Local cohomology & cellular approximations

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ICTS

Some references:

- Dwyer & Greenlees: Complete modules & torsion modules
- DG & I: Duality in algebra & topology.
- Hovey, Palmieri, Strickland: Axiomatic stable homotopy theory
- Benson, I., Krause: Various references

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My lectures are intended to complement those of Greenlees & Krause

(I) Cellular approximations:

Throughout R commutative noeth. ring.

Examples: $\mathbb{Z}$, polynomial rings, and quotients thereof.

R- complex:

$$M = \cdots \to M_{i+1} \to M_i \to M_{i-1} \to \cdots$$

- Lower grading.

$H(M) := \{H_i(M)\}_{i \in \mathbb{Z}}$

$D(R)$:= Derived category of all R-complexes

- Viewed as a triangulated category

Σ := Shift / translation

Besides this structure:

$- \otimes^L_R -$ & $R\mathrm{Hom}_R(-,-)$

All this makes $D(R)$ a compactly generated, symmetric monoidal category

i.e. a tensor-triangulated category

- This is the link to homotopy theory / stable module theory.

Often mainly interested in

$D^b(\mathrm{mod}\,R) = \{M \in R \,/\, H(M) \text{ f.g. } R\text{-module}\}$

i.e. $H_i(M)$ f.g. $\forall i$ and $= 0 \ \forall |i| \gg 0$

. $\mathrm{Hom}_D(M,N) = H_0(R\mathrm{Hom}_R(M,N))$

$$\mathrm{Hom}^*_D(M,N) = H^*(\mathrm{Hom}_R(M,N))$$

Fix $X \in D(R)$.

- $\mathrm{Loc}(X) :=$ smallest triangulated subcategory of $D(R)$ containing X and closed under arbitrary coproducts.
  - These are the R-complexes <u>built out of X</u>.

Example: $\mathrm{Loc}(R) = D(R)$

- Just the statement that each complex has a projective resln.

The X-cellular approximation of $M \in D(R)$ is a map $\mathrm{Cell}_X M \longrightarrow M$ s.t.

1. $\mathrm{Cell}_X M$ is built out of X;
2. The map is an X-equivalence:

$$\mathrm{Hom}^*_D(X, \mathrm{Cell}_X M) \xrightarrow{\cong} \mathrm{Hom}^*_D(X, M)$$

Equivalently:

$$\begin{tikzcd} \mathrm{Cell}_X M \arrow[r] & M \\ & \Sigma^i X \arrow[u, "f"'] \arrow[ul, dashed] \end{tikzcd}$$

← given

f factors uniquely through $\mathrm{Cell}_X M$. →

So "$\mathrm{Cell}_X M$" is the best approximation to M by objects built out of X.

Fact: $\boxed{\text{X-cellular approximations exist and are unique upto unique iso.}}$

- Due to Farjoun / Neeman, depending on who you ask.

Condition 2 equivalent to: $\mathrm{Hom}^*_D(X, L_X M) = 0$

where $\mathrm{Cell}_X M \longrightarrow M \longrightarrow L_X M \longrightarrow \Sigma\, \mathrm{Cell}_X M$.

- Underscores need to understand when $\mathrm{Hom}^*_D(M, N) = 0$.

Example: ① $R = \mathbb{Z}$ $\quad X = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

Then $0 \to \mathbb{Z} \hookrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}[1/p]/\mathbb{Z} \to 0$

gives

$$\Sigma^{-1} \frac{\mathbb{Z}[p^{-1}]}{\mathbb{Z}} \longrightarrow \mathbb{Z} \quad \text{in } D(\mathbb{Z})$$

- This is the $\mathbb{Z}/p$ - cellular approx. of $\mathbb{Z}$.

(justify later)

(2) $R = \mathbb{Z}$ (or any domain)

$\mathbb{Q}$ = rationals (field of fractions of R)

$$\mathrm{Cell}_{\mathbb{Q}} R = R\mathrm{Hom}_R(\mathbb{Q}, R)$$

- A complicated object.
  - Non-zero if $R = \mathbb{Z}$, or $k[x]$
  - zero if $R = k[[x]]$ k field.

(II) Local Cohomology: Fix $V \subseteq \mathrm{Spec}\, R$

closed, or even just specialization closed.

For any $M \in R\text{-Mod}$ set

$$\Gamma_V M = \mathrm{Ker}\left(M \longrightarrow \prod_{p \notin V} M_p\right)$$

- sections supported on M

Say $V = V(I)$ $I \subseteq R$ ideal. Then

$$\Gamma_V M = \bigcup_{n\geq 0} (0 :_M I^n) = \bigcup_{n\geq 0} \mathrm{Hom}_R(R/I^n, M)$$

For any $M \in D(R)$ injective resolution of M

$$R\Gamma_V M := \Gamma_V(\tilde{i}M)$$

Γ_V applied component-wise.

$R\Gamma_V(M) = \Gamma_V(iM) \hookrightarrow iM \simeq M$ so we get

$$R\Gamma_V(M) \longrightarrow M$$

the local cohomology of M supported on V.

Say $X \in D^b(\mathrm{mod}\, R)$

$$\mathrm{supp}_R X := \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid H(X)_{\mathfrak{p}} \neq 0 \}$$

- closed subset of $\mathrm{Spec}\, R$

$= V(\mathrm{ann}_R H(R))$

Fact: $\boxed{\text{With } V = \mathrm{supp}_R X, \text{ for any } M \in D(R)\ R\Gamma_V M \longrightarrow M \text{ is the } X\text{-cellular approx. of } M.}$

- So $\mathrm{cell}_X M$ only depends on $\mathrm{supp}_R X$. This holds in general.

Various methods exist to compute $R\Gamma_V M$:

Say $V = V(I)$ $I = (r_1, \dots, r_n)$

Set $K_\infty(\underline{r}) = \bigotimes_i \left[0 \to R \to R[\tfrac{1}{r_i}] \to 0\right]$

Stable Koszul cx. or "Extended" Čech cx.

$$0 \to R \to \oplus R[\tfrac{1}{r_i}] \to \oplus R[\tfrac{1}{r_i r_j}] \to \cdots \to R[\tfrac{1}{r_1 \cdots r_n}] \to 0$$

degree 0

One has $K^\infty(\underline{r}) \longrightarrow R$ (project onto degree 0)

This induces:

$$K^\infty(\underline{r}) \otimes_R M \longrightarrow M$$
$$\cong$$
$$R\Gamma_V M$$

Requires proof.

Example: $R\Gamma_{(p)} \mathbb{Z} = 0 \to \mathbb{Z} \to \mathbb{Z}[\tfrac{1}{p}] \to 0$

$$\simeq \Sigma^{-1} \mathbb{Z}[p^{-1}]/\mathbb{Z}$$

- This explains the earlier computation.

III Now suppose X is perfect:

$$X = 0 \to P_b \to \cdots \to P_a \to 0$$

Maybe only up to iso.

each P_i a f.g. projective R-module.

Set $E := \mathrm{End}_R(X) = \mathrm{Hom}_R(X, X)$

Viewed as a dg (= differential graded) algebra.

- product = composition of maps
- usual differential: $[\partial^X, -]$

$D(E^{op}) :=$ Derived category of right dg E-modules.

Note: X is a left E-module,

and this is compatible with R-action:

$R \longrightarrow E$ map of dg algebras

In particular, for any $N \in D(E^{op})$

$$N \overset{\mathbb{L}}{\underset{E}{\otimes}} X \in D(R)$$

Also $\quad \underset{R}{\mathrm{RHom}}(X,-) : D(R) \longrightarrow D(E^{op})$

One has an adjoint pair:

$$\begin{tikzcd} D(R) \arrow[r, shift left, "\underset{R}{\mathrm{RHom}}(X{,}-)"'] & D(E^{op}) \arrow[l, shift left, "- \overset{\mathbb{L}}{\underset{E}{\otimes}} X"'] \end{tikzcd}$$

One has a natural map:

$$\underset{R}{\mathrm{RHom}}(X, M) \overset{\mathbb{L}}{\underset{E}{\otimes}} X \longrightarrow M \quad \text{in } D(R)$$

$\forall \ M \in D(R)$

When X is perfect, this is the X-cellular approximation of M.

Sketch of proof: Since $\underset{R}{\mathrm{RHom}}(X, M) \in \mathrm{Loc}(E)$

(projective resolns exist)

$$\underset{R}{\mathrm{RHom}}(X, M) \overset{\mathbb{L}}{\underset{E}{\otimes}} X \in \mathrm{Loc}(\underbrace{E \overset{\mathbb{L}}{\underset{E}{\otimes}} X}_{= X})$$

Also
$$\begin{tikzcd}
\operatorname{RHom}_R(X, \operatorname{RHom}_R(X,M) \otimes_E X) \arrow[r, "2"] & \operatorname{RHom}_R(X, M) \\
\operatorname{RHom}_R(X,M) \otimes_E \operatorname{RHom}_R(\cancel{X}, X) \arrow[u, "1"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Always exists $\leftarrow$

When X is perfect, ① is an iso. in $D(R)$.

Thus ② is also an iso. $\square$

One gets another model for $R\Gamma_{\mathcal{V}} M$ $\quad \mathcal{V} = \operatorname{Supp}_R X$.

Example: $R = \mathbb{Z}$ $\quad X = \mathbb{Z}/p\mathbb{Z} \simeq 0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$

Then $E = \operatorname{End}_R(X)$, basically $M_2(\mathbb{Z})$

$$\operatorname{RHom}_R(\mathbb{Z}/p, \mathbb{Z}) \simeq \Sigma^{-1} \mathbb{Z}/p\mathbb{Z}.$$

So we get $\Sigma^{-1} \mathbb{Z}/p\mathbb{Z} \otimes^{L}_{E} \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}[p^{-1}]/\mathbb{Z}$

This is somewhat surprising because.

$$H^*(E) = \mathbb{Z}/p\mathbb{Z}[\eta] \qquad |\eta| = 1 \quad \text{(upper)}$$

$\uparrow$ Exterior algebra / $\mathbb{Z}/p\mathbb{Z}$.

So $p \cdot H^*(E) = 0$.

However, $p \cdot \underbrace{H^*(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbf{L}}_{E} \mathbb{Z}/p\mathbb{Z})}_{\mathrm{Tor}^E(\mathbb{Z}/p\mathbb{Z},\, \mathbb{Z}/p\mathbb{Z})} \neq 0$

Point is $p : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ is 0 in $D(\mathbb{Z})$

but not in $D(E)$.

Proxy-smallness: