ICTS Sept 12 2023

Lec 2 : Hecke operators (Ch. 5 of [DS]).

§ Double Coset Operators:

Lemma 1 : $\Gamma \subset SL_2(\mathbb{Z})$ cong. subgp $\Rightarrow \alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ also cong. subgp $\left(\alpha \in GL_2^+(\mathbb{Q})\right)$

Pf: Say $\Gamma(\bar{N}) \subset \Gamma$ for $\bar{N} \geq 1$.

WMA. $\bar{N} \gg 1$ so that $\bar{N}\alpha \in M_2(\mathbb{Z})$, $\bar{N}\alpha^{-1} \in M_2(\mathbb{Z})$.

Set $N = \bar{N}^3$

Then
$$\alpha\Gamma(N)\alpha^{-1} = \alpha(1+\bar{N}^3 M_2(\mathbb{Z}))\alpha^{-1} = 1 + \bar{N}\underbrace{\left(\bar{N}\alpha\, M_2(\mathbb{Z})\,\bar{N}\alpha^{-1}\right)}_{\subset M_2(\mathbb{Z})} \subset \Gamma(\bar{N})$$

$\therefore \quad \Gamma(N) \subset \alpha^{-1}\Gamma(\bar{N})\alpha \subset \alpha^{-1}\Gamma\alpha$ //.

Lemma 2 : Let $\Gamma_1, \Gamma_2 < SL_2(\mathbb{Z})$ be cong. subgps.

Let $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$, $\alpha \in GL_2^+(\mathbb{Q})$.

Then $\exists$
$$\Gamma_3\backslash\Gamma_2 \xleftrightarrow[\text{bijection}]{} \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$$
$$\{\Gamma_3\gamma_{2,j}\}_j \longmapsto \{\Gamma_1\alpha\gamma_{2,j}\}$$
("Multiply coset reps on left by α")

Pf. Clearly $\Gamma_2 \longrightarrow \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, $\gamma_2 \longmapsto \Gamma_1\alpha\gamma_2$ is surj.

Moreover: $\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2' \Leftrightarrow \gamma_2'\gamma_2^{-1} \in \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$

(Pf: $\exists\gamma_1\alpha\gamma_2 = \exists\gamma_1'\alpha\gamma_2' \Rightarrow \gamma_2'\gamma_2^{-1} = \alpha^{-1}\gamma_1'^{-1}\gamma_1\alpha$). //

Cor: $|\Gamma_1\backslash\Gamma_1\alpha\Gamma_2| < \infty$

Pf: Lem 2 $\Rightarrow$ ETS. $|\Gamma_3\backslash\Gamma_2| \overset{\text{obvious}}{\leq} |\alpha^{-1}\Gamma_1\alpha\cap SL_2(\mathbb{Z})\backslash SL_2(\mathbb{Z})| < \infty$

where $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$,

which is clear from Lemma 1 //

Recall: $$f\big|_{\beta}(z) := (\det\beta)^{k-1}(cz+d)^{-k} f(\beta z)$$

$z \in H$
$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$

Def (Double coset operator) For $f \in M_k(\Gamma_1)$, set

$$f\big|_{[\Gamma_1 \alpha \Gamma_2]}(z) := \sum_j f\big|_{\beta_j}(z)$$

w/ $$\Gamma_1 \alpha \Gamma_2 = \coprod_{\substack{j \\ \text{finite}}} \Gamma_1 \beta_j$$

Claim: $$\big|_{[\Gamma_1 \alpha \Gamma_2]} : \begin{array}{ccc} M_k(\Gamma_1) & \longrightarrow & M_k(\Gamma_2) \\ \cup & & \cup \\ S_k(\Gamma_1) & \longmapsto & S_k(\Gamma_2) \end{array}$$

Pf: • For $\gamma_2 \in \Gamma_2$ $$\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 \longleftrightarrow \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$$
$$\Gamma_1 \beta \longmapsto \Gamma_1 \beta \gamma_2$$
(right mult. by γ_2) is a bijection

$\therefore$ $\{\Gamma_1 \beta_j\}$ distinct cosets w/ union $\Gamma_1 \alpha \Gamma_2$ $\Leftrightarrow$ $\{\Gamma_1 \beta_j \gamma_2\}$ also distinct coset w/ union $\Gamma_1 \alpha \Gamma_2$

$$\therefore\ f\big|_{[\Gamma_1 \alpha \Gamma_2]}\big|_{\gamma_2} = \sum_j f\big|_{\beta_j}\big|_{\gamma_2} = \sum f\big|_{\beta_j \gamma_2} = f\big|_{[\Gamma_1 \alpha \Gamma_2]}$$

• holo at cusps / van at cusps obvious if Γ_1, Γ_2 are $\Gamma_1(N)$ or $\Gamma_0(N)$; easy check o.w. //.

Reinterpretation in terms of modular curves:

$$\begin{tikzcd}
{}_{\alpha^{-1}\Gamma_1\alpha\cap\Gamma_2}\backslash\mathcal{H} = {}_{\Gamma_3}\backslash\mathcal{H} = Y_3 \arrow[r, "\sim", "z \mapsto \alpha z"'] \arrow[d] & Y_3' = {}_{\Gamma_3'}\backslash\mathcal{H} = {}_{\alpha\Gamma_3\alpha^{-1}}\backslash\mathcal{H} = {}_{\Gamma_1\cap\alpha\Gamma_2\alpha^{-1}}\backslash\mathcal{H} \arrow[d] \\
{}_{\Gamma_2}\backslash\mathcal{H} = Y_2 & Y_1 = {}_{\Gamma_1}\backslash\mathcal{H}
\end{tikzcd}$$

Say $\Gamma_1\alpha\Gamma_2 = \coprod_j \Gamma_1\beta_j$ w/ $\beta_j = \alpha\gamma_{2,j}$

and ${}_{\Gamma_3}\backslash\Gamma_2 = \coprod_j \Gamma_3\gamma_{2,j}$

Then the above maps induce

$$\begin{array}{ccc}
\{\Gamma_3\gamma_{2,j}z\}_j & \xrightarrow{\ \cdot\alpha\ } & \{\Gamma_3'\alpha\gamma_{2,j}z = \Gamma_3'\beta_j z\}_j \\
\uparrow & & \downarrow \\
\Gamma_2 z & & \{\Gamma_1\beta_j z\}_j
\end{array}$$

counted w/ multiplicity = ram. index.

and therefore if $\mathrm{Div}(Y)$ = dir. gp of Y (formal sums of pts in Y w/ int coeffs)

get $[\Gamma_1\alpha\Gamma_2] : \mathrm{Div}(Y_2) \to \mathrm{Div}(Y_1)$

§ T_p operator w/ $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$ and $\alpha = \begin{pmatrix}1 & \\ & p\end{pmatrix}$

Need coset reps of ${}_{\Gamma_3}\backslash\Gamma_2$.

$$\Gamma_3 = \alpha^{-1}\Gamma_1\alpha\cap\Gamma_2 = \left\{\begin{pmatrix}1 & \\ & 1/p\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \\ & p\end{pmatrix}\right\}_{\in\,\Gamma_1(N)}\cap\Gamma_1(N)$$

$$= \left\{\begin{pmatrix}a & bp\\ c/p & d\end{pmatrix}\right\}\cap\Gamma_1(N)$$

$$= \Gamma^0(p)\cap\Gamma_1(N)$$

$\therefore$ Guess: ${}_{\Gamma_3}\backslash\Gamma_2 = \left\{\Gamma_3\overbrace{\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}}^{\gamma_{2,j}}\right\}_{j=0,1,\dots,p-1} \cup \left\{\Gamma_3\gamma_{2,\infty}\right\}$ " $\{\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\in SL_2(\mathbb{Z}) \mid b'\equiv 0\ (p)\}$

$\begin{pmatrix}mp & n\\ m & 1\end{pmatrix}$ (possibly).

Idea: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

→ Well: If $\overset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\overset{\shortparallel}{\gamma_2}} \in \Gamma_2$, then $\gamma_2 \in \Gamma_3\, \gamma_{2,j}$

$$\iff \gamma_2\, \gamma_{2,j}^{-1} \in \Gamma_3 = \Gamma^0(p) \cap \Gamma_1(N)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -aj+b \\ c & -cj+d \end{pmatrix}$$

$$\iff b \equiv aj \ (p) \underset{\boxed{p \nmid a}}{\iff} j \equiv ba^{-1}\ (p)$$

However if $p \mid a$, then $b \equiv 0\ (p)$ ⨳ $\left(\because\ ad - bc = 1\right)$.

Note: this only happens if $p \nmid N$ $\left(\text{Pf: } ad - b\underset{0\,(N)}{\underset{\shortparallel}{c}} = 1 \quad \therefore p \mid N,\ p \mid a \Rightarrow \text{⨳}\right)$

Need one more coset rep $\gamma_{2,\infty} = \begin{pmatrix} mp & n \\ N & 1 \end{pmatrix}$ (if $p \nmid N$).

w/ $mp - nN = 1$

→ Now if $\gamma_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$ w/ $p \mid a$, then $\gamma_2 \in \Gamma_3\, \gamma_{2,\infty}$

$$\iff \gamma_2\, \gamma_{2,\infty}^{-1} \in \Gamma_3$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ -N & mp \end{pmatrix} = \begin{pmatrix} \overset{\equiv 1\,(N)\ ✓}{a - bN} & \overset{\equiv 0\,(p)\ ✓}{-na + bmp} \\ \underset{\equiv 0\,(N)}{c - dN} & \underset{\equiv 1\,(N)}{-nc + dmp} \end{pmatrix}$$

$$\therefore\ \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 = \left\{ \Gamma_1\, \overset{\beta_j}{\overset{\shortparallel}{\alpha\gamma_{2,j}}} \right\} \cup \left\{ \Gamma_1\, \beta_\infty \right\}$$

$$\beta_j = \alpha\gamma_{2,j} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$$

$$\beta_\infty = \alpha\gamma_{2,\infty} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} mp & n \\ N & 1 \end{pmatrix} = \begin{pmatrix} mp & n \\ pN & p \end{pmatrix} = \begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

Get:

Prop: $N \geq 1$

$T_p = \left[\Gamma_1(N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_1(N)\right] : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$

is given by

$$T_p f = \sum_{j=0}^{p-1} f\Big|_{\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}} \ \underset{(\text{if } p \nmid N)}{+}\ f\Big|_{\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$$

→ can drop this for $\Gamma_0(N)$ //

Modular curve reinterpretation of T_p :

Claim: $T_p : \mathrm{Div}(S_1(N)) \longrightarrow \mathrm{Div}(S_1(N))$

$$[E, Q] \longmapsto \sum_{\substack{C<E \\ |C|=p \\ C\cap\langle Q\rangle = 0)}} [E/C,\ Q+C]$$

Pf: Recall $T_p : \mathrm{Div}(Y_1(N)) \longrightarrow \mathrm{Div}(Y_1(N))$

$$\Gamma_1(N)\tau \longmapsto \sum_j \Gamma_1(N)\,\beta_j\tau$$

including β_∞ when $p\nmid N$.

Lec 1 $\Rightarrow$ $[E, Q] = [\overset{E_\tau}{\overset{\shortparallel}{\mathbb{C}/\Lambda_\tau}},\ \frac{1}{N}+\Lambda_\tau]$ some $\tau\in H$.

$\leftrightarrow$ $\Gamma_1(N)\tau \in Y_1(N)$.

Now $\beta_j\tau \leftrightarrow [\mathbb{C}/\Lambda_{\beta_j\tau},\ \frac{1}{N}+\Lambda_{\beta_j\tau}]$

$j\neq\infty$: $\Lambda_{\beta_j\tau} = \mathbb{Z}\frac{\tau+j}{p}\oplus\mathbb{Z}\cdot 1 \supset \Lambda_\tau$ index p.

. $\mathbb{C}/\Lambda_{\beta_j\tau} = \dfrac{\mathbb{C}/\Lambda_\tau}{\Lambda_{\beta_j\tau}/\Lambda_\tau} = \dfrac{\mathbb{C}/\Lambda_\tau}{\langle\frac{\tau+j}{p}+\Lambda_\tau\rangle} = E_\tau/C_j$

. $\frac{1}{N}+\Lambda_{\beta_j\tau} \Leftrightarrow \underbrace{\frac{1}{N}+\Lambda_\tau}_{Q} + C_j$

$\therefore \beta_j\tau \leftrightarrow [\frac{E_\tau}{C_j},\ Q+C_j]$

$j=\infty$: $\Lambda_{\beta_\infty\tau} = \mathbb{Z}\begin{pmatrix} mp & n \\ Np & p\end{pmatrix}\tau \oplus\mathbb{Z} = \mathbb{Z}\left(\dfrac{mp\tau+n}{Np\tau+p}\right)\oplus\mathbb{Z}$

$$= \mathbb{Z}\left(\frac{m\tau+n/p}{N\tau+1}\right)\oplus\mathbb{Z}.$$

$\therefore (N\tau+1)\Lambda_{\beta_\infty\tau} = \mathbb{Z}(m\tau+n/p)\oplus\mathbb{Z}(N\tau+\overset{p/p}{\overset{\shortparallel}{1}})$

$$\underset{mp-Nn=1}{=} \mathbb{Z}\tau\oplus\mathbb{Z}\frac{1}{p}$$

$\therefore \beta_\infty\tau \leftrightarrow [\frac{E_\tau}{C_\infty},\ Q+C_\infty]$

$\therefore$. $\mathbb{C}/\Lambda_{\beta_\infty\tau} \xrightarrow[\sim]{\cdot N\tau+1} \mathbb{C}/\mathbb{Z}\tau\oplus\mathbb{Z}\frac{1}{p} = \dfrac{\mathbb{C}/\Lambda_\tau}{\mathbb{Z}\tau\oplus\frac{1}{p}\mathbb{Z}/\Lambda_\tau} = \dfrac{\mathbb{C}/\Lambda_\tau}{\langle\frac{1}{p}+\Lambda_\tau\rangle} = E_\tau/C_\infty$

. $\frac{1}{N}+\Lambda_{\beta_\infty\tau} \Leftrightarrow \frac{1}{N}+\Lambda_\tau+C_\infty$

But $\{C_j\}_{j=0,\dots,p-1} \cup \{C_\infty\}$ = all subgps of E_2 of $|\,| = p$

Indeed : all mutually disjoint (except for 0)

$\therefore \# |\bigcup C_j| = (p+1)(p-1)+1 = p^2 = |E[p]|$

Moreover: $C_j \cap \langle \overset{\varphi}{\overset{\shortparallel}{\tfrac{1}{N} + \Lambda_z}} \rangle = \begin{cases} 0 & 0 \le j < p \\ 0 & j = \infty,\ p \nmid N \end{cases}$

but $\neq 0 \quad j = \infty,\ p \mid N$

This proves the claim //.

Can define T_n for $n = 1, 2, \dots$

Diamond operator.

by $T_1 = 1$

$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}$

$T_n = \prod T_{p_i^{r_i}}$ if $n = \prod p_i^{r_i}$ = prime fact.

Facts: • $\langle\ \rangle$: Petersson inner prod. on $S_k(\Gamma_1(N))$

• T_n $(n, N) = 1$ self-adjoint / normal for $S_k(\Gamma_0(N))$ / $S_k(\Gamma_1(N))$

$\therefore \exists$ basis of eigenforms for $\forall T_n$ w/ $(n, N) = 1$.

• $\exists$ new subspace $S_k^{new}(\Gamma_1(N))$

on which $\exists$ basis of T_n, $\forall n$ . .

• If these forms are normalized s.t. $a_1 = 1$

then $\lambda_n = a_n$, $\forall n$.

• Let $\mathbb{T}$ = Hecke alg. = alg. generated by T_n in $S_k(\Gamma_1(N))$

Hecke e.v. $\Leftrightarrow$ norm. e.f. $f \Leftrightarrow \lambda_f : \mathbb{T} \to \mathbb{C}$, $T_n \mapsto a_n$

• In fact $\exists$ perfect pairing $S_k(\Gamma_1(N)) \times \mathbb{T} \longrightarrow \mathbb{C}$

$(f, T) \longmapsto a_1(Tf)$

$\therefore \dim_{\mathbb{C}} \mathbb{T} = \dim S_k(\Gamma_1(N))$.

• $a_n \in \bar{\mathbb{Z}}$

Need a $\mathbb{Z}$-structure on $S_k(\Gamma_1(N))$.

Pfs

1) Exp. basis

2) $H_1(X_1(N), \mathbb{Z})$ and $J_1(N)$

3) $S_k(\Gamma_1(N)) \to H^1_p(\Gamma_1(N), L(n, \mathbb{R}))$

4) E-S. iso.