IV. Properties of the SCET Lagrangian

Residual gauge invariance:

The SCET Lagrangian is <u>no longer</u> invariant under arbitrary local gauge transformations:

$$\psi(x) \;\to\; U(x)\,\psi(x)$$

$$A^\mu(x) \;\to\; U(x)\,A^\mu(x)\,U^\dagger(x) + \frac{i}{g_s}\,U(x)\left(\partial^\mu U^\dagger(x)\right)$$

$$A^{\mu,a}\,t^a \qquad = \frac{i}{g_s}\,U(x)\left(D^\mu(x)\,U^\dagger(x)\right)$$

The Fourier transform $\tilde{U}(p)$ could contain modes with arbitrary momenta (including hard ones), which would mix up the various modes in the EFT. However, the SCET Lagrangian is invariant (order by order in λ) under a set of collinear, anti-collinear and ultra-soft gauge transformations, which preserve the scaling properties of the various fields. (It is also invariant under global gauge transformations, of course.)

The appropriate form of the residual gauge transformations can be obtained by performing suitable expansions in the QCD relations shown above.

Collinear gauge transformations:

$$\xi_n(x) \xrightarrow{U_c} U_c(x)\,\xi_n(x)$$

$$\bar{n}\cdot A_c(x) \xrightarrow{U_c} U_c(x)\,\bar{n}\cdot A_c(x)\,U_c^\dagger(x) + \frac{i}{g_s} U_c(x)\left(\bar{n}\cdot\partial\, U_c^\dagger(x)\right)$$

$$A^\mu_{c\perp}(x) \xrightarrow{U_c} U_c(x)\,A^\mu_{c\perp}(x)\,U_c^\dagger(x) + \frac{i}{g_s} U_c(x)\left(\partial^\mu_\perp U_c^\dagger(x)\right)$$

$$n\cdot A_c(x) + n\cdot A_{us}(x_-)$$

$$\xrightarrow{U_c} U_c(x)\left(n\cdot A_c(x) + n\cdot A_{us}(x_-)\right)U_c^\dagger(x) + \frac{i}{g_s} U_c(x)\left(n\cdot\partial\, U_c^\dagger(x)\right)$$

The last relation can be rewritten in the form:

$$n\cdot A_c(x) \xrightarrow{U_c} U_c(x)\,n\cdot A_c(x)\,U_c^\dagger(x) + \frac{i}{g_s} U_c(x)\left[n\cdot D_{us}(x_-),\, U_c^\dagger(x)\right]$$

The ultra-soft fields do not transform at all:

$$\left.\begin{aligned} q_{us}(x) &\xrightarrow{U_c} q_{us}(x) \\ A^\mu_{us}(x) &\xrightarrow{U_c} A^\mu_{us}(x) \end{aligned}\right\}$$

The same is true for anti-collinear fields.

The collinear Wilson line transforms as:

$$W_c(x) \xrightarrow{U_c} U_c(x)\,W_c(x)\,U_c^\dagger(-\infty\bar{n}) = U_c(x)\,W_c(x)$$

consider gauge transformations that vanish at infinity

Ultra-soft gauge transformations:

$$\xi_n(x) \xrightarrow{U_{us}} U_{us}(x_-)\,\xi_n(x)$$

$$A_c^\mu(x) \xrightarrow{U_{us}} U_{us}(x_-)\,A_c^\mu(x)\,U_{us}^\dagger(x_-)$$

The collinear gluon field transforms as a background field.
The ultra-soft fields transform in the usual way:

$$q_{us}(x) \xrightarrow{U_{us}} U_{us}(x)\,q_{us}(x)$$

$$A_{us}^\mu(x) \xrightarrow{U_{us}} U_{us}(x)\,A_{us}^\mu(x)\,U_{us}^\dagger(x) + \frac{i}{g_s}\,U_{us}(x)\left(\partial^\mu U_{us}^\dagger(x)\right)$$

Note that the collinear Wilson line transforms as:

$$W_c(x) \xrightarrow{U_{us}} U_{us}(x_-)\,W_c(x)\,U_{us}^\dagger(x_-)$$

This follows from the fact that the gluon fields in the path-ordered exponential all live at the same value of x_-:

$$A_c^\mu(\underbrace{x+t\bar{n}}) \xrightarrow{U_{us}} U_{us}(x_-)\,A_c^\mu(x+t\bar{n})\,U_{us}^\dagger(x_-)$$

$$(x+t\bar{n})_- = \frac{n}{2}\,\bar{n}\cdot(x+t\bar{n}) = \frac{n}{2}\,\bar{n}\cdot x = x_-$$

It is straightforward to show that $\mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{c+us}$ is invariant under these residual gauge transformations. Note, in particular, that (as differential operators):

$$
\begin{aligned}
i D_{c\perp}^{\mu} &\xrightarrow{U_c} U_c(x)\, i D_{c\perp}^{\mu}\, U_c^{\dagger}(x) \\
&\xrightarrow{U_{us}} U_{us}(x_-)\, i D_{c\perp}^{\mu}\, U_{us}^{\dagger}(x_-) \\
i n\cdot D_c + g_s n\cdot A_{us}(x_-) &\xrightarrow{U_c} U_c(x)\left(i n\cdot D_c + g_s n\cdot A_{us}(x_-)\right) U_c^{\dagger}(x) \\
&\xrightarrow{U_{us}} U_{us}(x_-)\left(i n\cdot D_c + g_s n\cdot A_{us}(x_-)\right) U_{us}^{\dagger}(x_-)
\end{aligned}
$$

Nonrenormalization theorem:

There is an important difference between SCET and HQET. In the absence of ultra-soft interactions, the effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_c(x) = {} & \bar{\xi}_n \frac{\not{\bar{n}}}{2}\, i n\cdot D_c\, \xi_n(x) \\
& + \left(\bar{\xi}_n\, i \not{D}_c^{\perp} W_c\right)(x)\, \frac{\not{\bar{n}}}{2}\, i \int_{-\infty}^{0} dt \left(W_c^{\dagger}\, i \not{D}_c^{\perp}\, \xi_n\right)(x+t\bar{n}) \\
& + (\text{pure glue terms})
\end{aligned}
$$

is <u>exact to all orders in λ</u>, i.e. it does not receive any power corrections! In fact, this Lagrangian does not contain any small ratio of scales, unlike in HQET, where $iv\cdot D_s/m_Q \ll 1$. By a Lorentz boost, a set of highly boosted collinear particles with momenta $p_c^\mu \sim (\lambda^2, 1, \lambda)$ can be transformed into a set of particles with momenta $\Lambda^\mu{}_\nu p_c^\nu \sim (\lambda, \lambda, \lambda)$. Thus, the Lagrangian $\mathcal{L}_c$ is, in fact, completely equivalent to the QCD Lagrangian, and as a result its vertices do not receive any hard matching corrections.

Remarkably, this statement remains true when ultra-soft interactions are included.

(see: Beneke, Chapovsky, Diehl, Feldmann 2002)

Reparameterization invariance:

The choice of the light-like reference vectors $n^\mu, \bar{n}^\mu$ ($n^2 = 0 = \bar{n}^2$, $n\cdot\bar{n} = 2$) in the construction of SCET is not unique. For instance, one can rescale:

$$\boxed{n^\mu \to \xi\, n^\mu\,, \qquad \bar{n}^\mu \to \xi^{-1}\, \bar{n}^\mu} \qquad \text{(III)}$$

(with $\xi \sim \lambda^0$)

One can also rotate n^μ or $\bar{n}^\mu$ by small amounts away from the z-axis. In infinitesimal form:

$$\boxed{\begin{aligned} n^\mu &\to n^\mu + \Delta_\perp^\mu \\ \bar{n}^\mu &\to \bar{n}^\mu \end{aligned}} \quad (\mathrm{I}) \qquad\qquad \boxed{\begin{aligned} n^\mu &\to n^\mu \\ \bar{n}^\mu &\to \bar{n}^\mu + \bar{\Delta}_\perp^\mu \end{aligned}} \quad (\mathrm{II})$$

$$(\Delta_\perp \sim \lambda) \qquad\qquad\qquad\qquad (\bar{\Delta}_\perp \sim \lambda)$$

The most general transformation is a combination of these three.

The SCET Lagrangian is invariant under these reparameterization transformations as a consequence of Lorentz invariance. The transformations (I) and (II) relate the coefficients of operators arising at different orders in λ. Invariance under type-III transformations, on the other hand, must be satisfied order by order in λ.

(see: Manohar, Mehen, Pirjol, Stewart 2002)