collapses to $\quad H^r_{\mathfrak{m}}(H^*BG) = \Sigma^r H_*BG$

(NB: The suspension is homological $(\Sigma^r M)_s = M_{s-r}$.)

Proof of functional equation

① We have seen for $P = k[x_1, \dots, x_r]$ that $(H^r_m P)^\vee = \Sigma_d P$

& $[P](\frac{1}{t}) = (-1)^r t^d [P](t)$

(where $[M](t) = \sum_s \dim_k (M^s) t^s$)

& x_i is of codegree d_i

② If R is CM then by the Auslander–Buchsbaum formula
$R = P \otimes F$ for F a finite dim'l graded vector space

& $(H^r_m R)^\vee \cong \Sigma_r R$

Thus $R = \Sigma_r (H^r_m R)^\vee = \Sigma_r (H^r_m P)^\vee F^\vee = \Sigma_{r-d} P F^\vee$

& hence
$$[R](t) = t^{d-r}[P](t)[F](\tfrac{1}{t}) = (-1)^r t^{d-r} t^{-d} [P](\tfrac{1}{t})[F](\tfrac{1}{t}) = (-t)^{-r}[R](1/t)$$

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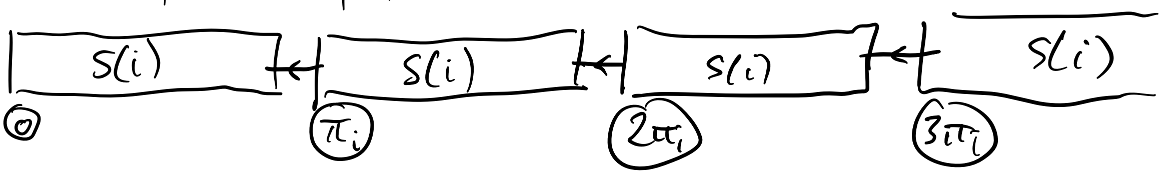
Proof: We assume that there is a free resolution of k over kG in the form

$$E(1)\otimes_k E(2)\otimes_k \cdots \otimes_k E(n)$$

where $E(i)$ is periodic of period π_i

$E(i) =$ 

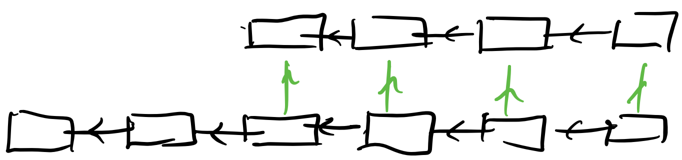
ie $\quad 0 \leftarrow k \leftarrow \boxed{S_0(i) \leftarrow \cdots \leftarrow S_{\pi_i-1}(i)} \leftarrow k \leftarrow 0$ is exact.

This is n-multigraded. We view it as formed by stacking boxes in the positive orthant. If $A \subseteq \{1,\ldots,n\}$ we may consider

$L(A)$ = stacked boxes allowing negatives in the i direction for $i \in A$ (in effect $L(A) \simeq \varprojlim \Sigma^{-?} E$)

Example $(n=1)$ $L(\phi) = E$

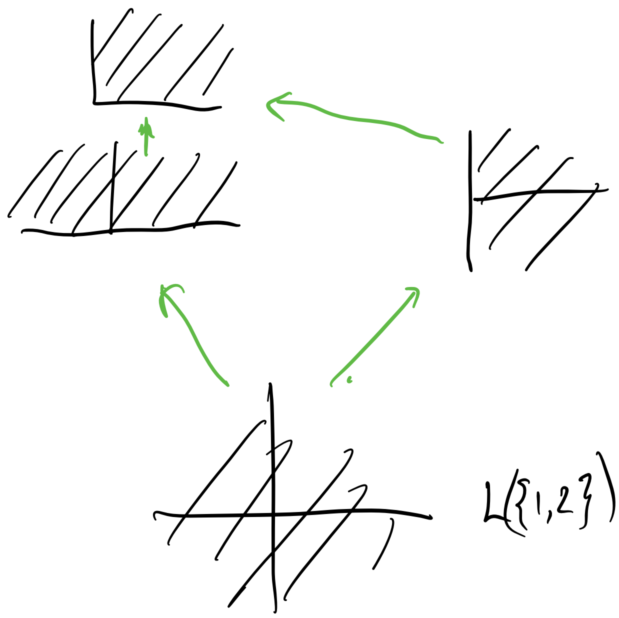
$L(1) =$



$(n=2)$

$L(\phi) =$

$L(1) =$



$L(2)$

$L(\{1,2\})$

Now form the total complex $L_s = \bigoplus_{|A|=s} L(A)$ under inclusions

There are two SSs for
$$H^* \mathrm{Hom}_{kG}(L_\bullet, k)$$

If we take L-cohomology first we obtain $H^*(\mathrm{Hom}_{kG}(\overset{\text{boxes all negative}}{H_*}L_\bullet, k))$

$$= H^*(\mathrm{Hom}_{kG}(\Sigma^{-n} E^\vee, k))$$

$$= H^*(\Sigma^n E \otimes_{kG} k)$$

$$= \Sigma^n H_* BG$$

If we take E-cohomology first we note $L(A) = \varinjlim(\Sigma^\bullet E, x_A)$

so $H^* \mathrm{Hom}_{kG}(LA, k) = \varinjlim (H^*BG, x_A) = H^*BG[\frac{1}{x_A}]$

& the SS reads $H^*_I(H^*BG) \Rightarrow H_*BG$.

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It remains to describe resolutions of the type described above.

Method 1 (Works for p-groups)

One may find complex rep^{ns} $V_1, V_2, \dots, V_n$ so that G acts freely on $S(V_1) \times \cdots \times S(V_n)$.

Then we may take

$$E(i) = C_*(S(\infty V_i))$$

[Pf. on for $1 \to Z \to G \to \bar{G} \to 1$, $Z = \langle g \rangle$; on $\bar{G}$ by induction on $S(\bar{V}_1) \times \cdots \times S(\bar{V}_n)$

& if W is a free action of Z

then $V = \mathrm{ind}_Z^G W$ will do as V_n V_1

If $x \in G$ is of order p either nontriv in $\bar{G}$ & acts freely on T or else $\langle x \rangle = Z$ & use $V\downarrow = W^{\oplus N}$. //

This fails for A_4 for example. //

Method 2 (Benson - Carlson) Supports.

Choose $\widetilde{R} = k[\zeta_1, \dots, \zeta_n] \subseteq H^* BG$.

Now represent $\zeta_i \in H^d BG = \mathrm{Ext}^d_{kG}(k, k)$ by

$E(\zeta) =$

$0 \leftarrow k \leftarrow \boxed{P_0 \leftarrow \cdots \leftarrow P_{d-1}} \leftarrow k \leftarrow 0$ (where $P_0, \dots, P_{n-2}$ are proj).

Then support $E(\zeta) = V(\zeta)$

& Support $(E(\zeta_1) \otimes \cdots \otimes E(\zeta_n)) = V(\zeta_1) \cap \cdots \cap V(\zeta_n) = \emptyset$

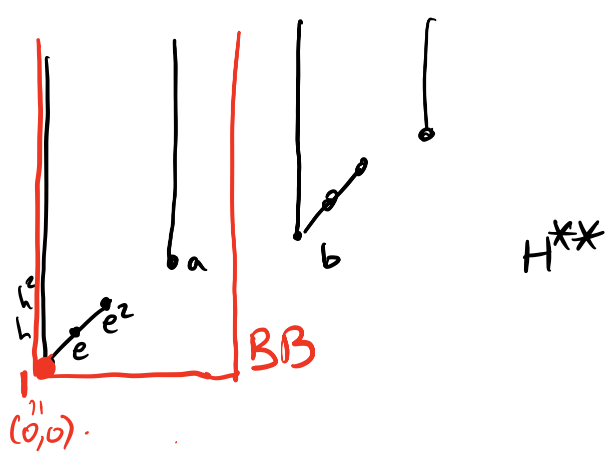
Hence $E(\zeta_1) \otimes \cdots \otimes E(\zeta_n)$ is free //

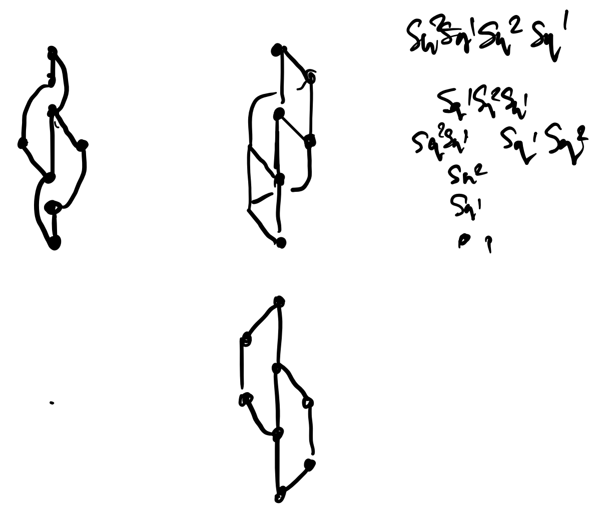
Theorem (Bruner-G): If A is a finite dim'l sub Hopf algebra of the Steenrod algebra of formal dim^n f then there is a CCT:

$$H^*_{m} H^{**}(A) \Rightarrow \Sigma^{(f,0)} H^{**}(A)^\vee$$

Proof: Exactly as above but use the Adams-Margolis criterion for a module to be free.

Example: $A(1) = \langle Sq^1, Sq^2 \rangle$

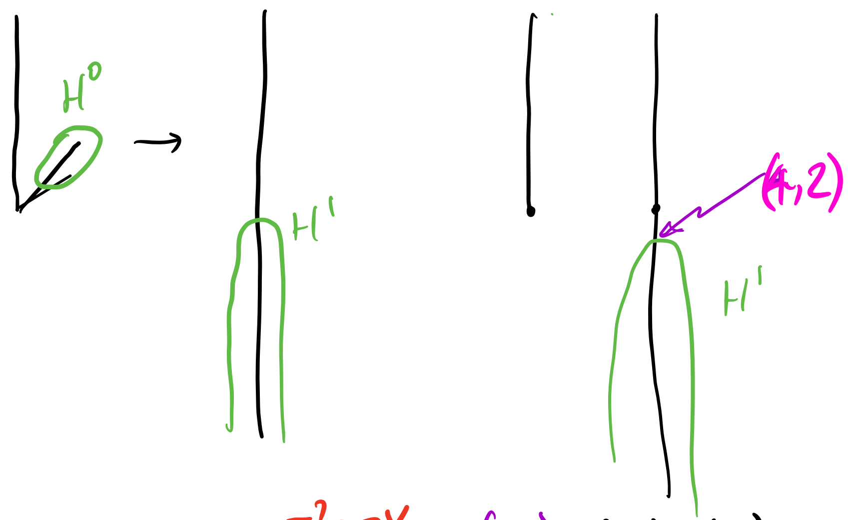




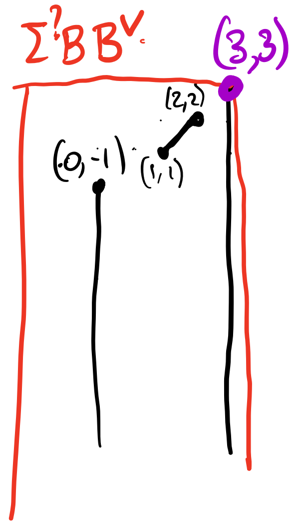
$$H^{**}(A(1)) = BB[b], \quad m = (h, b)$$

$H^*_{(h,b)} \Leftarrow H^*_h H^*_b$. But $H^*_{(b)}(BB[b]) = \dfrac{BB[b, b^{-1}]}{BB[b]} = \bigoplus_{i<0} BB \cdot b^i$

Now



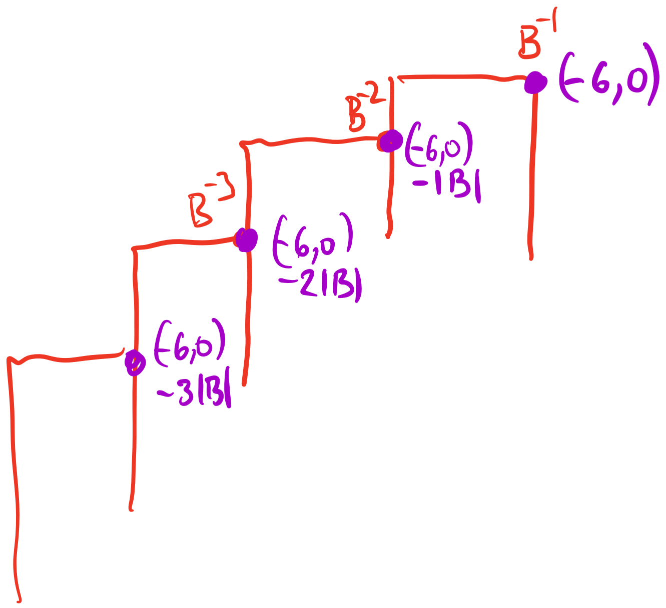
Hence $H^*_{(h)}(BB) =$



$(3,3) = (4,2) + (-1,1)$

Moving to $H^1_{(b)}$ adds a further $(-1,1)$

Get to $(2,4)$



$(-6,0) = (2,4) - (8,4)$