Step 5: The effective action $S_Q^{eff} = \int d^Dx \, \mathcal{L}_Q^{eff}(x)$ contains non-localities at the scale $\Delta x \sim \frac{1}{2m_Q}$, as expected (inverse derivative). Since derivatives acting on h_v produce the residual momentum $k^\mu = \mathcal{O}(\Lambda_{QCD})$, and since the covariant derivative only contains the soft gluon field $A_s^{\mu,a} = \mathcal{O}(\Lambda_{QCD})$ – see below –, we can expand:

$$\frac{1}{2m_Q + iv\cdot D_s} = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv\cdot D_s}{2m_Q} \right)^n$$

This leads the local effective Lagrangian:

$$\boxed{\mathcal{L}_Q^{eff} = \bar{h}_v \, iv\cdot D_s \, h_v + \frac{1}{2m_Q} \bar{h}_v \, i\not{D}_s^\perp \, i\not{D}_s^\perp \, h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right)}$$

(H. Georgi: PLB 240 (1990) 447)

Note also that:

$$H_v = \underbrace{\frac{1}{2m_Q} \, i\not{D}_s^\perp}_{\mathcal{O}\left(\frac{\Lambda_{QCD}}{m_Q}\right)} \, h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

is power-suppressed relative to the field h_v.

New symmetries of QCD in the heavy-quark limit ($m_Q \to \infty$):

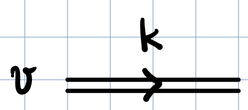
$$\mathcal{L}_{HQET} = \bar{h}_v \, i v \cdot D_s \, h_v = \bar{h}_v \left(i v \cdot \partial + g_s \, v \cdot A_s^a \, t_a \right) h_v$$

$\rightarrow$ independent of the heavy-quark mass (m_Q) and spin (no Dirac matrices):

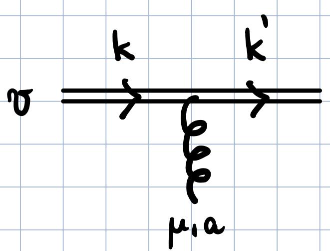
$$\boxed{SU(2 n_Q) \text{ spin-flavor symmetry}}$$

(N. Isgur, M.B. Wise: PLB 232 (1989) 113
" 237 (1990) 527)

Feynman rules of HQET:



$$= \frac{i}{v \cdot k + i\epsilon}$$ propagator



$$= i g_s v^\mu t_a$$ vertex

First-order power corrections and hard gluon effects:

Using that:

$$\gamma_\mu \gamma_\nu = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} + \frac{1}{2}[\gamma_\mu, \gamma_\nu] = g_{\mu\nu} - i\sigma_{\mu\nu}$$

we find:

$$\boxed{\begin{aligned} \mathcal{L}_Q^{\text{eff}} &= \bar{h}_v\, i v\cdot D_s\, h_v \\ &+ \frac{1}{2m_Q}\left[\bar{h}_v (iD_s^\perp)^2 h_v + \frac{g_s}{2}\bar{h}_v \sigma_{\mu\nu} G_s^{\mu\nu} h_v\right] + \ldots \end{aligned}}$$

use: $[iD_s^{\perp\mu}, iD_s^{\perp\nu}] = ig_s G_s^{\mu\nu}$

nonrelativistic kinetic energy $\vec{k}^2/2m_Q$

chromomagnetic interaction with heavy-quark spin

↳ $\mathcal{O}\left(\frac{1}{m_Q}\right)$ terms break the spin-flavor symmetry!

Implications for spectroscopy:

$$\frac{m_{B^*}}{m_B} = 1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right)$$

exp.: $1 + (\lesssim 1\%)$

$$\frac{m_{D^*} - m_D}{m_{B^*} - m_B} = \frac{m_b}{m_c} \approx 3$$

exp.: $\frac{142\ \text{MeV}}{46\ \text{MeV}}$

$$m_{\Lambda_b} - m_B \simeq m_{\Lambda_c} - m_D \quad \text{etc.}$$

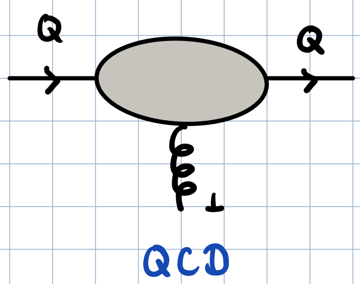
↳ all works well!

Hard-gluon corrections can introduce nontrivial Wilson coefficients for the subleading terms, but Lorentz invariance ensures that the kinetic operator is not renormalized. The chromomagnetic operator gets multiplied by:

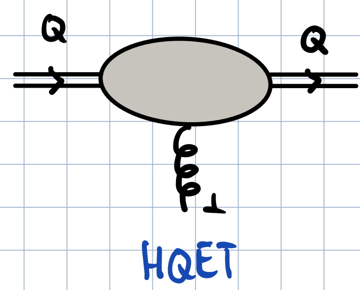
$$C_{mag}(\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}\left[-C_A \ln\frac{m_Q^2}{\mu^2} + 2(C_A + C_F)\right] + \mathcal{O}(\alpha_s^2)$$

(E. Eichten, B. R. Hill: PLB 243 (1990) 427)

→ perturbative matching calculation:



$\stackrel{!}{=} C_{mag}(\mu)$



$+ \mathcal{O}\left(\frac{1}{m_Q^2}\right)$

HQET power counting:

It is useful to derive counting rules in the HQET expansion parameter $\lambda = \frac{\Lambda_{QCD}}{m_Q} \ll 1$, by expressing all dimensionful quantities in powers of the hard scale m_Q. E.g.:

residual momentum: $k^\mu \sim \Lambda_{QCD} = \lambda\, m_Q \sim \lambda$

$$\Rightarrow \quad \boxed{\partial^\mu \sim \lambda}$$

For the fields, the power counting follows from the propagators:

$$\langle 0 | T\{ h_v(x)\, \bar{h}_v(0) \} | 0 \rangle = \underbrace{\int \frac{d^4k}{(2\pi)^4}}_{\sim \lambda^4} \underbrace{e^{-ik\cdot x}}_{\sim 1} \underbrace{\frac{i}{v\cdot k + i\epsilon}}_{\sim \frac{1}{\lambda}} \sim \lambda^3$$

$$\Rightarrow \quad \boxed{h_v \sim \lambda^{3/2}}$$

and for the soft gluons:

$$\langle 0 | T\{ A_s^{\mu,a}(x)\, A_s^{\nu,b}(0) \} | 0 \rangle = \underbrace{\int \frac{d^4k}{(2\pi)^4}}_{\sim \lambda^4} \underbrace{\frac{i\delta^{ab}}{k^2 + i\epsilon}}_{\sim \frac{1}{\lambda^2}} \underbrace{\left[-g^{\mu\nu} + (1-\xi) \frac{k^\mu k^\nu}{k^2} \right]}_{\sim \lambda^0} \sim \lambda^2$$

$$\Rightarrow \quad \boxed{A_s^\mu \sim \lambda}$$

This ensures a homogeneous power counting for the covariant derivative $iD_s^\mu \sim \lambda$. These rules are those implied by naive dimensional analysis.

It follows that the first few terms in the effective HQET Lagrangian scales as:

$$\mathcal{L}_{eff}^{Q} \sim \lambda^4 + \frac{1}{m_Q} \lambda^5 + \ldots$$

$$\Rightarrow \quad S_{HQET} = \underbrace{\int d^4x}_{\sim \lambda^{-4}} \mathcal{L}_{eff}^{Q}(x) \sim \lambda^0 + \frac{1}{m_Q} \lambda + \mathcal{O}(\lambda^2)$$

↳ as it should be

Decoupling transformation:

Let us now reconsider the leading-order Lagrangian of HQET:

$$\boxed{\mathcal{L}_{HQET} = \bar{h}_v \, i v \cdot D_s \, h_v}$$

Whenever the coupling to the gauge field involve the covariant derivative dotted into a 4-vector (here: v^μ), this coupling can be removed by a <u>field redefinition</u>.

Let us define:

new field

$$h_v(x) = S_v(x) \, h_v^{(0)}(x)$$

$$S_v(x) = \mathcal{P} \exp\left(i g_s \int_{-\infty}^{0} dt \, v \cdot A_s(x + vt) \right) \qquad \text{unitary operator}$$

↳ soft Wilson line along direction v^μ (time-like)

The symbol "P" means that gauge fields are ordered from left to right in the order of decreasing t values.
Using that

$$x^\mu = (v\cdot x)\, v^\mu + x_\perp^\mu \qquad (\text{with } v\cdot x_\perp = 0)$$

we have:

$$S_v(x) = P \exp\left(i g_s \int_{-\infty}^{v\cdot x} dt'\, v\cdot A_s(x_\perp + v t') \right)$$

$$\Rightarrow \quad i v\cdot\partial_x\, S_v(x) = -g_s\, v\cdot A_s(x)\; S_v(x)$$

or: $$\boxed{\left[i v\cdot D_s\; S_v(x) \right] = 0}$$ (important!)

It follows that:

opposite ordering

$$\mathcal{L}_{HQET} = \bar{h}_v^{(0)}\, S_v^\dagger\; i v\cdot D_s\; S_v\, h_v^{(0)}$$

$$= \bar{h}_v^{(0)}\, \underbrace{S_v^\dagger\, S_v}_{\mathbb{1}}\; i v\cdot\partial\, h_v^{(0)} = \bar{h}_v^{(0)}\; i v\cdot\partial\; h_v^{(0)}$$

↳ free theory!

What does this mean? In the heavy-quark limit, soft gluons can be decoupled from the heavy quark by means of a field redefinition. The new fields $h_v^{(0)}$ no longer interact; they are sterile. But how, then, do heavy quark interact with other particles?

In fact, they interact like Wilson lines, e.g.:

$$\bar{q}_s \not{A}_s h_v = (\bar{q}_s \not{A}_s S_v)\, h_v^{(0)}$$

$\Rightarrow$ heavy quark behave like Wilson lines along the time-like direction v^μ (Eichten, Hill 1990)

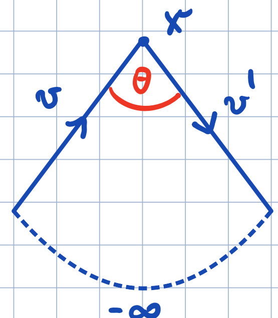
A particularly interesting example are flavor-changing heavy-quark currents such as $\bar{c}\gamma^\mu(1-\gamma_5)b$, which mediate semileptonic decays such as $B(v) \to D^{(*)}(v')\,\ell\,\bar{\nu}$. In HQET these currents match onto:

$$\bar{c}\gamma^\mu(1-\gamma_5)b \longrightarrow \sum_i \underset{\substack{\uparrow \\ \text{pert. QCD}}}{C_i(m_b, m_c, \mu)}\; \bar{h}_{v'}\, \Gamma_i^\mu\, h_v \qquad \text{(MN 1991)}$$

After soft-gluon decoupling the effective currents are:

$$\bar{h}_{v'}\, \Gamma_i^\mu\, h_v = \bar{h}_{v'}^{(0)}\, \Gamma_i^\mu\, \underbrace{(S_{v'}^\dagger S_v)}_{\substack{\text{non-trivial coupling to gluons,} \\ \text{unless } v=v' \text{ (zero recoil)}}}\, h_v^{(0)}$$

Note:

$S_{v'}^\dagger S_v =$ 

↳ closed Wilson loop with a cusp at x with angle θ

$$\boxed{\cosh\theta = \frac{v\cdot v'}{\sqrt{v^2 v'^2}} = v\cdot v'}$$

In 1987 (several years before HQET), Korchemsky & Radyushkin showed that Wilson loops with cusps require UV renormalization, governed by a universal cusp anomalous dimension:

$$\Gamma_{cusp}(\alpha_s, \theta) = \frac{C_F \alpha_s}{\pi} (\theta \coth\theta - 1) + \mathcal{O}(\alpha_s^2)$$

$\Rightarrow$ $\Gamma_{cusp}(\alpha, \theta = 0) = 0$ to all orders $(v = v')$

(non-renormalization theorem)

In the context of HQET, this "velocity-dependent anomalous dimension" of heavy-quark currents was calculated by Falk, Georgi, Grinstein & Wise in 1990.

Final comment:

We will see that Wilson lines and the cusp anomalous dimension also play an important role in SCET. There we will encounter light-like Wilson lines, for which $v^\mu \to n^\mu$ with $n^2 = 0$.