Exercises

1) For d-regular G and $\bar{A} = \frac{1}{d} A$, show that for any $f, g : V \to \mathbb{R}$

$$\langle f, \bar{A} g \rangle = \mathop{\mathbb{E}}_{(u,v) \in E} f(u)\, g(v)$$

2) Show the above for non-regular graphs with $\bar{A} = D^{-1} A$ and $\langle f, g \rangle = \mathop{\mathbb{E}}_{v \sim \mu} f(v)\, g(v)$

where $\mu(v) = \dfrac{\deg(v)}{2|E|}$

3) Let $\bar{A} = D^{-1} A$ and μ be as above.

Prove that $\forall f, g : V \to \mathbb{R}$

$$\langle f, \bar{A} g \rangle_\mu = \langle \bar{A} f, g \rangle_\mu$$

4) Let G be a d-regular graph. and $\lambda = \max\{\lambda_2, -\lambda_n\}$

Using $\mathrm{Tr}(\bar{A}^2) = \sum \lambda_i^2$, show that $\lambda \geq \frac{1}{\sqrt{d}}$.

5) Let $G = (L, R, E)$ be a bipartite graph with biadjacency matrix B defined as

$$B(l, r) = 1 \iff (l, r) \in E$$

for $l \in L$, $r \in R$

if $\sigma_1 \ldots \sigma_m$ are the non-zero singular values of B, prove that the eigenvalues of the adjacency matrix $A = \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)$ are

$\{\pm \sigma_1, \ldots, \pm \sigma_m\}$