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ICTS
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# Lec 4: Int. Models & q-exp principle (Ch 8, 9, 12 of [DI])

§ Affine models

$F_1(N) : \mathbb{Z}[1/N]$ - schemes $\longrightarrow$ Sets

Objects: $S \longmapsto \{$ iso classes of $(\mathcal{E}, P) \}$

$P : S \to \mathcal{E}$ is a "pt of exact order $N$", i.e. $\forall$ geom pts $s : \mathrm{Spec}(k) \to S$, $P \circ s$ has exact order $N$ in $\mathcal{E}(k)$

"elliptic curve over $S$"

// smooth proper gp. scheme whose geom. fibers are ell. curves

Morphisms:

If $f : S \longrightarrow T$ is a morphism of $\mathbb{Z}[\frac{1}{N}]$-schemes

define $F_1(N)(f) : F_1(N)(T) \longrightarrow F_1(N)(S)$

$$(\mathcal{E}, P) \longmapsto (\mathcal{E}_S, P_S)$$

w/

$$\begin{array}{ccc} \mathcal{E}_S & \to & \mathcal{E} \\ \downarrow & & \downarrow \\ S & \to & T \end{array} \qquad \begin{array}{ccc} S & \to & T \\ \downarrow{\scriptstyle P_S} & & \downarrow{\scriptstyle P} \\ \mathcal{E}_S & \to & \mathcal{E} \end{array}$$

Thus $F_1(N)$ is a functor.

Thm 8.2.1 $(N > 3)$. $\exists$ smooth $\mathbb{Z}[1/N]$-scheme $Y_1(N)$ representing $F_1(N)$, i.e.

$$Y_1(N)(S) := \mathrm{Hom}(S, Y_1(N)) \underset{\text{bij, funct. in } S}{=} F_1(N)(S)$$

[Rmk: $Y_1(N)$ is smooth of rel. dim 1 over $\mathbb{Z}[1/N]$ w/ irred. geometric fibers]

Pf: Igusa, Katz-Mazur.

Taking $S = \mathbb{C}$ get

$$\mathcal{Y}_1(N)(\mathbb{C}) \underset{\text{bij}}{=} Y_1(N)$$

$\therefore$ $\mathcal{Y}_1(N)$ is a model of $Y_1(N)$ over $\mathbb{Z}[1/N]$.

Taking $\mathrm{id} : \underset{\overset{\|}{S}}{\mathcal{Y}_1(N)} \longrightarrow \mathcal{Y}_1(N)$ get $\left(\mathcal{E}^{\mathrm{univ}}, P^{\mathrm{univ}}\right)_{/\mathcal{Y}_1(N)}$.

<u>Property:</u> Every $(\mathcal{E}, P)$ over $T$ is obtained by base change

$$\begin{tikzcd} \mathcal{E} \arrow[r, dashed] \arrow[d, dotted] & \mathcal{E}^{\mathrm{univ}} \arrow[d] \\ T \arrow[r] & \mathcal{Y}_1(N) \end{tikzcd}$$

<u>Variant 8.2.2</u>: Rmk: $\underset{\substack{\text{section of exact}\\ \text{order } N}}{P : S \to \mathcal{E}} \underset{\text{equiv.}}{\longleftrightarrow} \underset{\substack{\text{closed immersion}\\ \text{of gp. schemes}\\ \text{over } S}}{(\mathbb{Z}/_N)_S \hookrightarrow \mathcal{E}/_S}$

- Some other authors consider instead $i : (\mu_N)_{/S} \hookrightarrow \mathcal{E}$

Resulting moduli problem is represented by by a smooth affine scheme $\mathcal{Y}_\mu(N)/\mathbb{Z}$.

- So . $\mathcal{Y}_\mu(N)(\mathbb{C}) \underset{\text{bij}}{=} Y_1(N)$

Turns out $\left(E_\tau, i : \underset{\underset{\mu_N}{\cap}}{e^{2\pi i/N}} \mapsto \tfrac{1}{N} + \Lambda_\tau\right) \leftarrow\!\shortmid \left(E_\tau = \mathbb{C}/\Lambda_\tau, \tfrac{1}{N} + \Lambda_\tau\right)$ $\tau \in \mathcal{H}$

$\Sigma$ These models are iso over $\mathbb{Z}[1/N, G_N]$ (not over $\mathbb{Z}[1/N]$).

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Rmk:

- |||ly $\mathcal{Y}_0(N)_{/\mathbb{Z}[1/N]}$ : "coarse moduli scheme" $\mathcal{Y}_0(N)(\mathbb{C}) \underset{\text{bij}}{=} Y_0(N)$ if $N > 3$.
- For $N \leq 3$ get similar "~~coarse~~ coarse mod. schemes" $\mathcal{Y}_1(N)$ $\mathcal{Y}_0(N)$.

<u>Rmk</u>: (§ 8.3) Can define models for $T_p$ (as was done/$\mathbb{Q}$)

— . —

§ <u>Gen. ell. curves</u>: <u>Ch 9</u> of [DI].

<u>Motivation</u>: $\mathbb{C}/\Lambda_\tau \simeq \mathbb{C}^*/\langle q\rangle \qquad q = e^{2\pi i\tau}$

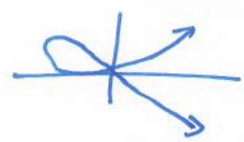
Moreover an equation for LHS is obtained by substituting $q = e^{2\pi i \tau}$ in the <u>Tate curve</u>:
$$E_q : Y^2Z + XYZ = X^3 + a_4 XZ^2 + a_6 Z^3 \Big/ \operatorname{Spec} \mathbb{Z}[[q]]$$

w/
$$a_4 = -5 \sum_{n\geq 1} \frac{n^3 q^n}{1-q^n}$$
$$a_6 = -\frac{1}{12} \sum_{n\geq 1} \frac{(7n^5 + 5n^3)q^n}{1-q^n}$$

As $\tau \to \infty$, $q \to 0$, $E_\tau$ degenerates to the deg. ell. curve $\boxed{C : Y^2Z + XYZ = X^3}$

Sing: $X = Y = 0$ ord. double point

Write $C^{reg}$ = smooth locus of $C$



$y^2 + xy = x^3$

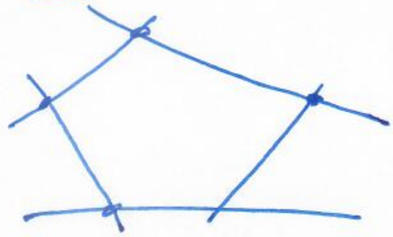
Get $+ : C^{reg} \times C \to C$

(by putting $q = 0$ in gp. law of Tate curve).

Moreover have:
$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ \big\downarrow\wr \quad \big\downarrow{\scriptstyle \text{extends to norm.}} & \circlearrowleft & \big\downarrow \\ C^{reg} \times C & \longrightarrow & C \end{array}$$

<u>Def</u> The pair $(C, +)$ is a Néron 1-gon.
More generally define a Néron $N$-gon for $N \geq 1$ / alg. closed field as:

(22)

$C_N :=$  $=$ $N$ irred components, each iso to $\mathbb{P}^1$ indexed by $\mathbb{Z}/N$

Require: ① Normalization $\coprod_{\mathbb{Z}/N} \mathbb{P}^1 \to C_N$ + $N$ ord. double points.

to send $(\infty)_i \longmapsto$ same
$(0)_{i+1} \longmapsto$ point.

Let $+ : C_N^{reg} \times C_N \longrightarrow C_N$ be char. by comm. of diagram

$$\begin{tikzcd}
\coprod_{i \in \mathbb{Z}/N} \mathbb{G}_m \arrow[d, "s"] & \times & \coprod_{i \in \mathbb{Z}/N} \mathbb{P}^1 \arrow[d, "\text{norm.}"] \arrow[r] & \coprod_{i \in \mathbb{Z}/N} \mathbb{P}^1 \arrow[d, "\text{norm.}"] \\
C_N^{reg} & \times & C_N \arrow[r] & C_N
\end{tikzcd}$$

$(x)_i \qquad (y)_j \longmapsto (xy)_{i+j}$

<u>Def</u> A <u>generalized elliptic curve</u> over $S$ is a pair $(E,+)$ w/ · $E$ a scheme over $S$

· $+ : \quad E^{reg} \times E \to E$

s.t. • $+$ makes $E^{reg}$ a comm. gp. scheme$/S$ acting on $E$

• geom. fibers of $(E,+)$ are · elliptic curves
· Néron $N$-gons.

<u>Facts</u>: · sm $\Rightarrow$ ell. curve
· $S = \mathrm{Spec}(k)$ $k$ alg closed $\Rightarrow$ ell. curve or Néron $N$-gon.

<u>Eg (9.2.1)</u> : Consider $E_q$ (Tate Curve) over $\underset{S}{\mathrm{Spec}\, \mathbb{Z}[[q]]}$

$E_q^{reg} =$ comp. of $X = Y = q = 0$.

Gp law extends to $+ \; E_q^{reg} \times E_q \longrightarrow E_q$ over $S$.

$s: \mathbb{Z}[[q]] \to k \xrightarrow{s(q)=0} (E_{q,k}, +_k) =$ Néron 1-gon

geom point $s \longrightarrow C$ ... ell. curve (if $k = \mathbb{C}$, $s(q) = e^{2\pi i \tau}$ $\Rightarrow C_\tau \cong E_{...}$

# § Canonical models revisited (§ 9.3)

Regard $X_1(N)$ as a moduli space.

$\leftrightarrow [E, P]$    $E$ : gen. ell. curve/$\mathbb{C}$
$P \in E^{reg}$ has exact order $N$
(and $\langle P \rangle$ meets every component).

Turns out the map (cf. Lec 3).

$$\underset{\mathbb{Z}/N}{[(c, \underset{(\mathbb{Z}/(c,N))^*}{d})]} \longleftrightarrow \text{Néron } N\text{-gon}_{/(c,N)} \ (E, P)$$

$$P = \left(e^{2\pi i d/N}\right)_{c/(c,N)}$$

Rmk $(d, (c,N)) = 1 \Rightarrow$
$P$ has exact order $N$.

component.

**Def** $\mathcal{Y}_1(N)$ : $\mathbb{Z}[1/N]$-Schemes $\longrightarrow$ Sets
$S \mapsto \{$ iso. classes of $(E, P)$ $\}$

gen. ell. curve$/S$
$P: S \to E^{reg}$ section of exact order $N$
$+$
$\forall$ geom points $s: \mathrm{Spec}(k) \to S$ image of resulting immersion $(\mathbb{Z}/N)_k \hookrightarrow \mathcal{E}_k^{reg}$ meets every component.

**Thm** (N 74). $\mathcal{Y}_1(N)$ is representable by $\mathcal{X}_1(N)$ over $\mathbb{Z}[1/N]$.

$\therefore \ \mathcal{X}_1(N)(\mathbb{C}) \underset{bij}{=} X_1(N)$.

Rmk: • llly "coarse mod. schemes" $\mathcal{X}_0(N)$ (N 74)
• $\mathcal{X}_1(N)$, $\mathcal{X}_0(N)$ for $N \leq 4$.

E.g. $E_q \in \mathcal{Y}_0(1)(\mathbb{Z}[[q]])$ $\therefore$ induces $\mathrm{Spec}(\mathbb{Z}[[q]]) \to \mathcal{X}_0(1)$
$\therefore \ \mathrm{Spf}\, \mathbb{Z}[[q]] \to \hat{\mathcal{X}}_0(1)$.

Variant 9.3.6: Again using $(\mu_N)_S \hookrightarrow \mathcal{E}_S^{reg}$ (N 74).

get a model $\mathcal{X}_\mu(N)/\mathbb{Z}$ of $X_1(N)$.

Convenient for $q$-exp. principle.

Rmk: Again $\exists$ model for $T_p$.

§ q-exp principle: (§12.3).

$\exists$ map $M_k(\Gamma_1(N)) \hookrightarrow \mathbb{C}[[q]]$

$f \longmapsto$ q-exp of $f$

Let $M_k(\Gamma_1(N), \mathbb{Z})$ = pre-image of $\mathbb{Z}[[q]]$ $\hookrightarrow \mathbb{Z}[[q]]$

Let $M_k(\Gamma_1(N), A) = M_k(\Gamma_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} A$ = "mod. forms w/ coeffs in A" $\hookrightarrow A[[q]]$.

↑ ring.

To make this space precise need alg. def of mod. forms:

Let $\omega$ = pull back along zero sec $\mathcal{X}_\mu(N) \to \mathcal{E}_{univ}$ of $\Omega^1_{\mathcal{E}_{univ}/\mathcal{X}_\mu(N)}$

**Def** $\mathcal{M}_k(\Gamma_1(N), A) = H^0\left(\mathcal{X}_\mu(N)_A, \omega_A^{k}\right)$

= "modular forms over A".

Facts
- $\mathcal{M}_k(\Gamma_1(N), \mathbb{C}) = M_k(\Gamma_1(N))$ (compare line bundles)
- $\mathcal{M}_k(\Gamma_1(N), A) \otimes_A B \simeq \mathcal{M}_k(\Gamma_1(N), B)$ if $k \geq 1$, $\frac{1}{N} \in B$.

Recall: Consider $s_\infty$: Spec $\mathbb{Z} \to \mathcal{X}_\mu(N)$ corr. to gen. ell. curve. $(\mathbb{P}^1_{/\mathbb{Z}}, \mu_N \hookrightarrow \mathbb{G}_m \mathbb{P}^1_{/\mathbb{Z}})$

closed immersion.

$q=0$

In fact, factors as Spec $\mathbb{Z}[[q]]$ corr. to $(E_q, \mu_N \hookrightarrow E_q)$ Tate curve.

Get an induced map of formal schemes

$\operatorname{Spf} \mathbb{Z}[[q]] \xrightarrow{\sim} \hat{\mathcal{X}}_\mu(N)$ "completion along image of $s_\infty$.

Let $\hat{\omega}$ = completion of $\omega$. $\longrightarrow \hat{\omega}$ free $\mathbb{Z}[[q]]$-mod. w/ can. gen $\omega_{can}$.

Now $\omega_{can}^{\otimes k} \longleftrightarrow \hat{\omega}^{\otimes k}$.

$\therefore$ Get $\phi_{\infty,A}: H^0(\mathcal{X}_\mu(N), \omega_A^{\otimes k}) \to H^0(\hat{\mathcal{X}}_\mu(N), \hat{\omega}_A^{\otimes k}) = A[[q]] \cdot \omega_{can}^{\otimes k}$

q-expansion map. (for a ring A)

**Thm 12.3.4** 1. $\Phi_{\infty, A}$ is injective.

2. $$\begin{tikzcd} \mathcal{M}_k(\Gamma_1(N), A) \arrow[r, hook, "\Phi_{\infty,A}"] \arrow[d, hook, "\text{inj}"'] \arrow[dr, phantom, "\circlearrowright \Phi_{\infty,B}"] & A[[q]] \arrow[d, hook] \\ \mathcal{M}_k(\Gamma_1(N), B) \arrow[r, hook] & B[[q]] \end{tikzcd}$$

$\therefore$ Image of $\mathcal{M}_k(\Gamma_1(N), A)$ in $\mathcal{M}_k(\Gamma_1(N), B)$ consists exactly of the set of mod. forms w/ q-exp at $\infty$ in A.

**Cor:** Im of $\mathcal{M}_k(\Gamma_1(N), \mathbb{Z})$ in $\mathcal{M}_k(\Gamma_1(N), \mathbb{C}) = M_k(\Gamma_1(N))$ is exactly $M_k(\Gamma_1(N), \mathbb{Z})$ earlier.

**Thm:** For all $N \geq 1$, $k \geq 1$, $M_k(\Gamma_1(N))$ has a basis over $\mathbb{Z}$.

**Pf.:** $N \leq 3$ Check directly

$N \geq 4$ $\mathcal{M}_k(\Gamma_1(N), \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\sim} \mathcal{M}_k(\Gamma_1(N), \mathbb{C}) = M_k(\Gamma_1(N))$

$\mathcal{M}_k(\Gamma_1(N), \mathbb{Z}) \overset{\text{Cor}}{=} M_k(\Gamma_1(N), \mathbb{Z})$ ∎