Beyond discrete-time and QND measurement

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Part I: continuous-time QND measurement

Stochastic stability

Definition. Let x_t^z be a diffusion process on the metric state space X, started at $x_0 = z$ and let $\tilde{z}$ denote an equilibrium position of the diffusion, i.e. $x_t^{\tilde{z}} = \tilde{z}$. Then

- the equilibrium $\tilde{z}$ is said to be stable in probability if

$$\lim_{z \to \tilde{z}} \mathbb{P}\left(\sup_{0 \le t < \infty} ||x_t^z - \tilde{z}|| \ge \epsilon \right) = 0 \quad \forall \epsilon > 0.$$

- the equilibrium $\tilde{z}$ is globally stable if it is stable in probability and additionally

$$\mathbb{P}(\lim_{t \to \infty} x_t^z = \tilde{z}) = 1, \quad \forall z \in X.$$

Stochastic Lyapunov theory

Stochastic differential equation: $dx_t = b(x_t)dt + \sigma(x_t)dw_t$ with x_0 as initial condition.

Infinitesimal generator: $\mathcal{L} := b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2}$.

Lyapunov function:

$$\mathbb{E}\left(\frac{dV(x_t)}{dt}\right) = \mathcal{L}V(x_t) = \frac{\partial V}{\partial x}(x_t)b(x_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x_t)\sigma(x_t)^2 \leq 0.$$

Kushner's theorem: Convergence in probability towards the invariant set is included in $\mathcal{L}(V) = 0$.

Problem presentation for quantum spin-$\frac{1}{2}$ systems

State space: $\mathcal{S}_2 = \{\rho \in \mathbb{C}^{2\times 2} | \, \rho \geq 0, \; \rho = \rho^*, \; \mathrm{Tr}(\rho) = 1\}$.

$$d\rho_t = -iu_t[\sigma_y, \rho_t]\, dt + \frac{1}{2}(2\sigma_z\rho_t\sigma_z - \sigma_z^2\rho_t - \rho_t\sigma_z)\, dt + (\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}(\sigma_z\rho_t)\rho_t)dW_t.$$

- Two equilibriums are $\rho_g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- Main problem: To stabilize deterministically one of these states.

Consider the Lyapunov function: $V(\rho_t) = 1 - \mathrm{Tr}(\rho_t\rho_e)^2$

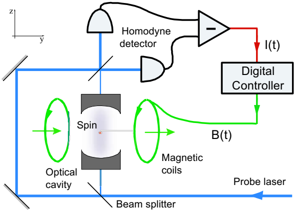
$$\mathcal{L}V(\rho_t) = 2u_t\mathrm{Tr}(i[\sigma_y, \rho_t]\rho_e) - 4\mathrm{Tr}(\rho_t\rho_e)^2(1 - \mathrm{Tr}(\sigma_z\rho_t))^2.$$

$$u_t = -\mathrm{Tr}(i[\sigma_y, \rho_t]\rho_e).$$

Previous results

- R. Van Handel, J. K. Stockton, and H. Mabuchi, Feedback control of quantum state reduction, IEEE TAC, 50(6), 768–780, 2005; (2-level, continuous)
- M. Mirrahimi and R. Van Handel, Stabilizing feedback controls for quantum systems, SIAM Journal on Control and Optimization, 46(2), 445–467, 2007; (N-level, switching)
- K. Tsumura, Global stabilization at arbitrary eigenstates of N-dimensional quantum spin systems via continuous feedback, ACC, 4148–4153, 2008; (N-level, continuous)

Van Handel, Stockton, Mabuchi, 2005



Bloch sphere: $\rho = \dfrac{I + x\sigma_x + y\sigma_y + z\sigma_z}{2}$

$$dx_t = (B(t)z_t - \frac{1}{2}Mx_t)dt - \sqrt{M\eta}\,x_t z_t dw_t$$

$$dy_t = -\frac{1}{2}My_t dt - \sqrt{M\eta}\,y_t z_t dw_t$$

$$dz_t = -B(t)x_t dt + \sqrt{M\eta}(1 - z_t^2)dw_t$$

Aim: stabilizing $(x, z) = (0, 1) \Longrightarrow B(t) = -\lambda x_t - \mu(1 - z)$ with $\lambda > 0$ and a Lyapunov function

$$V(x, z) = (\alpha + \beta z - x)(1 - z)$$

Mirrahimi and Van Handel, 2007

$$d\rho_t = -iu_t[\sigma_y, \rho_t]\,dt + \frac{1}{2}(2\sigma_z\rho_t\sigma_z - \sigma_z^2\rho_t - \rho_t\sigma_z)\,dt$$
$$+ (\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}(\sigma_z\rho_t)\rho_t)dW_t.$$

Theorem (M. Mirrahimi and R. van Handel, 2007.) Consider the following control law

- $u_t = -\,\mathrm{tr}\left(i[\sigma_y, \rho_t]\rho_e\right)$ if $\mathrm{tr}\left(\rho_t\rho_e\right) \geq \gamma$
- $u_t = 1$ if $\mathrm{tr}\left(\rho_t\rho_e\right) \leq \gamma/2$
- If $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \mathrm{tr}\left(\rho\rho_e\right) < \gamma\}$, then $u_t = -\,\mathrm{tr}\left(i[\sigma_y, \rho_t]\rho_e\right)$ if the last entry of ρ_t into $\mathcal{B}$ has been via the boundary $\mathrm{tr}\left(\rho\rho_e\right) = \gamma$ and $u_t = 1$ if not. Then there exists a $\gamma > 0$ s.t. $u(t)$ globally stabilizes the system around ρ_e and $\mathbb{E}(\rho_t) \to \rho_e$ as $t \to \infty$.

Main ideas of the proof

Consider $V(\rho) = 1 - \mathrm{tr}\,(\rho\rho_e)$ and for $\alpha \in [0, 1]$ define the set $\mathcal{S}_\alpha = \{\rho \in \mathcal{S}_2 : \ V(\rho) = \alpha\}$, $\mathcal{S}_{>\alpha}$, $\mathcal{S}_{\geq\alpha}$, $\mathcal{S}_{<\alpha}$, and $\mathcal{S}_{\leq\alpha}$.

- Step 1. When the initial state is in the set $\mathcal{S}_1$, the control $u = 1$ ensures the exit of the trajectories in expectation from $\mathcal{S}_1$.
- Step 2. There exists a $\gamma > 0$ such that whenever the initial state lies inside the set $\mathcal{S}_{>1-\gamma}$ and the control field is taken to be $u = 1$, the expectation value of the first exit time from this set takes a finite value.

Main ideas of the proof

- Step 3. Whenever the initial state lies inside the set $S_{\leq 1-\gamma}$ and the control is given by the feedback law $u(t) = -\operatorname{tr}\left(i[\sigma_y, \rho_t]\rho_e\right)$, the sample paths never exit the set $S_{<1-\gamma/2}$ with a probability uniformly larger than a strictly positive value. Then, almost all paths that never leave $S_{<1-\gamma/2}$ converge to the equilibrium point ρ_e.
- Step 4. There is a unique solution ρ_t under the control $u(t)$ by piecing together the solutions with fixed controls $u(t) = 1$ and $u(t) = -\operatorname{tr}\left(i[\sigma_y, \rho_t]\rho_e\right)$.

K. Tsumura, 2008

Theorem (Tsumura, 2008). Consider the quantum spin$-1/2$ system evolving in the set $\mathcal{S}_2$, then

$$u(t) = -\alpha \operatorname{tr}\left(i[\sigma_y, \rho_t]\rho_e\right) + \beta(1 - \operatorname{tr}(\rho\rho_e))$$

globally stabilizes the system evolution of quantum spin$-1/2$ system around ρ_e and $\mathbb{E}(\rho_t) \to \rho_e$ as $t \to \infty$ when $\frac{\beta^2}{8\alpha\eta} < 1$.

Main ideas of the proof

- $\rho = \rho_e$ is stable in probability.
- there exists $0 < \gamma < 1$ and almost all sample paths which never leave the domain $S_{<1-\gamma}$ converge to ρ_e.
- for almost all sample paths there exists a finite time T and after it, they never leave $\mathcal{S}_{<1-\gamma}$.

Exponential stabilization of quantum spin-1/2 systems

Bures distance

The Bures distance [1] between two quantum states ρ_a and ρ_b lying in $\mathcal{S}_2$ is given by

$$d_B(\rho_a, \rho_b) := \sqrt{2 - 2\mathrm{Tr}\left(\sqrt{\sqrt{\rho_b}\rho_a\sqrt{\rho_b}}\right)}$$

which is equal to for the 2-dimensional state space

$$d_B(\rho_a, \rho_b) = \sqrt{2 - 2\sqrt{\mathrm{Tr}(\rho_a\rho_b) + 2\sqrt{\det(\rho_a)\det(\rho_b)}}}.$$

Also, the Bures distance between a quantum state ρ_a and a set $E \subseteq \mathcal{S}_2$ is

$$d_B(\rho_a, E) = \min_{\rho \in E} d_B(\rho_a, \rho).$$

Given $E \subset \mathcal{S}_2$, we define the neighborhood $B_r(E)$ of E as

$$B_r(E) = \{\rho \in \mathcal{S}_2 : d_B(\rho, E) < r\}.$$

[1] I. Bengtsson, K. Zyczkowski, Cambridge University Press, 2017.

Stochastic stability[2]

Definition. Consider the SDE: $d\rho_t = f(\rho_t)dt + g(\rho_t)dW_t$.
Equilibrium: $\bar{\rho}$

- stable in probability for every $\epsilon \in (0,1)$ and $r > 0$ $\forall\, \rho_{t_0} \in B_\delta(\bar{\rho})$, there exists a $\delta = \delta(\epsilon, r, t_0)$ s.t. $\mathbb{P}\left\{\rho_t \in B_r(\bar{\rho}) \text{ for } t \geqslant t_0\right\} \geqslant 1 - \epsilon$.
- exponential stable in mean there exists $\alpha, \beta > 0$, $\forall\, \rho_{t_0} \in \mathcal{S}_2$ s.t. $\mathbb{E}\left[d_B(\rho_t, \bar{\rho})\right] \leqslant \alpha\, d_B(\rho_{t_0}, \bar{\rho}) e^{-\beta(t-t_0)}$.
- almost surely exponentially stable $\forall\, \rho_{t_0} \in \mathcal{S}_2$ $\limsup_{t\to\infty} \frac{1}{t} \log d_B(\rho_t, \bar{\rho}) < 0, \quad a.s.$

[2] H. K. Khalil, 1996 and X. Mao, Elsevier, 2007.

Itô's formula

Suppose that the function $V(\rho, t) : \mathcal{S} \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuously twice differentiable in ρ and once in t. The infinitesimal generator $\mathscr{L}$ associated with $d\rho_t = F_N(\rho_t)dt + G_N(\rho_t)dW_t$ is

$$\mathscr{L} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \rho} F_N(\rho_t) + \frac{1}{2}\frac{\partial^2}{\partial \rho^2} G_N^2(\rho_t)$$

If $\mathscr{L}$ acts on such $V(\rho, t)$ and by Itô's formula, then

$$dV(\rho, t) = \mathscr{L} V(\rho, t)dt + \frac{\partial V(\rho, t)}{\partial \rho} G_N(\rho_t)dW_t.$$

Hence, $d\mathbb{E}(V(\rho, t))/dt = \mathbb{E}(\mathscr{L} V(\rho, t))$.

2-level quantum systems: spin-$\frac{1}{2}$ systems

The state of 2-level quantum system can be represented by

$$\mathcal{S}_2 = \{\rho \in \mathbb{C}^{2\times 2} : \rho = \rho^*, \mathrm{Tr}(\rho) = 1, \rho \geqslant 0\}$$

Bloch sphere coordinates:

$$\rho = \frac{\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2} = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

where $\sigma_{x,y,z}$ are the Pauli matrices. The vector (x, y, z) belongs to the ball,

$$B(\mathbb{R}^3) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1\}$$

Two orthonormal states of 2-level quantum system:

- **Ground state**: $\rho_g = |0\rangle\langle 0| \longleftrightarrow (0, 0, 1)$ of energy ω_g
- **Excited state**: $\rho_e = |1\rangle\langle 1| \longleftrightarrow (0, 0, -1)$ of energy ω_e

Stochastic master equation

Time evaluation of the quantum state of the quantum spin-$\frac{1}{2}$ systems under the imperfect continuous measurement is described by:

$$d\rho_t = \left(-i\frac{\omega_{eg}}{2}[\sigma_z, \rho_t] + \frac{M}{4}(\sigma_z\rho_t\sigma_z - \rho_t) - i\frac{u_t}{2}[\sigma_y, \rho_t]\right) dt$$
$$+ \frac{\sqrt{\eta M}}{2}[\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}(\sigma_z\rho_t)\rho_t]dW_t$$

- W_t is the 1-dimensional standard Wiener process.
- u_t scalar control input.
- $\eta \in [0, 1]$ is determined by the efficiency of the photo-detectors, and $M > 0$ is the strength of the interaction between the light and the atoms, $\omega_{eg} = \omega_e - \omega_g$.

Evolution in Bloch sphere respresentation

$$
\begin{aligned}
dx_t &= \left(-\omega_{eg} y_t - \frac{M x_t}{2} + u_t z_t\right) dt - \sqrt{\eta M} x_t z_t dW_t \\
dy_t &= \left(\omega_{eg} x_t - \frac{M y_t}{2}\right) dt - \sqrt{\eta M} y_t z_t dW_t \\
dz_t &= -u_t x_t dt + \sqrt{\eta M}\left(1 - z_t^2\right) dW_t.
\end{aligned}
$$

Strook-Varadhan support theorem

Consider the Ito SDE,

$$dx_t = b(x_t)dt + \sigma(x_t)dw(t)$$

and the associated deterministic controlled equation

$$\frac{dx_t^u}{dt} = b(x_t^u) - \frac{1}{2}\nabla\sigma(x_t^u)x_t^u + u(t)\sigma(x_t^u)$$

Consider $\mathcal{U}$ the set of all piecewise constant functions from $\mathbb{R}^+$ to $\mathbb{R}$ and define

$$\Gamma_x = \overline{\{x^u :\ u \in \mathcal{U}\}}$$

the set of all controlled trajectories starting at x. The set Γ_x is the smallest closed set of the continuous trajectories starting at x such that

$$\mathbb{P}(\{\omega \in \Omega |\ x(\omega) \in \Gamma_x\}) = 1.$$

Never reach lemma

Never reach lemma 1 (without feedback)

Assume that $\rho_{t_0} \notin \bar{E}_2$ with $\bar{E}_2 := \{\rho_e, \rho_g\}$ and that $u_t = 0$. Then

$$\mathbb{P}\{\rho_t \notin \bar{E}_2, \forall t \geqslant t_0\} = 1$$

Never reach lemma 2 (with feedback)

Assume that $\rho_{t_0} \neq \bar{\rho}$ with $\bar{\rho} \in \bar{E}_2$ and that u_t is continuous, continuously differentiable in $\mathcal{S}_2 \setminus \bar{\rho}$ and $|u_t| \leqslant C\sqrt{1 - \mathrm{Tr}(\rho_t \bar{\rho})}$ for some $C \in \mathbb{R}_+$. Then

$$\mathbb{P}\{\rho_t \neq \bar{\rho}, \forall t \geqslant t_0\} = 1$$

Remark. The above lemmas are inspired by analogous results in [3] [4].

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[3] X. Mao, Elsevier, 2007.

[4] R. Khasminiskii, Springer, 2011.

2-level quantum state reduction

Theorem (Liang, A., Mason, 2018). When $u_t = 0$ and $\rho_{t_0} \in \mathcal{S}_2$, the set $\bar{E}_2 := \{\rho_e, \rho_g\}$ is exponentially stable in mean and a.s. exponentially stable with the rate $\frac{\eta M}{2}$. Moreover, the probability of convergence to $\bar{\rho} \in \bar{E}_2$ is $\mathrm{Tr}(\rho_{t_0}\bar{\rho})$.

Proof:

Step 1:

$V(\rho_t) = \sqrt{1 - \mathrm{Tr}^2(\sigma_z \rho_t)} \Rightarrow \mathscr{L} V(\rho_t) = -\frac{\eta M}{2} V(\rho_t) \Rightarrow$

$\mathbb{E}[V(\rho_t)] = V(\rho_{t_0}) e^{-\frac{\eta M}{2}(t-t_0)}$

Step 2: $C_1 d_B(\rho_t, \bar{E}_2) \leqslant V(\rho_t) \leqslant C_2 d_B(\rho_t, \bar{E}_2) \Rightarrow$

$\mathbb{E}[d_B(\rho_t, \bar{E}_2)] \leqslant \frac{C_2}{C_1} d_B(\rho_{t_0}, \bar{E}_2) e^{-\frac{\eta M}{2}(t-t_0)}$

$\Rightarrow \limsup_{t\to\infty} \frac{1}{t} \log d_B(\rho_t, \bar{E}_2) \leqslant -\frac{\eta M}{2}, \quad a.s.$

Step 3:

$\rho_\infty := P_e \rho_e + P_g \rho_g$ and $\mathscr{L}\mathrm{Tr}(\rho_t \bar{\rho}) = 0$, then $\mathrm{Tr}(\rho_t \bar{\rho})$ is a positive martingale, and $P_e = \mathbb{E}[\mathrm{Tr}(\rho_\infty \rho_e)] = \mathrm{Tr}(\rho_{t_0} \rho_e)$, $P_g = \mathbb{E}[\mathrm{Tr}(\rho_\infty \rho_g)] = \mathrm{Tr}(\rho_{t_0} \rho_g)$.

Numerical simulation

Recall:

$$d\rho_t = \left(-i\frac{\omega_{eg}}{2}[\sigma_z, \rho_t] + \frac{M}{4}(\sigma_z\rho_t\sigma_z - \rho_t) - i\frac{u_t}{2}[\sigma_y, \rho_t]\right) dt + \frac{\sqrt{\eta}M}{2}[\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}(\sigma_z\rho_t)\rho_t]dW_t$$

In order to guarantee ρ_t remains in $\mathcal{S}_2$, we rewrite

$$\rho_t + d\rho_t = \frac{\mathbb{M}_{dY_t}\rho_t\mathbb{M}^*_{dY_t} + \frac{1-\eta M}{4}\sigma_z\rho_t\sigma_z dt}{\mathrm{Tr}\left(\mathbb{M}_{dY_t}\rho_t\mathbb{M}^*_{dY_t} + \frac{1-\eta M}{4}\sigma_z\rho_t\sigma_z dt\right)}$$

where

$$\mathbb{M}_{dY_t} = \mathbb{1} - \left[\frac{i}{2}(\omega_{eg}\sigma_z + u_t\sigma_y) + \frac{M}{8}\mathbb{1}\right] dt + \frac{\sqrt{\eta M}}{2}\sigma_z dY_t$$

$$dY_t = dW_t + \sqrt{\eta M}\mathrm{Tr}(\sigma_z\rho_t)dt$$

Numerical simulation

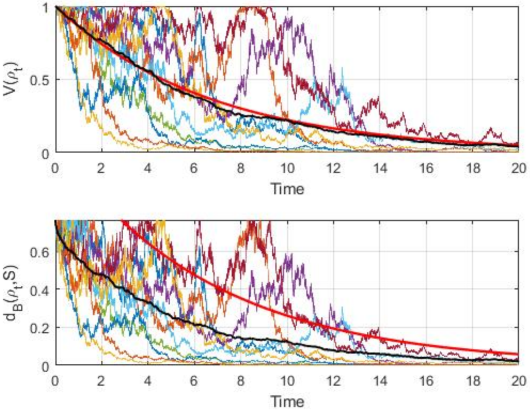


Figure: Quantum state reduction starting at $(0, 0, 0)$, when $\omega_{eg} = 0$, $\eta = 0,3$, $M = 1$. The black curve represents the mean value of the 10 samples, the red curve represents the exponential reference.

Almost surely global exponential stabilization

Theorem, Liang, A., Mason, 2018. Assume that the feedback law u_t satisfies the condition of Never reach lemma and $u_t = 0$ iff $\rho_t = \bar{\rho}$. Suppose that there exists a function $V(\rho)$, which is continuous on $\mathcal{S}_2$ and twice continuously differentiable on the set $\mathcal{S}_2 \setminus \bar{E}_2$, and positive constants C_1, C_2 and positive function $C(r)$ such that

(i) $C_1\, d_B(\rho, \bar{\rho}) \leqslant V(\rho) \leqslant C_2\, d_B(\rho, \bar{\rho}), \forall\, \rho \in \mathcal{S}_2 \setminus \bar{E}_2$

(ii) $\mathscr{L} V(\rho) \leqslant -C(r)\, V(\rho), \forall\, r \in (0, \sqrt{2})\,, \forall\, \rho \in B_r(\bar{\rho}) \setminus \bar{\rho}$.

Then $\bar{\rho}$ is a.s. exponentially stable.

Almost surely global exponential stabilization

Proof:

Step 1: $\bar{\rho}$ is stable in probability;

Step 2: For almost all sample path, there exists $T < \infty$ such that, for all $t \geqslant T$, $\rho_t \in B_r(\bar{\rho})$;

Step 3: $\bar{\rho}$ is almost surely exponentially stable.

Other related works

- Liang, A., Mason, On exponential stabilization of N-level quantum angular momentum systems, Siam journal on Control and Optimization 2019.
- Liang, A., Mason, Robust feedback stabilization of N-level quantum spin systems, Siam journal on Control and Optimization, 2021.
- Liang, A., Mason, Feedback exponential stabilization of GHZ states of multiqubit systems, IEEE Transactions on Automatic Control, 2021.
- Liang, A., Model robustness for feedback stabilization of open quantum systems, Automatica, 2024.
- A., Mason, Ramadan, Feedback stabilization via a quantum projection filter, Siam journal on Control and Optimization, 2025.

Part II: Discrete-time generic (non-QND) measurement

Discrete-time quantum trajectories

State space :

$$S = \{\rho \in \mathbb{C}^{d\times d} |\, \rho = \rho^*,\ \rho \geq 0, \mathrm{tr}\,(\rho) = 1\}$$

quantum channel:

$$\Phi(X) = \sum_i V_i X V_i^\dagger \quad \textit{with} \quad \sum_i V_i^\dagger V_i = Id,\ V_i \in \mathbb{C}^{d\times d}$$

The open quantum system, whose transitions are described by Φ, is a **Markov chain** defined by:

$$\rho_{n+1} = \frac{V_{i_n}\rho_n V_{i_n}^\dagger}{\mathrm{tr}\left(V_{i_n}\rho_n V_{i_n}^\dagger\right)} \quad \textit{with} \quad \mathbb{P}(i_n = i) = \mathrm{tr}\left(V_i \rho_n V_i^\dagger\right)$$

Here $\Phi(\rho) = \sum_{i=1}^m V_i \rho V_i^\dagger$ and $\mathbb{E}(\rho_{n+1}|\rho_n) = \Phi(\rho_n)$

QND measurements

A measurement is Quantum Non Demolition (QND) if there exists a basis $\{|\alpha\rangle\langle\alpha|\}$ of the Hilbert space s.t.

$$\forall i, \quad \frac{V_i |\alpha\rangle\langle\alpha| V_i^\dagger}{\mathrm{tr}\left(V_i |\alpha\rangle\langle\alpha| V_i^\dagger\right)} = |\alpha\rangle\langle\alpha| .$$

$\longrightarrow$ The elements $|\alpha\rangle\langle\alpha|$ are called pointer states.

Example: LKB photon box where $|n\rangle\langle n|$ are pointer states

Selection of the pointer state

Theorem [5] [6] Suppose that $\forall\, \alpha \neq \beta$, $p(i|\alpha) \neq p(i|\beta)$ for some $i \in \{1, \cdots, m\}$. Then,

- there exists a random variable Υ (among pointer states)

$$\lim_{n\to\infty} \rho_n = |\Upsilon\rangle\langle\Upsilon|\,.$$

- $\mathbb{P}(\Upsilon = \alpha) = \mathrm{tr}\left(|\alpha\rangle\langle\alpha|\,\rho_0\right).$

[5] A., Rouchon, Mirrahimi, 2011.
[6] Bauer and Bernar, 2011.

Beyond QND case: invariant subspaces

Recall : $\mathcal{S} = \{\rho \in \mathbb{C}^{d\times d} | \, \rho = \rho^*, \, \rho \geq 0, \mathrm{tr}\,(\rho) = 1\}$

Invariant state space: $\mathcal{D} = \mathcal{S} \cap \{\Phi(X) = X \,|\, X \in \mathcal{B}(\mathbb{C}^d)\}$

Decomposition of $\mathbb{C}^d$: into a recurrent subspace $\mathcal{R} = \sup\{supp(\rho) | \, \rho \in \mathcal{D}\}$ and a transient subspace $\mathcal{T} = \mathcal{R}^{\perp}$

Decomposition of the Hilbert space

Extreme invariant states: extreme points of the set of fixed points of Φ, which is a convex set

Baumgartner, Narnhofer, 2012: $\mathcal{R} = \bigoplus_{u=1}^{N} \mathcal{H}_u$, where

$\mathcal{H}_u$: support of an extreme invariant state ρ_∞^u

QND case:

$$\mathcal{H} = \mathcal{R} = \bigoplus_{\alpha=1}^{\ell} \mathbb{C}|e_\alpha\rangle$$

$\{|e_\alpha\rangle,\ \alpha = 1, \ldots, \ell\}$: set of the pointer states.

Form of the Kraus operators

QND

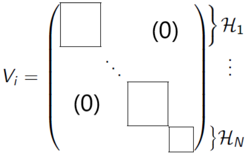
$$V_i = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix}$$

General case

$$V_i = \begin{pmatrix} \boxed{(*)} & (0) & (0) & (*) \\ (0) & \boxed{(*)} & (0) & (*) \\ (0) & (0) & \ddots & (*) \\ (0) & (0) & (0) & \boxed{(*)} \end{pmatrix}$$

Identifiability hypothesis

In absence of transient part, the Kraus operators have the form

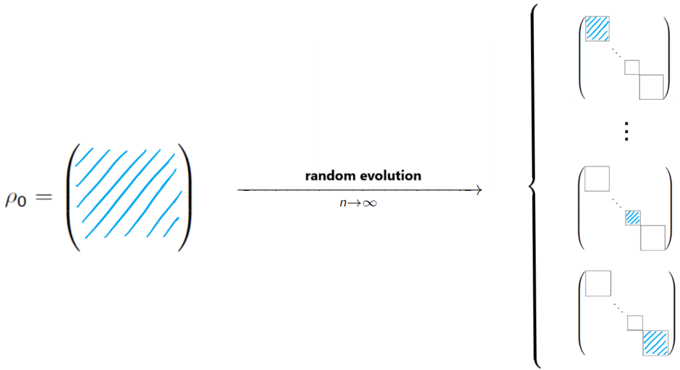


ID Hypothesis: Let $\rho_\infty^u \neq \rho_\infty^v$ be two distinct extreme invariant states. Then there exists a sequence $(i_1, ..., i_l) \in \{1, ..., m\}^l$ s. t.

$$\mathrm{tr}\left(V_{i_l} \ldots V_{i_1} \rho_\infty^u V_{i_1}^* \ldots V_{i_l}^*\right) \neq \mathrm{tr}\left(V_{i_l} \ldots V_{i_1} \rho_\infty^v V_{i_1}^* \ldots V_{i_l}^*\right)$$

Random selection of a minimal subspace

$\Longrightarrow$ Under ID, quantum trajectories become supported in one of its minimal invariant subspaces.[7]



Question: What is the speed of convergence ?

[7] A., Bompais, Pellegrini, 2021.

Exponential convergence in mean

Take the operator M_α as the orthogonal projector onto $\mathcal{H}_\alpha$.

Lyapunov function:

$$W(\rho) = \frac{1}{2} \sum_{\alpha \neq \beta} \sqrt{\operatorname{tr}(M_\alpha \rho) \operatorname{tr}(M_\beta \rho)}$$

Theorem (A., Bompais, Pellegrini, 2024) Under ID hypothesis, we have

$$\mathbb{E}(W(\rho_n)) \leqslant C e^{-\gamma n}$$

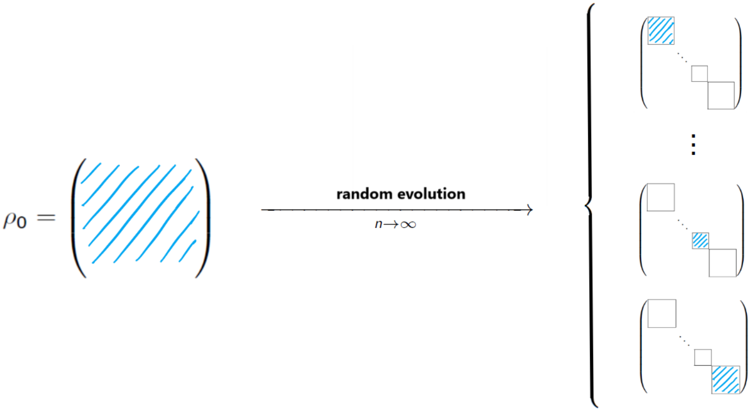
Key point: ID implies the identifiability of all states supported by different minimal invariant subspaces (uniform ID)

- Uniform ID ensures that there exists a length $N \in \mathbb{N}$ such that $\forall \rho^{(\alpha)}$ with $\operatorname{supp} \rho^{(\alpha)} \subset \mathcal{H}_\alpha$, for all $\rho^{(\beta)}$ with $\operatorname{supp} \rho^{(\beta)} \subset \mathcal{H}_\beta$, $\alpha \neq \beta$, there exists a word $I_N \in \mathcal{O}^N$ such that $\mathbb{P}_{\rho^{(\alpha)}}(I_N) \neq \mathbb{P}_{\rho^{(\beta)}}(I_N)$.
- Compute the increment:
$\mathbb{E}[W(\rho_{k+N}) \mid \rho_k]$

$$\leq \underbrace{\sup_{\alpha \neq \beta} \sup_{\rho \in \mathcal{A}_{\alpha,\beta}} \sum_{I \in \mathcal{O}^N} \sqrt{\operatorname{tr}\left(V_I \tilde{\rho}_k^{(\alpha)} V_I^\dagger\right) \operatorname{tr}\left(V_I \tilde{\rho}_k^{(\beta)} V_I^\dagger\right)}}_{\kappa} W(\rho_k)$$

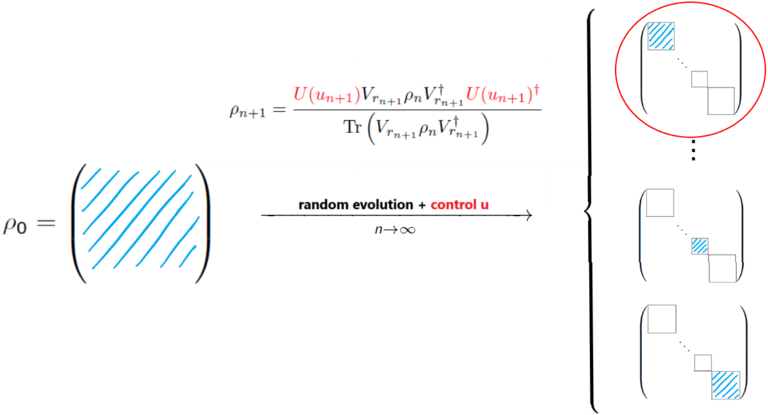
- Uniform ID gives $\kappa < 1$.

Feedback stabilisation of a minimal subspace



Feedback stabilisation of a minimal subspace

$$\rho_{n+1} = \frac{U(u_{n+1}) V_{r_{n+1}} \rho_n V_{r_{n+1}}^\dagger U(u_{n+1})^\dagger}{\mathrm{Tr}\left(V_{r_{n+1}} \rho_n V_{r_{n+1}}^\dagger\right)}$$



Stabilization of an invariant subspace

Goal: stabilize the target minimal subspace $\bar{\alpha}$

Lyapunov function:

$$\begin{aligned} Z(\rho) &= V(\rho) + \varepsilon R(\rho) \\ &= \sqrt{1 - \operatorname{tr}(M_{\bar{\alpha}}\rho)} + \varepsilon \sum_{\beta \neq \bar{\alpha}} \sqrt{\operatorname{tr}(M_{\beta}\rho)} \end{aligned}$$

Take the feedback control

$$u_n = \begin{cases} \underset{u \in [-\bar{u}, \bar{u}]}{\arg\min}\; Z(U(u)\rho_{n+\frac{1}{2}} U(u)^{\dagger}) & \text{if } n = qN - 1,\ q \in \mathbb{N} \\ 0 & \text{if } n \neq qN - 1,\ q \in \mathbb{N} \end{cases}$$

Here $\rho_{n+\frac{1}{2}}$ is the intermediate state after measurement

Stabilization of an invariant subspace

Assumption: Let $\{|\phi_j\rangle, j = 1, \ldots, d_{\bar{\alpha}}\}$ be an orthonormal basis of $\mathcal{H}_{\bar{\alpha}}$.

$$\mathrm{Vect}\{H^k |\phi_j\rangle,\ k = 1, \ldots, d,\ j = 1, \ldots, d_{\bar{\alpha}}\} \supset \bigoplus_{\beta \neq \bar{\alpha}} \mathcal{H}_\beta$$

A., Bompais, Pellegrini, 2024: there exists $\bar{\varepsilon}$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, there exist $\bar{C} > 0$ and $\bar{\gamma} > 0$ (depending on ε) such that for all $n \geqslant 0$,

$$\mathbb{E}\big(Z(\rho_n)\big) \leqslant \bar{C} e^{-\bar{\gamma} n}$$

In particular, for all $n \geqslant 0$,

$$\mathbb{E}\sqrt{1 - \mathrm{tr}\,(M_{\bar{\alpha}} \rho_n)} \leqslant \bar{C} e^{-\bar{\gamma} n}$$

Proof idea:

$S_{<\delta} := \{\rho \in \mathcal{S}(\mathcal{H}) \mid \operatorname{tr}(M_{\bar{\alpha}}\rho) < \delta\}$ for $\delta > 0$. Similar notation shall be used with $>$, $\leq$ and $\geq$.

region \ function	$V(\rho)$	$R(\rho)$	$Z(\rho) = V(\rho) + \varepsilon R(\rho)$
$S_{<\delta}$	✓	X	✓
$S_{\geq\delta} \cap S_{\leq 1-\delta}$	X	✓	✓
$S_{>1-\delta}$	X	✓	✓

Table: Exponential decay of the function every N steps, depending on the region.

Conclusion and perspective

- We have shown exponential convergence in mean for discrete-time quantum trajectories
- We have shown exponential convergence almost surely for continuous-time quantum trajectories with QND measurements
- See the work in [8], considering the transient part and imperfections
- Work in progress for continuous-time quantum trajectories with generic measurements
- Application of results in stabilization of subspaces in quantum error correction
- Finding physical examples and implementation

[8] Benoist, Greggio, Pellegrini, 2024.

Thank you!