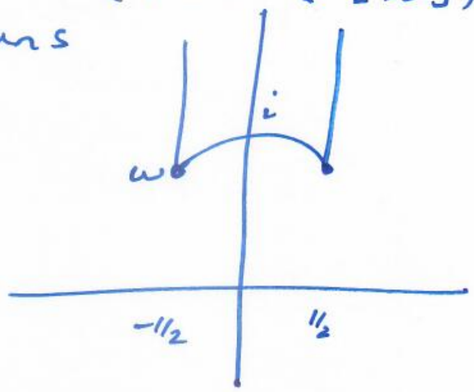
 ICTS 13/9/23

(Ch 6 of [DS])

Lec 3 : Jacobians

§ Compactifications :

$SL_2(\mathbb{Z})\backslash H$ $\underline{\underline{\quad}}$ fund domain



$$SL_2(\mathbb{Z})_z = \begin{cases} \langle -1 \rangle & z \neq i, \omega \\ \left\langle \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\rangle & z = i \\ \left\langle \begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix} \right\rangle & z = \omega \end{cases}$$ elliptic points.

$$\text{Cusps} = SL_2(\mathbb{Z})\backslash \mathbb{P}^1(\mathbb{Q}) = [1,0] = \infty .$$

$$X(SL_2(\mathbb{Z})) = SL_2(\mathbb{Z})\backslash H \cup \mathbb{P}^1(\mathbb{Q})$$: compact Riemann surface $\underset{j}{\simeq} \mathbb{P}^1(\mathbb{C})$.

Similarly have $X_0(N) = \Gamma_0(N)\backslash H \cup \mathbb{P}^1(\mathbb{Q})$

Mod. Thm (Ver $X_{\mathbb{C}}$) : $X_0(N) \xrightarrow{\exists \text{ holo map}} E/\mathbb{C}$, $j(E) \in \mathbb{Q}$.
of compact Riemann surfaces

Similarly have $X_1(N) = \Gamma_1(N)\backslash H \cup \mathbb{P}^1(\mathbb{Q})$

Rmk: cusps of $X_1(N) \underset{\text{bij}}{\longleftrightarrow} \left\{ (c,d) \in \mathbb{Z}/N \times \left(\mathbb{Z}/(c,N)\right)^* \right\} / \pm 1$

$\Gamma_1(N)\frac{a}{c} \mapsto (c,d)$ w/ d chosen s.t. $ad \equiv 1 \ ((c,N))$.

(NB: $(a,N)=1$, $(d,c)=1$ $\Rightarrow (a,(c,N))=1$.

Thm (genus). $\Gamma < SL_2(\mathbb{Z})$ cong. subgp.

Say $X(\Gamma) \longrightarrow X(SL_2(\mathbb{Z}))$ has degree d

w/ e_2, e_3 = # of ell. points of order 2; 3 of $X(\Gamma)$

e_∞ = # of cusps of $X(\Gamma)$

Then $$g(X(\Gamma)) = 1 + \frac{d}{12} - \frac{e_2}{4} - \frac{e_3}{3} - \frac{e_\infty}{2}.$$

· Pf: Hurwitz genus formula.
· $g(X(SL_2(\mathbb{Z}))) = 1 + \frac{1}{12} - \frac{1}{4} - \frac{1}{3} - \frac{1}{2}$

Turns out $\quad S_2(\Gamma) \simeq \Omega^1(X(\Gamma))\ \left(= g(X(\Gamma))\right)$ $\therefore \dim S_2(\Gamma)$

$$f \longmapsto f(q)\,\frac{dq}{q}$$

Riemann Roch gives:

Thm $\dim M_k(\Gamma) = \begin{cases} (k-1)(g-1) + \lfloor \frac{k}{4} \rfloor \varepsilon_2 + \lfloor \frac{k}{3} \rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty, & k \geq 2 \text{ even} \\ 1, & k = 0 \\ 0, & k \text{ odd and } -1 \in \Gamma \\ (k-1)(g-1) + \lfloor \frac{k}{3} \rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty^{reg} + \frac{k-1}{2}\varepsilon_\infty^{irreg}, & k \geq 3 \text{ odd } -1 \notin \Gamma. \\ \frac{1}{2}\varepsilon_\infty^{reg}, & k = 1,\ -1 \notin \Gamma,\ \varepsilon_\infty^{reg} > 2g-2 \\ \geq \frac{1}{2}\varepsilon_\infty^{reg}, & k = 1,\ -1 \notin \Gamma,\ \varepsilon_\infty^{reg} \leq 2g-2. \end{cases}$

Similar formulas for $S_k(\Gamma)$ $\qquad$ cf. [GK] TIFR Proc.

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§ Jacobians: Let X = compact Riemann surface.

$\Omega^1(X)$ = space of holo. diff's on X.
($\dim \Omega^1(X) = g(X)$).

Let $\Omega^1(X)^\wedge = \mathrm{Hom}_{\mathbb{C}}(\Omega^1(X), \mathbb{C})$

NB: $H_1(X, \mathbb{Z}) \hookrightarrow \Omega^1(X)^\wedge$

$$\sigma \longmapsto \left(\omega \longmapsto \int_\sigma \omega\right)$$

Def $\mathrm{Jac}(X) = \dfrac{\Omega^1(X)^\wedge}{H_1(X,\mathbb{Z})}$ = Jacobian of X.

Another formulation:

$\mathrm{Div}^0(X) := \left\{ \sum_{x \in X} n_x \cdot x \;\middle|\; \begin{array}{l} n_x \in \mathbb{Z} \\ n_x = 0 \ \forall' x. \\ \sum n_x = 0 \end{array} \right\} \subset \mathrm{Div}(X)$.

(15)

$$\mathrm{Div}^{\ell}(X) := \left\{ \sum n_x x \in \mathrm{Div}^0(X) \;\middle|\; \sum n_x x = \overset{\mathrm{div} f}{\overset{\shortparallel}{(f)}} \right\}$$

divisor of a mero fn on X (i.e. $f \in \mathbb{C}(X)$).

$$\mathrm{Pic}^0(X) = \frac{\mathrm{Div}^0(X)}{\mathrm{Div}^{\ell}(X)}.$$

<u>Thm (Abel-Jacobi)</u> The map $\mathrm{Div}^0(X) \to \Omega'(X)^{\wedge}$

$$\sum n_x \cdot x \mapsto \sum n_x \int_{x_0}^{x}$$

fixed base point.

induces an <u>isomorphism</u> of ab. gps:

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Jac}(X).$$

[E.g.: If $g = 0$, RHS $= 0 \Rightarrow$ LHS $= 0 \Rightarrow$ familiar fact that every deg 0 divisor is principal].

<u>Mod Thm (Ver $J_{\mathbb{C}}$)</u>: $J_0(N) = \mathrm{Jac}(X_0(N))$

$$\exists \quad J_0(N) \underset{\text{of cx. tori.}}{\xrightarrow{\text{holo homo}}} E/_{\mathbb{C}} \to \text{i.e. } j(E) \in \mathbb{Q}$$

§ <u>Maps between Jacobians</u> (on Picard gp. side).

$h : X \to Y$ non-const. holo. map of compact Riemann surfaces

<u>hp</u>: $\quad N : \mathbb{C}(X) \to \mathbb{C}(Y)$

$$N(f)(y) = \prod_{x \in h^{-1}(y)} f(x)^{e_x}$$

$v_x, v_y =$ ord. of van. at x, y.

<u>Turns out</u> $v_y(N(f)) = \sum_{x \in h^{-1}(y)} v_x(f)$

$$\therefore \mathrm{div}\, N(f) = \sum_y v_y(N(f))\, y = \sum_y \left(\sum_{x \in h^{-1}(y)} v_x(f) \right) y$$

$$= \sum v_x(f) \cdot h(x). \longrightarrow (*)$$

Define: $h_p : \mathrm{Div}^0(X) \longrightarrow \mathrm{Div}^0(Y)$

$$\sum n_x x \longmapsto \sum n_x h(x)$$

Then $(f) \longmapsto (N(f))$ by comp. (*) above.

$\therefore$ get $\boxed{h_p : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(Y)}$

h^P: $h^* : \mathbb{C}(Y) \longrightarrow \mathbb{C}(X)$

Clearly $v_x(h^* g) = e_x \, v_{h(x)}(g)$

$$\begin{aligned}\therefore \operatorname{div} h^* g &= \sum_{x \in X} v_x(h^* g) \cdot x \\ &= \sum_{x \in X} e_x \, v_{h(x)}(g) \cdot x \\ &= \sum_{y \in Y} v_y(g) \sum_{x \in h^{-1}(y)} e_x \cdot x \quad \text{(**)}\end{aligned}$$

Define: $h^P : \mathrm{Div}^0(Y) \longrightarrow \mathrm{Div}^0(X)$

$$\sum_{y \in Y} n_y y \longmapsto \sum_{y \in Y} n_y \sum_{x \in h^{-1}(y)} e_x \cdot x$$

NB $\sum n_y = 0 \Rightarrow \sum n_y \underbrace{\sum e_x}_{d} = d \sum n_y = 0$

Then $(g) \longmapsto (h^* g)$

$\therefore$ get $\boxed{h^P : \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(X)}$

NB: $\boxed{h_p \circ h^P = \deg h}$

Pf:
$$\begin{aligned}(h_p \circ h^P)\left(\sum n_y y\right) &= h_p\left(\sum_y n_y \sum_{x \in h^{-1}(y)} e_x \cdot x\right) \\ &= \sum_y n_y \underbrace{\sum_{x \in h^{-1}(y)} e_x}_{=d} \cdot \underbrace{h(x)}_{=y} \\ &= d \sum n_y \cdot y. \quad //\end{aligned}$$

§ Hecke operators on modular Jacobians

Now: $\Omega^1_{hol}(X(\Gamma)) \cong S_2(\Gamma)$

$$\therefore \quad \mathrm{Pic}^0(X(\Gamma)) = \mathrm{Jac}(X(\Gamma)) = \frac{S_2(\Gamma)^\wedge}{H_1(X(\Gamma), \mathbb{Z})}$$

Now: Lec 2 $\Rightarrow [\Gamma_1 \alpha \Gamma_2] : \mathrm{Div}^0(Y_2) \to \mathrm{Div}^0(Y_1)$

$$\begin{array}{ccc} \mathrm{Div}^0(Y_2) & \to & \mathrm{Div}^0(Y_1) \\ \cap & & \cap \\ \text{extends to} \quad \mathrm{Div}^0(X_2) & \to & \mathrm{Div}^0(X_1) \end{array}$$

In terms of

$$\begin{array}{ccc} X_3 & \xrightarrow{\sim} & X_3' \\ \downarrow \pi_2 & & \downarrow \pi_1 \\ X_2 & & X_1 \end{array}$$

$$[\Gamma_1 \alpha \Gamma_2] = (\pi_1)_* \circ \alpha \circ \pi_2^*$$

Prop: $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, $\alpha = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$, $[\Gamma_1 \alpha \Gamma_2] = T_p$

$J_1(N) = \mathrm{Jac}(X_1(N))$ $\quad T_p$ induces $T_p : J_1(N) \longrightarrow J_1(N)$

$$[\varphi] \longmapsto [\varphi \circ T_p]$$

for $\varphi \in S_2(\Gamma_1(N))^\wedge$.

Now:

$$\begin{array}{ccccccccc} 0 & \to & H_1(X_1(N), \mathbb{Z}) & \to & S_2(\Gamma_1(N))^\wedge & \to & J_1(N) & \to & 0 \\ & & \therefore \downarrow T_p & & \downarrow T_p & \circlearrowright & \downarrow T_p & & \\ 0 & \to & H_1(X_1(N), \mathbb{Z}) & \to & S_2(\Gamma_1(N))^\wedge & \to & J_1(N) & \to & 0 \end{array}$$

$\therefore$ T_p preserves an int. st. on $S_2(\Gamma_1(N))^\wedge$

$\therefore$ eigenvalues are alg!!

Let $\mathbb{T}_{\mathbb{Z}}$ = alg $/\mathbb{Z}$ gen. by T_n in $S_2(\Gamma_1(N))$

f normalized eigenform $\leftrightarrow$ $\lambda_f : \mathbb{T}_{\mathbb{Z}} \to \overline{\mathbb{Z}}$
of level N_f Lexed

$T_n \mapsto a_n$

Let $I_f = \ker(\lambda_f) \subset \mathbb{T}_{\mathbb{Z}}$: I

Now $\mathbb{T}_{\mathbb{Z}} \curvearrowright J_1(N_f)$

$\therefore$ $I_f J_1(N_f)$ makes sense.

Def $A_f = J_1(N_f) / I_f J_1(N_f)$ $\circlearrowleft$ $\mathbb{T}_{\mathbb{Z}}/I_f$ is $\mathbb{Z}[a_n]$ f.g.

Thm: $J_1(N) \underset{\text{isog.}}{\simeq} \prod_{[f] \in S_2(\Gamma_1(N_f))^{\text{new}}} A_f^{m_f}$

w/ m_f = # of div of N/N_f

[ ] = equiv. class up to Gal. conjugacy!

Rmks: i) $L(s, A_f) = \prod_{\sigma : \mathbb{Q}(a_n) \hookrightarrow \mathbb{C}} L(s, f^\sigma)$

$X \longmapsto (x_v)$

ii) $\operatorname{End}(A_f) \otimes \mathbb{Q} \in {}_2\mathrm{Br}(F) \hookrightarrow \bigoplus_v {}_2\mathrm{Br}(F_v)$

$F = \mathbb{Q}(a_n)$, ${}_2\mathrm{Br}(F_v) = \mathbb{Z}/2$

[BG] AIF, [GGQ] MRN, [BG] IJM, [BG] PAMS

X_v ess. controlled by $v(a_p^2 \varepsilon(p)^{-1})$ (for $p \nmid N_f$).

iii) All abelian varieties of GL_2-type $/\mathbb{Q}$ are "modular" i.e. isog. to A_f

? Is there a "Serre Uniformity Conjecture" here?
YES cf. [G-P] BLMS if one fixes dim A_f.