INTERNATIONAL CENTRE for THEORETICAL SCIENCES
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Infosys-ICTS Chandrasekhar Lectures, 2024 Aug.
ICTS Program
"Quantum Information, Quantum Field Theory and Gravity"

Gravity duals of CFTs on manifolds with boundaries (Lecture 2)

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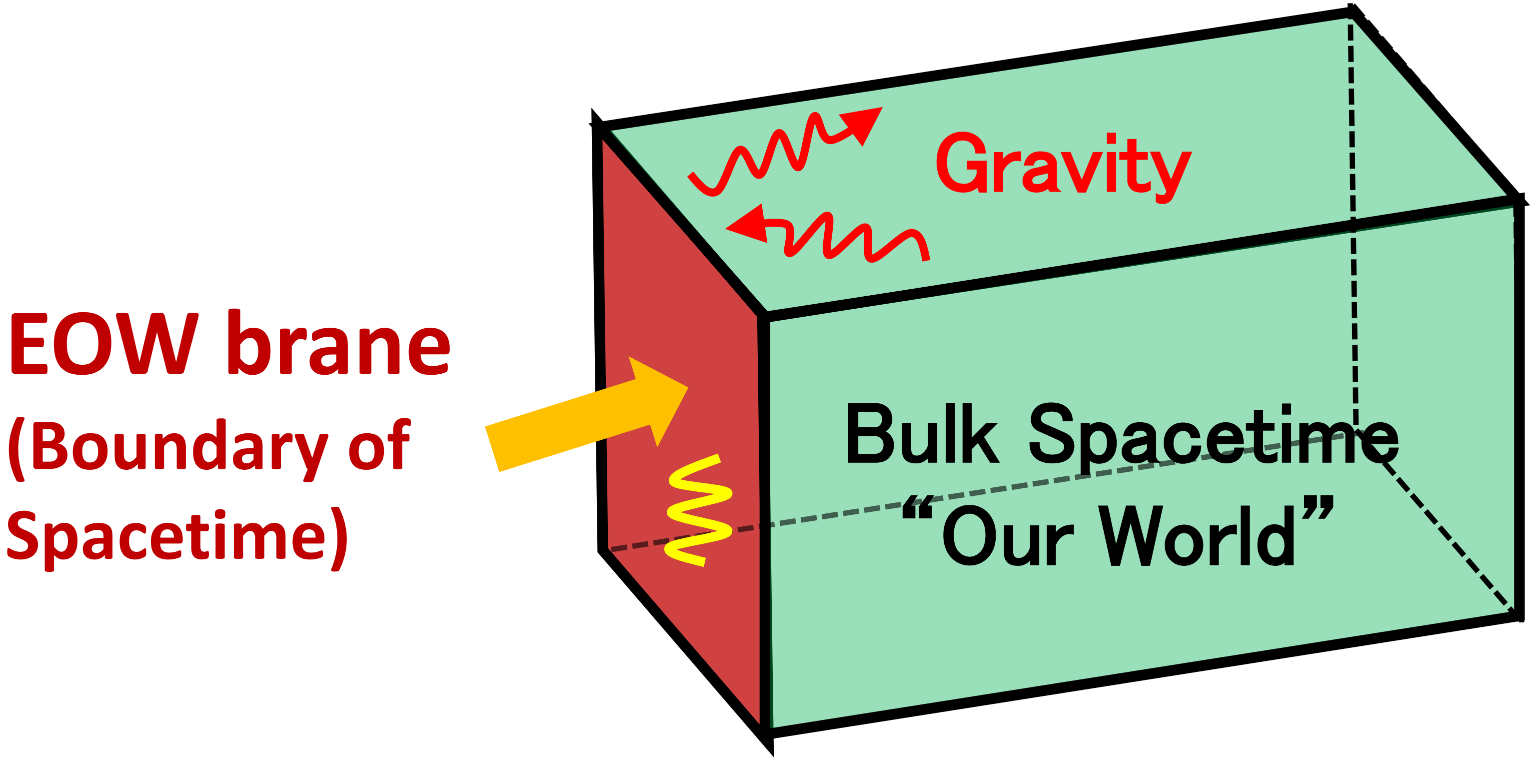
Grant-in-Aid for Transformative Research Areas A
Extreme Universe

① Introduction

End-of-the-world brane (EOW brane)

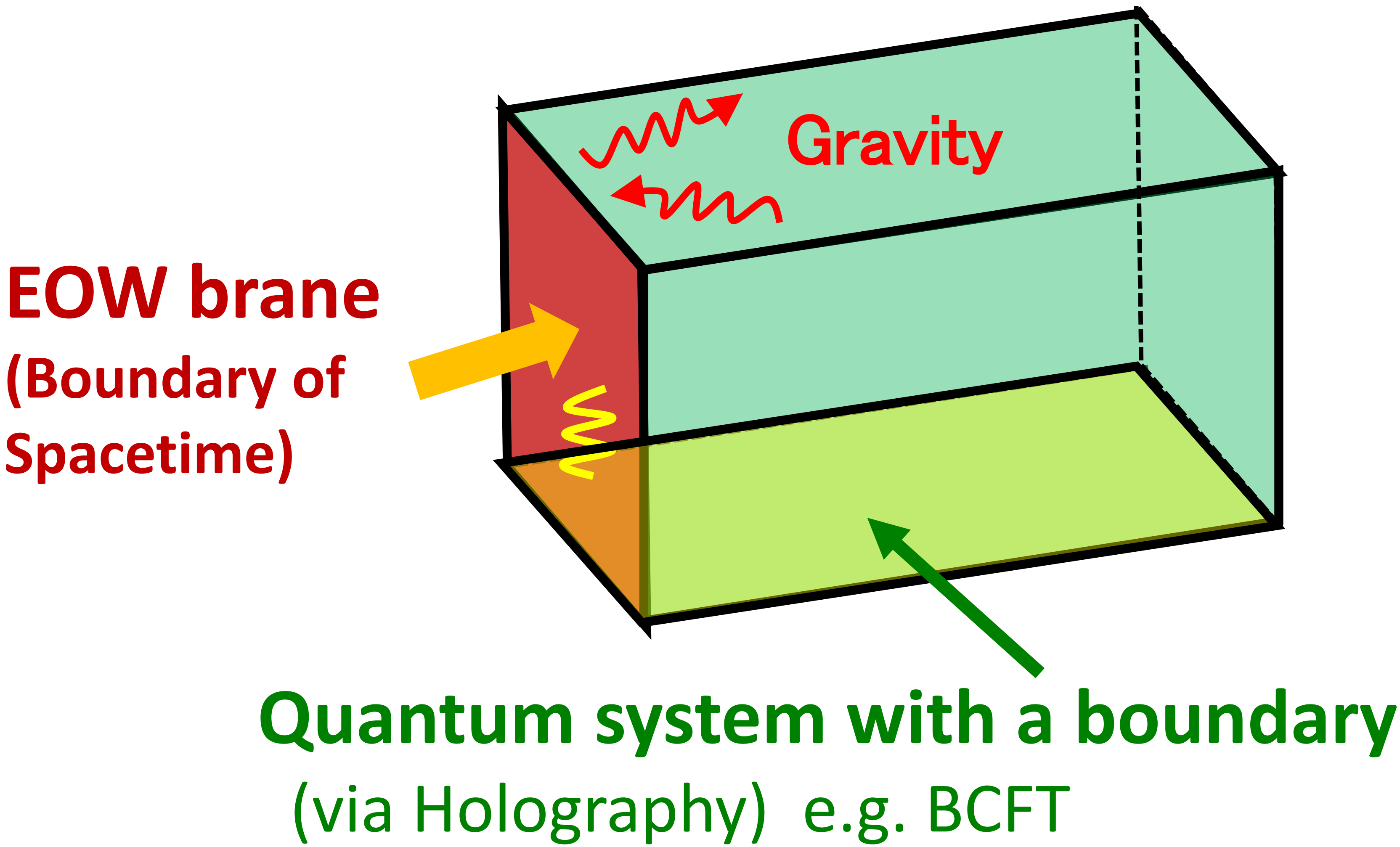


Recently, EOW branes play crucial roles in holography.

This is much like how D-branes are important in string theory.

① Introduction

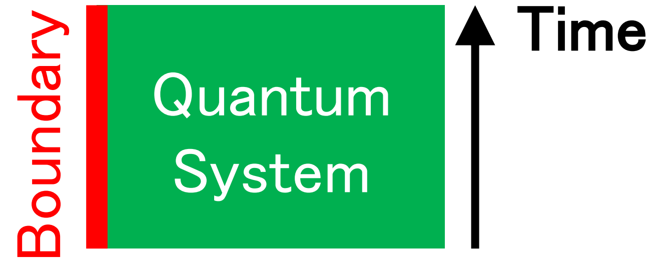
End-of-the-world brane (EOW brane)



Examples of Applications of EOW branes in holography

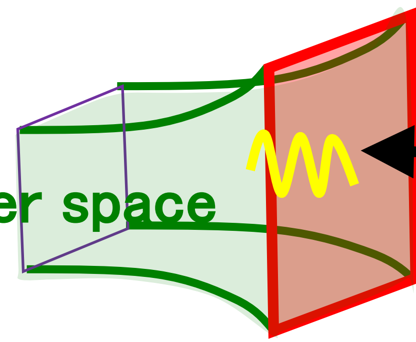
[1] Quantum systems with boundaries

(Boundary conformal field theory: BCFT)



[2] Black hole information problem

(AdS/BCFT + brane-world)

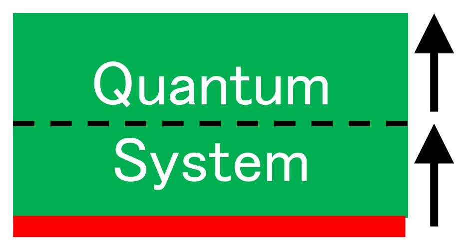


[Randall-Sundrum 1999]

[3] Non-equilibrium dynamics

(Quantum quenches,….

Entanglement Transition)



[Calabrese-Cardy 2005, Hartman-Maldacena 2013]

This talk is mainly based on

PRL133 (2024) 031501 [arXiv:2403.19934] (g-theorem from SSA)

with Jonathan Harper, Hiroki Kanda and Kenya Tasuki (YITP, Kyoto)

We will also mention

JHEP 03 (2023) 105 [arXiv: 2302.03895] (AdS/BCFT with localized scalar)

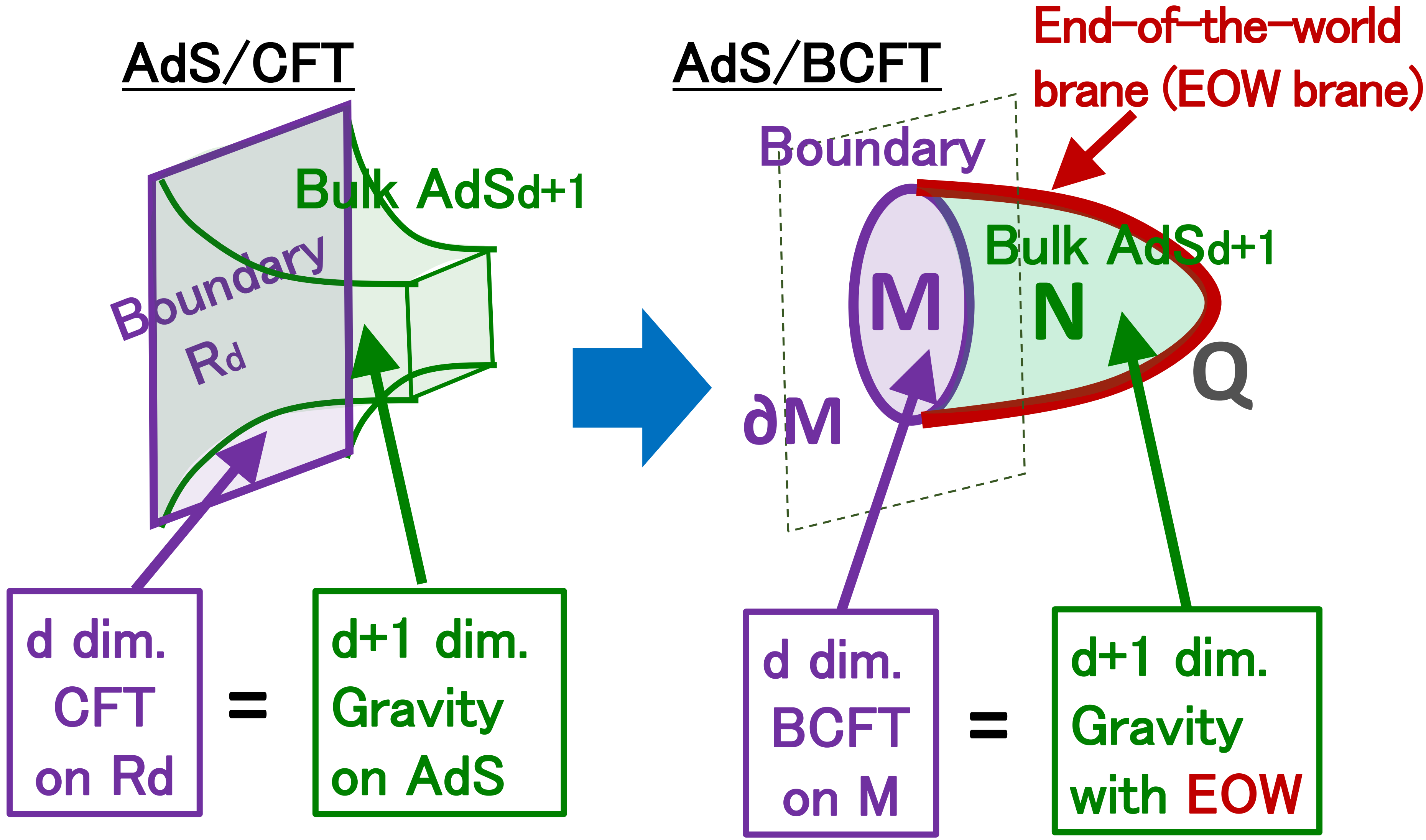
JHEP 03 (2024) 060 [arXiv:2311.13201] (Hol. entanglement transition)

with Hiroki Kanda, Taishi Kawamoto, Masahide Sato, Yu-ki Suzuki,

Kenya Tasuki (YITP, Kyoto) and Zixia Wei (Harvard).

Contents

② Overview of AdS/BCFT



The gravity action in Euclidean signature looks like

Gibbons -Hawking term

$$I=\frac{1}{16\pi G_N}\int_N \sqrt{-g}(R-2\Lambda+L_{matter})+\frac{1}{8\pi G_N}\int_Q \sqrt{-h}(K+L^Q_{matter}).$$

Bulk matter fields

Matter fields localized on Q

The coordinate of Q and its induced metric are x^a and h^{ab} .

We define the extrinsic curvature and its trace

$K_{ab}=\nabla_a n_b, \quad K=h^{ab}K_{ab}$. (n^a is a unit vector normal to Q.)

e.g. Gaussian normal coordinate: $ds^2=d\rho^2+h_{ab}(\rho,x)dx^a dx^b$

➡ $$K_{ab}=\frac{1}{2}\partial_\rho h_{ab}(\rho,x).$$

Variation: $$\delta I = \frac{1}{16\pi G_N}\int_Q \sqrt{-h}\,(K_{ab} - Kh_{ab} - T^Q_{ab})\,\delta h^{ab}.$$

At the AdS boundary M, we impose the Dirichlet boundary condition $\delta h^{ab} = 0$ following the standard AdS/CFT argument.

On the other hand, at the new boundary Q, we argue to require the Neumann b.c. :

$$K_{ab} - Kh_{ab} - T^Q_{ab} = 0$$

`boundary Einstein eq.'

Why Neumann b.c. (brane-world type) ?

(1) Keep the boundary dynamical. New data at Q should not be required.

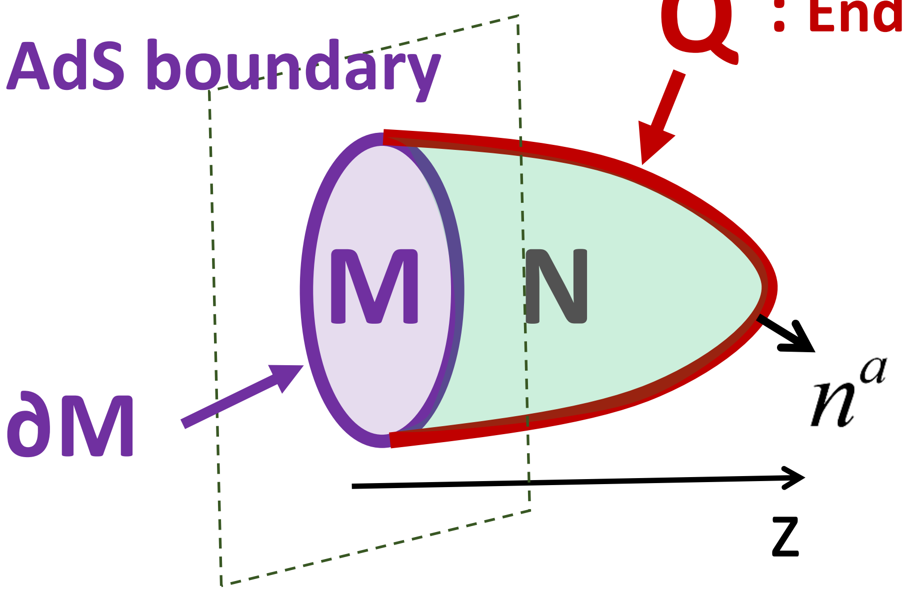
(2) Orientifolds in string theory lead to this condition.

In general, this AdS/BCFT description is a hard wall approximation.

AdS/BCFT construction

[TT 2011, See also Karch-Randall 2001]

CFT on a manifold M with a boundary ∂M = Gravity on an asymptotically AdS space N, s.t. ∂N=M∪Q



Q : End of the world (EOW) brane

We impose Neumann b.c.:

$$K_{ab} - K h_{ab} - T^{Q}_{ab} = 0$$

Depend on types of EOW brane.

Extrinsic curvature:

$$K_{ab} = \nabla_a n_b, \quad K = h^{ab} K_{ab}$$

BCFT (Boundary Conformal Field Theory)

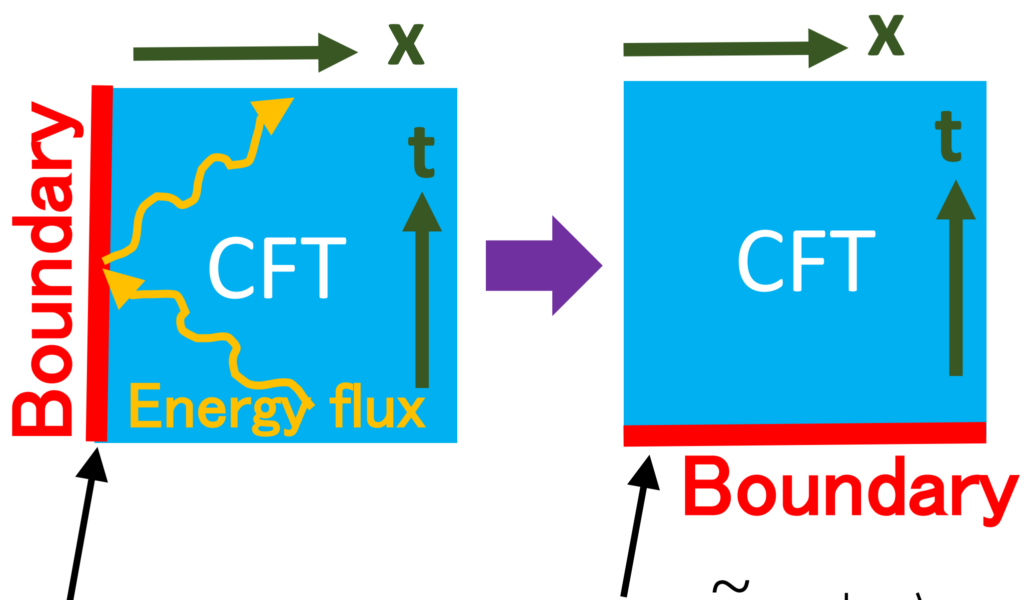
For special boundary conditions, a part of conformal symmetries are preserved, called the boundary conformal field theory (BCFT).

[Cardy 1984, .., McAvity–Osborn 1995, ⋯ ; Cond-mat application: Kondo effect]

d dim CFT : SO(2,d)

U

d dim. BCFT: SO(2,d−1)

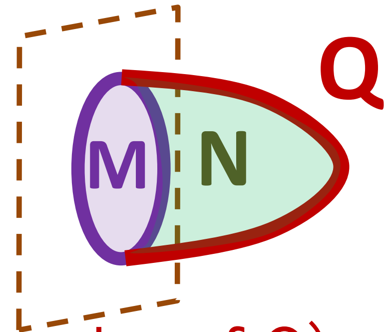


$$T_{tx} \propto \left[T(z) - \bar{T}(\bar{z})\right]_{Bdy} = 0$$

$$(L_n - \tilde{L}_{-n})\,|B\rangle = 0$$

Boundary state

Holographic Dual of BCFT



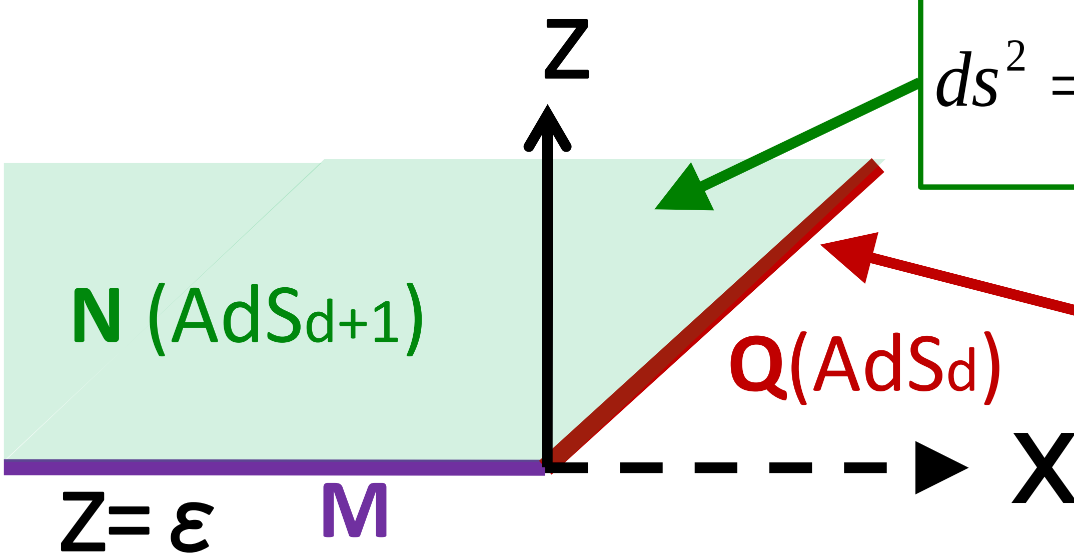
To preserve the BCFT symmetry, we choose

$$T^{Q}_{ab} \propto h_{ab} \quad \Rightarrow \quad T^{Q}_{ab} = -T\, h_{ab}$$ (T is the tension of Q).

The Neumann b.c. looks like

$$K_{ab} = (K - T)\, h_{ab}$$.

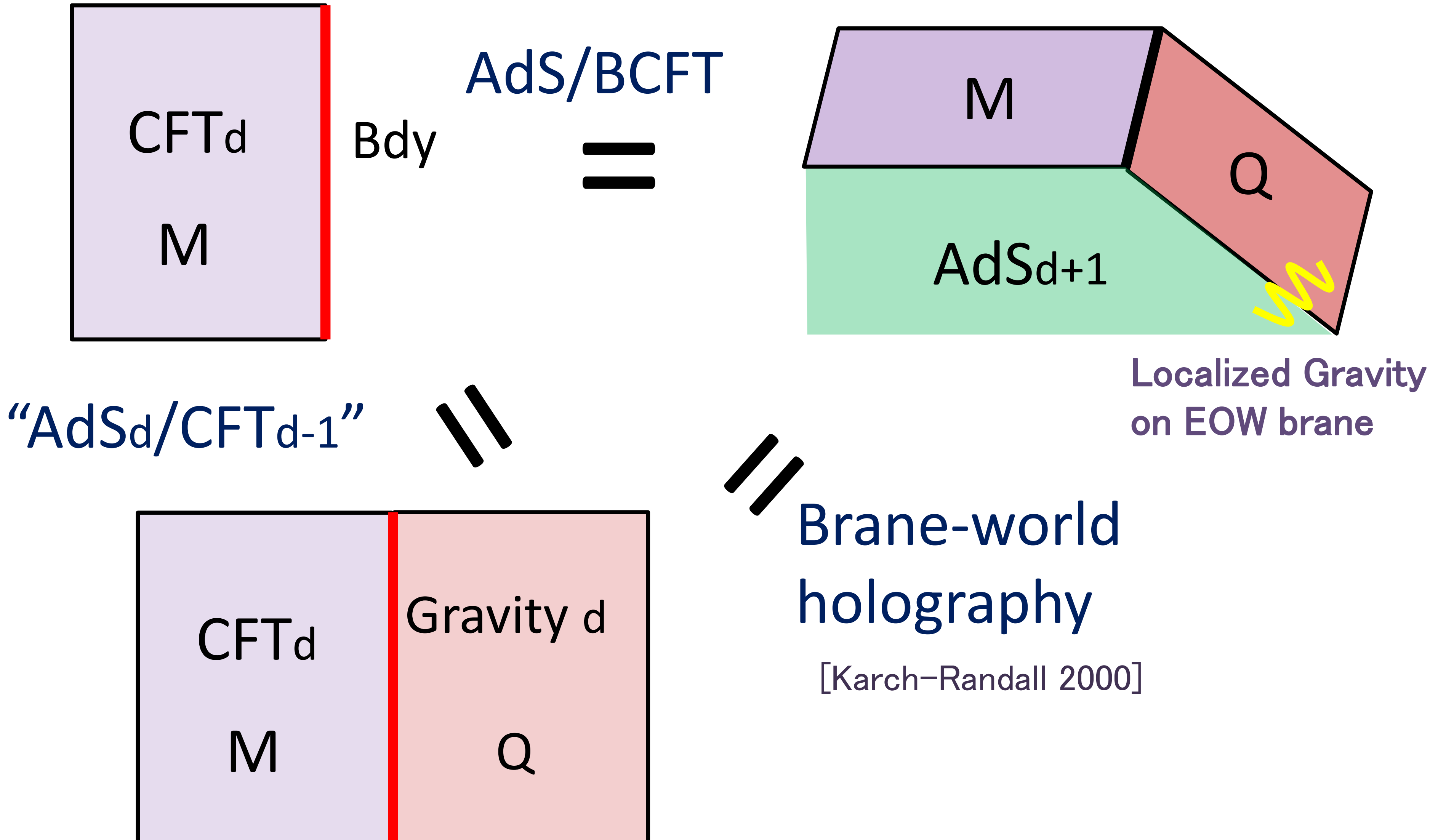
Example: Dual of BCFT on a half space



$$ds^2 = \frac{-dt^2 + dz^2 + dx^2 + d\vec{w}^2}{z^2}$$

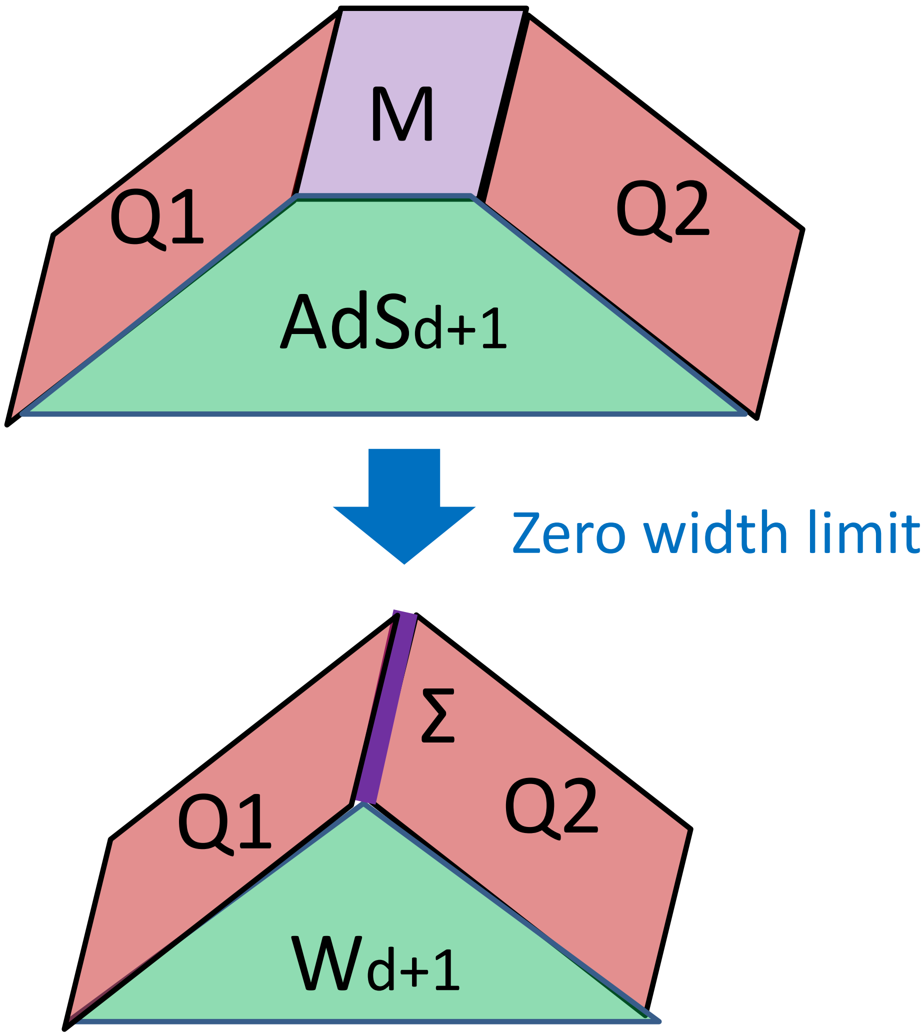
$$z = \frac{\sqrt{1-T^2}}{T}\, x$$

Double Holography



Wedge Holography

[Akal–Kusuki–Wei–TT 2020, Bousso–Wildenhain 2020]



d+1 dim. Classical Gravity on the wedge Wd+1

||

d dim. Quantum Gravity on EOW branes Q1 ∪ Q2

||

d−1 dim. CFT on Σ

⬇

Codimension Two Holography !

Holographic Entanglement Entropy (HEE)

For static states in CFTs, S_A is computed from the minimal area surface Γ_A:

$$S_A = \min_{\Gamma_A}\left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N}\right]$$

[Ryu-TT 2006]

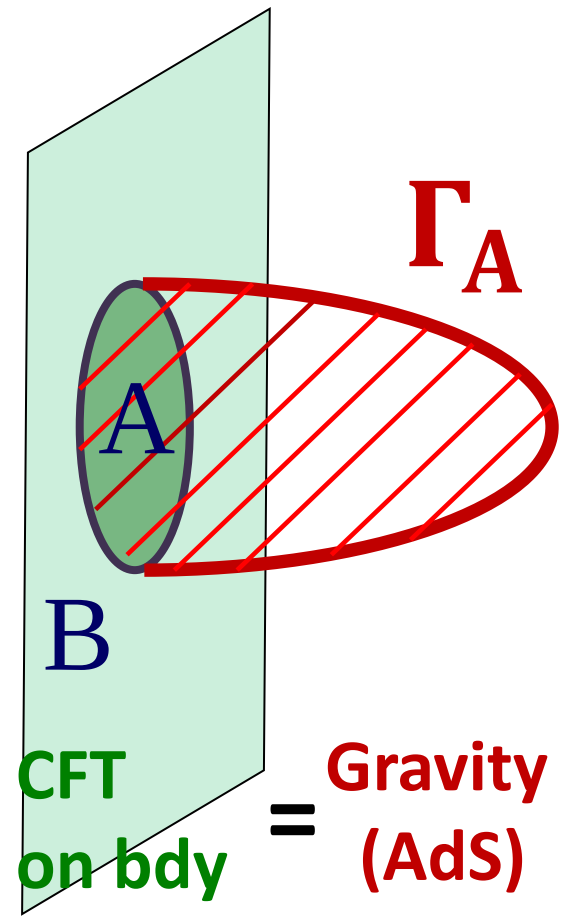
For time-dependent states in CFTs,

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t)$$

S_A is found from the extremal surface area:

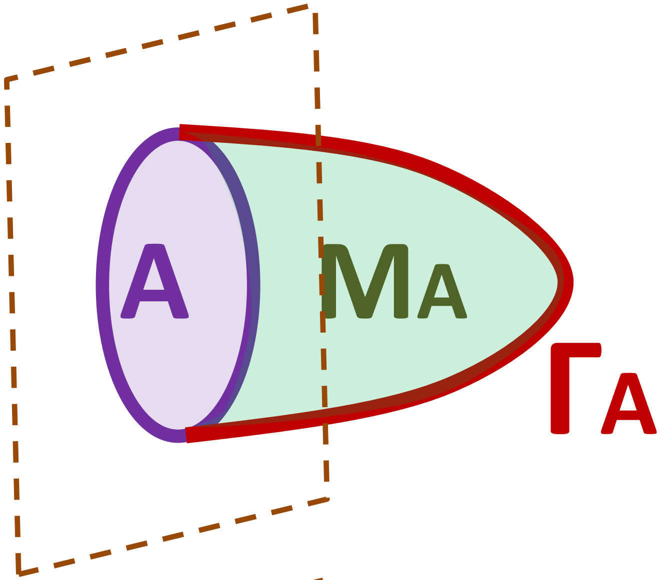
$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

[Hubeny-Rangamani-TT 07]



Note: $\partial\Gamma_A = \partial A$ and Γ_A is homologous to A.

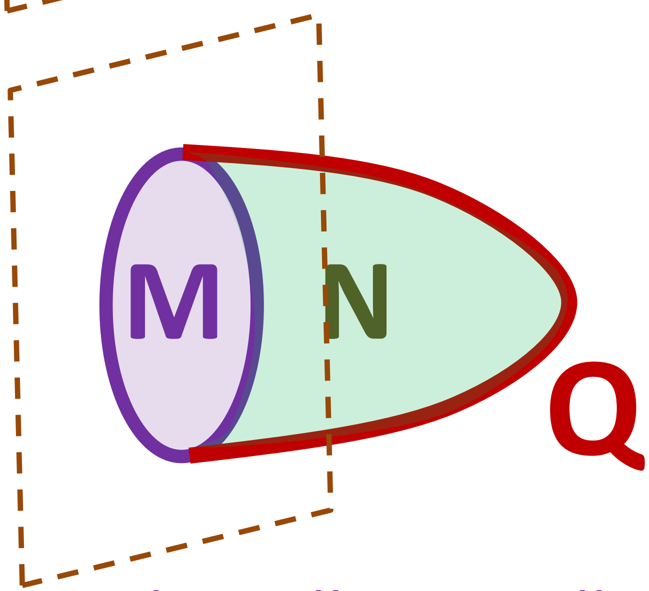
Differences between two "subregion/subregion duality"



[1] Entanglement Wedge

⇒ ΓA is extremal surface.
(no back-reactions)

$$h^{ab}K_{ab}=0$$



[2] AdS/BCFT

⇒ Q is totally geodesic surface
or it generalizations.

$$K_{ab}=\text{fixed}$$

⇒ Surface Q back-reacts !

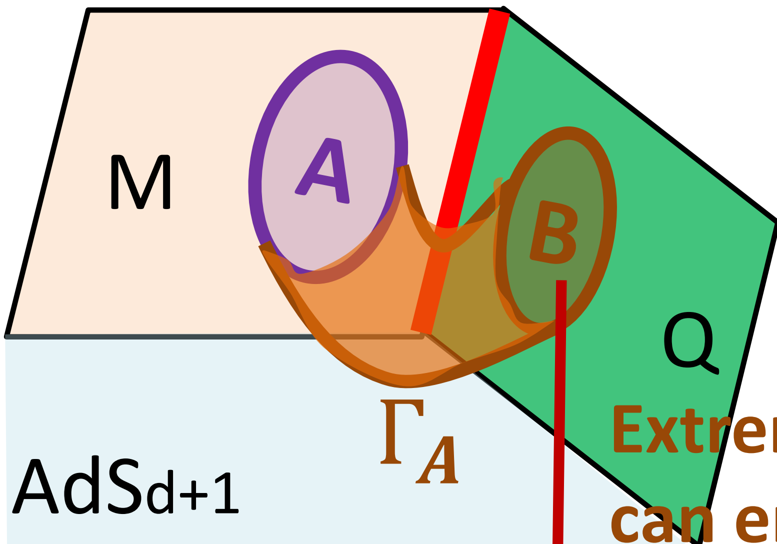
In this talk, we will see interesting interplay between them.

Holographic Entanglement Entropy in AdS/BCFT

[TT 2011, Fujita-Tonni-TT 2011]

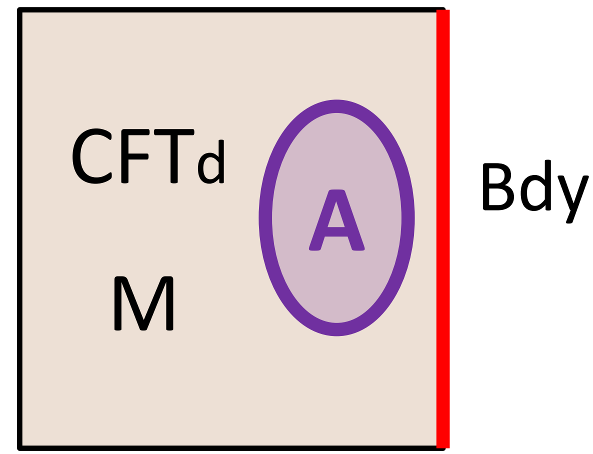
$$S_A = \underset{\Gamma_A, B}{\text{Min Ext}} \left[\frac{\text{Area}(\Gamma_A)}{4G_N} \right]$$

$$\partial\Gamma_A = \partial A \cup \partial B$$



AdS/BCFT

=



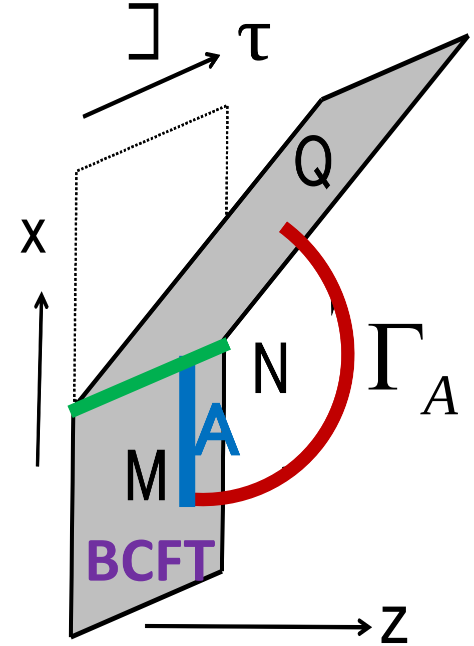
This region B is now known as an **Island** !

Island formula: $S_A = \text{Min}\left[\frac{\text{Area}(\Sigma)}{4G_N} + S_{A\cup\Sigma}\right]$

HEE in AdS3/BCFT2

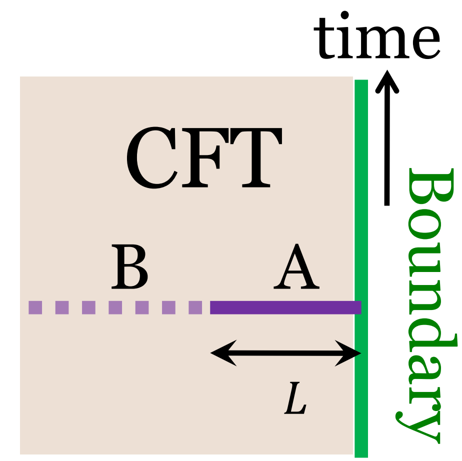
The holographic EE is obtained as

$$S_A = \frac{\text{Length}}{4G_N} = \frac{c}{6}\log\frac{2L}{\varepsilon} + \frac{c}{12}\log\frac{1+T}{1-T}\ .$$



cf. CFT Result

$$S_A = \underbrace{\frac{c}{6}\log\frac{2L}{\varepsilon}}_{\text{Bulk Part}} \quad + \quad \underbrace{\log g}_{\text{Boundary Entropy (g-function)}}\ .$$



③ New Entropic g-theorem

(3-1) Entropic c-theorem

Entanglement entropy

→ a measure of degrees of freedom in quantum systems

In two dimensional CFTs, $$S_A = \frac{c}{3}\log\frac{l}{\varepsilon}.$$

Central charge ~ # of fields

<u>C-theorem in 2d CFT</u>

The central charge c monotonically decreases under the RG flow.

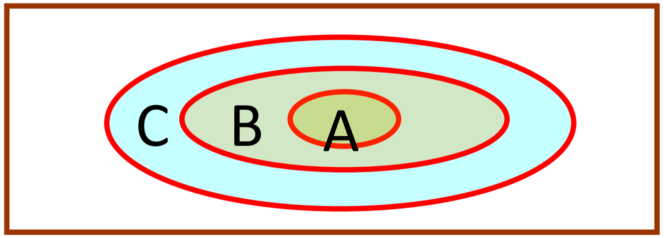
[Zamolodchikov1986]

➡ Quantum information gives an entropic proof of c-theorem!

[Casini–Huerta 2004]

Strong subadditivity (SSA) [Lieb-Ruskai 1973]

$$H_{tot} = H_A \otimes H_B \otimes H_C \otimes H_{other}$$



$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

$$S_{AB} \equiv S_{A \cup B}$$

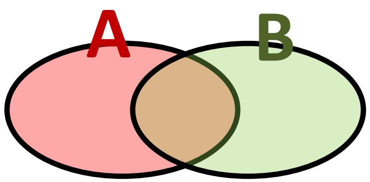
- The most fundamental inequality of EE.
- Analogous to 2nd law of thermodynamics.
- SSA follows from monotonicity of relative entropy.

$$S(\rho|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$$S(\rho_{ABC}|\rho_A \otimes \rho_{BC}) \geqq S(\rho_{AB}|\rho_A \otimes \rho_B) \Leftrightarrow \mathbf{SSA}$$

Tracing out C

Entropic c-theorem [Casini–Huerta 2004]

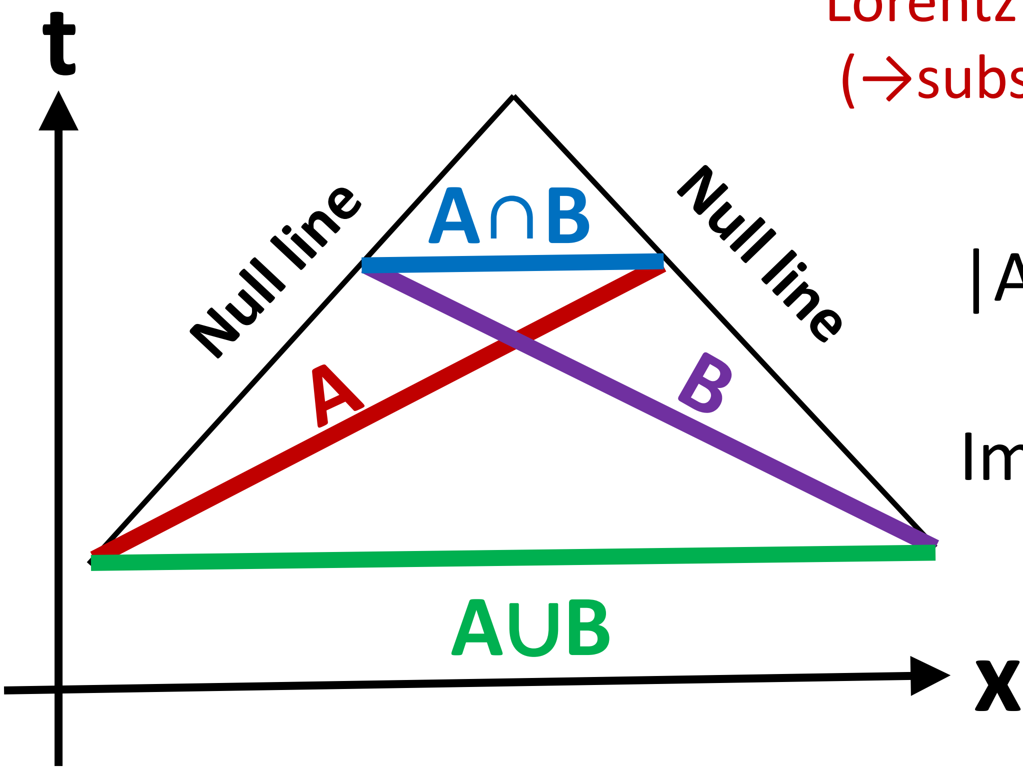


We rewrite SSA as:

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

We apply SSA to a 1+1 dim relativistic field theory vacuum.

Lorentz inv. and translational invariant
(→subsystem can be boosted)



|A|=Lorentz invariant length

Important geometric relation:

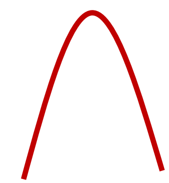
$$|A|\cdot|B| = |A\cup B|\cdot|A\cap B|$$

By introducing $|A \cup B| = e^{x}$ and $|A \cap B| = e^{y}$,

we obtain $|A| = |B| = e^{\frac{x+y}{2}}$.

Then the SSA $S_A + S_B \geq S_{A \cup B} + S_{A \cap B}$

leads to $2S\left(\frac{x+y}{2}\right) \geq S(x) + S(y).$ **Concave**



$\Rightarrow$ $S''(x) \leq 0.$

$\Downarrow$

Entropic c-theorem: $C'(x) \leq 0$

For 2d CFT, we have

$$S_A = \frac{c}{3} \log \frac{e^{x}}{\varepsilon}$$

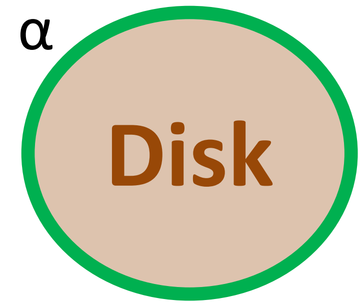
Here we introduced entropic c-function: $C(x) = 3S'(x)$

(3-2) g-theorem

Definitions of g-function (boundary Entropy) [Affleck-Ludwig 1991]

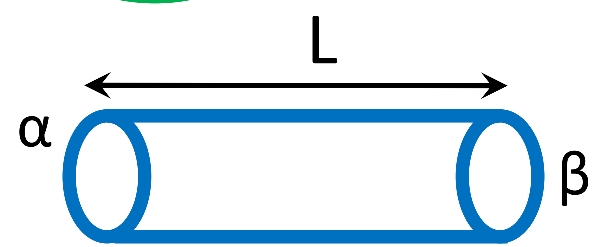
Def 1 (Disk Amplitude)

$$S_{bdy(\alpha)} = \log g_{\alpha}\,, \qquad g_{\alpha} \equiv \langle 0 | B_{\alpha} \rangle.$$



Def 2 (Cylinder Amplitude)

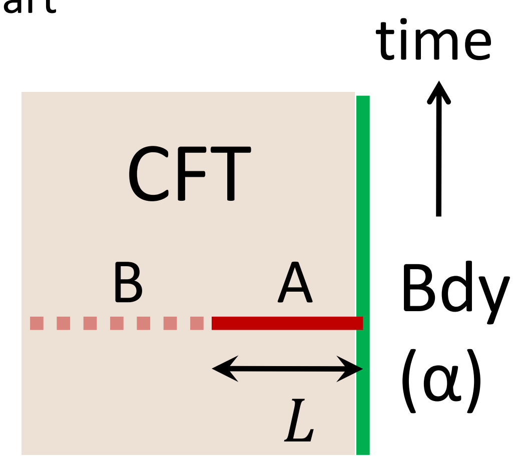
$$Z^{cylinder}_{(\alpha,\beta)} = \langle B_{\alpha} | e^{-HL} | B_{\beta} \rangle \underset{L\to\infty}{\approx} \underbrace{g_{\alpha} g_{\beta}}_{\text{Boundary Part}} \underbrace{e^{-E_0 L}}_{\text{Bulk Part}}.$$



Def 3 (Entanglement Entropy)

In 2d BCFT, the EE behaves like

$$S_A = \underbrace{\frac{c}{6}\log\frac{2L}{\varepsilon}}_{\text{Bulk Part}} + \underbrace{\log g_{\alpha}}_{\text{Boundary Part}}\ .$$



[Calabrese-Cardy 2004]

g-theorem

g-function (boundary entropy) monotonically decreases under boundary RG flow.

[Affleck-Ludwig 1991]

g-function = (UV regularized) disk partition function

boundary RG flow: $\int dt\, O(t, x = 0)$

Known proofs of g-th

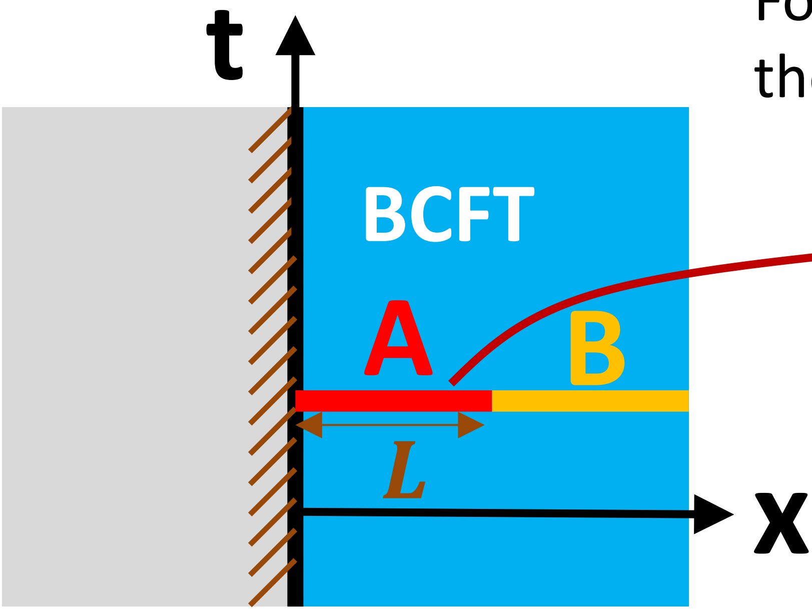
(i) Field theoretic proof [Friedan-Konechny 2003]

(ii) QI proof using relative entropy [Casini-Landea-Torroba 2016,2022]

(iii) Proof from symmetry argument [Cuomo-Komargodski-Raviv-Moshe 2021]

→ We will give the simplest direct proof of g-th from SSA which give a more geometric insight ! [Harper-Kanda-Tasuki-TT 2024]

(3-3) EE in 2d BCFT and g-function



For BCFTs (conformal b.c.),
the replica method calculation leads to

$$S_A = \frac{c}{6}\log\frac{2L}{\varepsilon} + \log g$$

[Calabrese-Cardy 2004]

A constant fixed by its bdy condition

When the bdy breaks conformal invariance, though the bulk is CFT, the **bdy entropy log g** depends on the size of A, i.e. |A|=L.

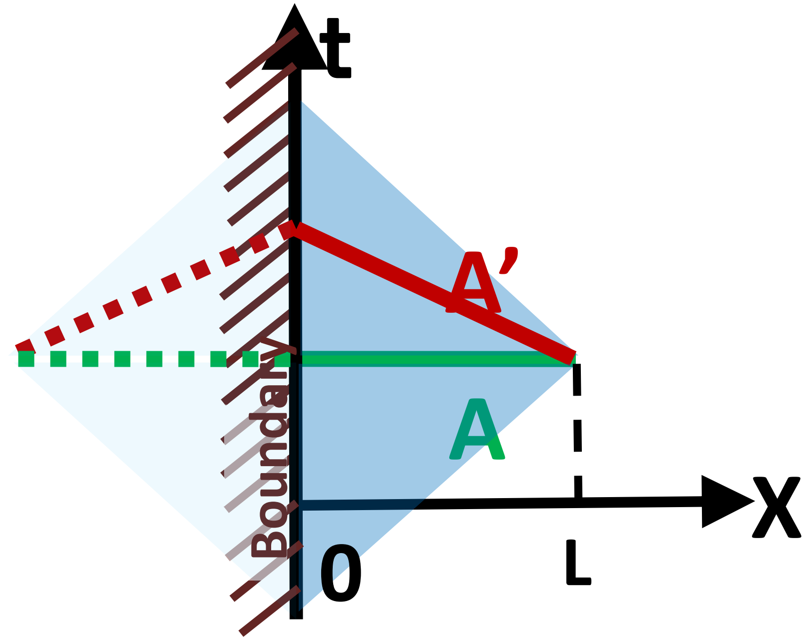
Below, we will show log g monotonically decreases as a function of L.

A few useful properties

(i) Due to the complete reflection at the boundary, we find

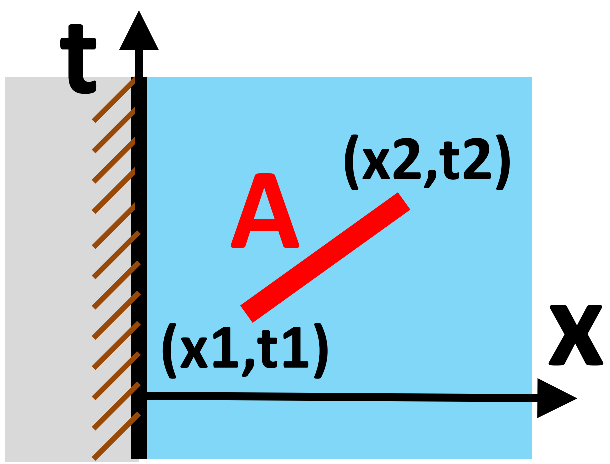
$$S_{A'} = S_A$$

We simply call this $S_{dis}(L)$.



(ii) In general, EE non-trivially depends on the two end points:

$$S(x_1, t_1; x_2, t_2)$$

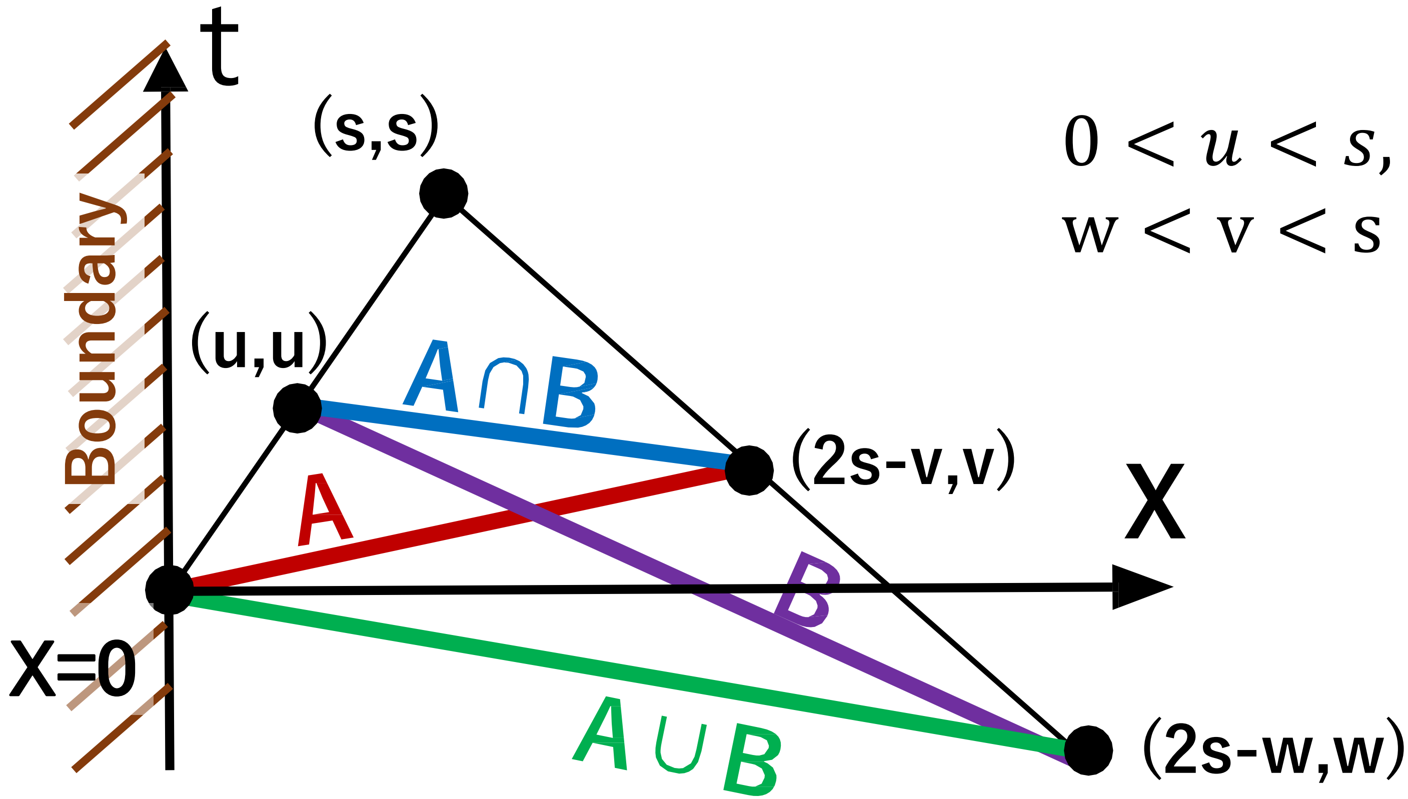


However, when A gets closer to light-like,

$$S(x_1, t_1; x_2, t_2) \approx \frac{c}{3} \log \sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2} / \varepsilon.$$

(3-4) Proving entropic g-theorem from SSA

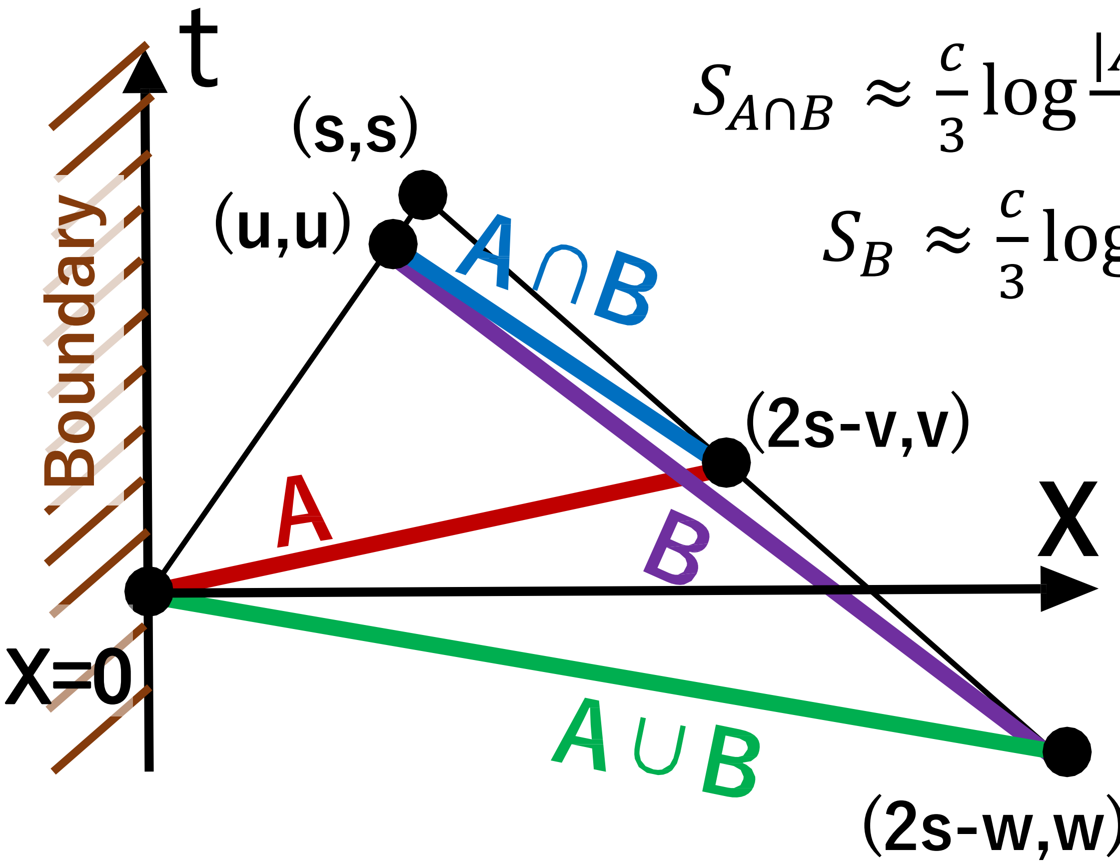
We consider the following setup of SSA:



$0 < u < s,$
$w < v < s$

$$S_{A\cup B} = S_{dis}\ (2s - w), \qquad S_A = S_{dis}\ (2s - v),$$
$$S_{A\cap B} = S(u, u; 2s - v, v), \qquad S_B = S(u, u; 2s - w, w)$$

Now **we take the limit u→s**, where A∩B and B become light-like.



$$S_{A\cap B} \approx \frac{c}{3}\log\frac{|A\cap B|}{\varepsilon} = \frac{c}{6}\log\frac{4(s-u)(s-v)}{\varepsilon^2}.$$

$$S_B \approx \frac{c}{3}\log\frac{|B|}{\varepsilon} = \frac{c}{6}\log\frac{4(s-u)(s-w)}{\varepsilon^2}.$$

$$\Delta S \equiv S_A + S_B - S_{A\cap B} - S_{A\cup B}$$
$$\approx S_{dis}(2s-v) - S_{dis}(2s-w) + \frac{c}{6}\log\frac{s-w}{s-v}.$$

By taking the limit v→w, the SSA inequality ΔS≧0 leads to

$$\frac{dS_{dis}(L)}{dL} \leq \frac{c}{6} \cdot \frac{1}{L-s} \quad , \quad (L \equiv 2s - w).$$

Since L and s are arbitrary, we can choose L>>s, which leads to

$$L\,\frac{dS_{dis}(L)}{dL} \leq \frac{c}{6} \quad .$$

By introducing the entropic g-function as

$$\log g(L) \equiv S_{dis}(L) - \frac{c}{6}\log\frac{2L}{\epsilon},$$

we are now able to derive the entropic g-theorem:

$$L\frac{d}{dL}\log g(L) \leq 0.$$

(3-5) Entropic g-theorem for Interface CFTs

Consider a 2d CFT on a plane with a defect line at x=0.

[Oshikawa-Affleck 1996, Bachas -de Boer-Dijkgraaf-Ooguri 2001]

Its EE looks like $S_A = \frac{c}{3}\log\frac{L}{\varepsilon} + \log \mathrm{g}(\mathrm{L})$.

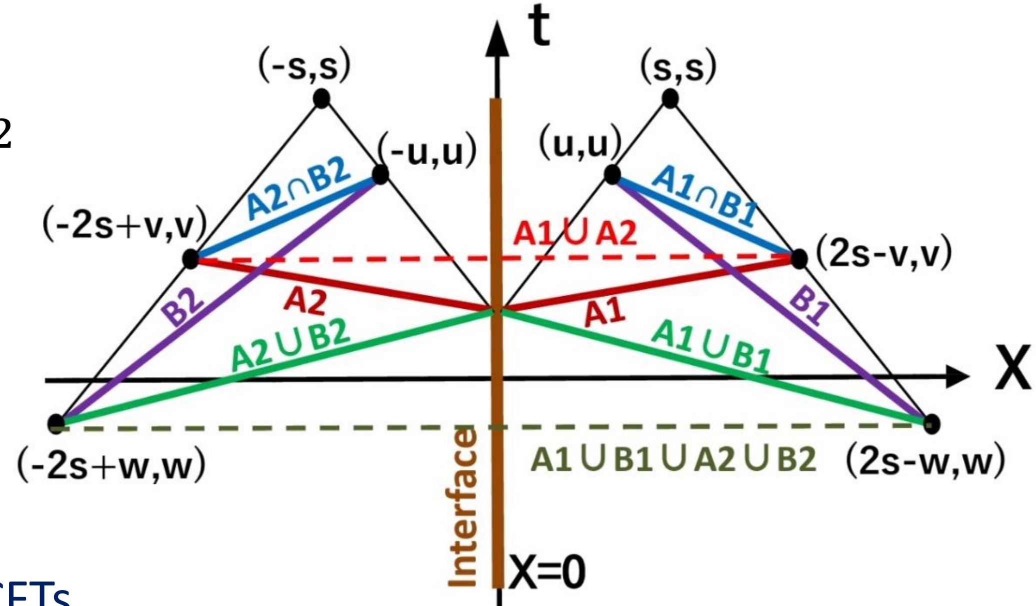
[Azeyanagi-Karch-Thompson-TT 2007]

Again we can derive the entropic g-theorem from the SSA, by doubling the setup:

$$A = A_1 \cap A_2, \quad B = B_1 \cap B_2$$

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

$$L\frac{d}{dL}\log g(L) \leq 0.$$



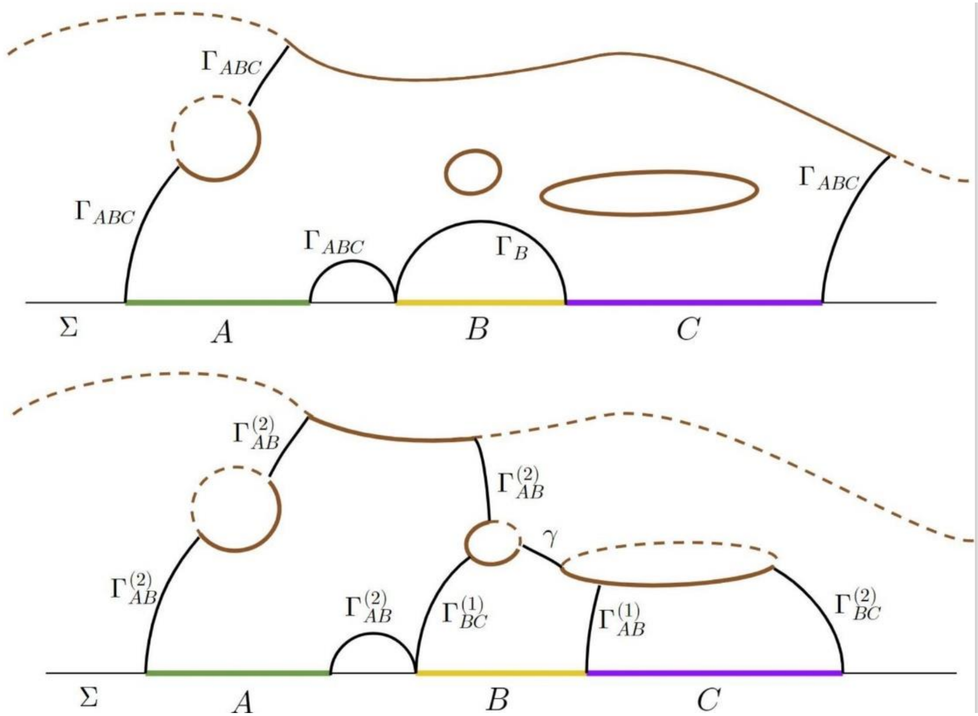
Note: for Interface of two different CFTs, we can simply take c=(c1+c2)/2 and the Proof follows similarly.

[cf. Other types of constraint from SSA Karch-Kusuki-Ooguri-Sun-Wang 2023]

④ Holographic g-theorem

(4-1) SSA in a static AdS/BCFT

SSA in static setups of AdS/BCFT (or a more generally SSA on A time symmetric slice) is satisfied for any shapes of EOW branes.



The minimal surfaces Γ_* are all on the same time slice.

↓

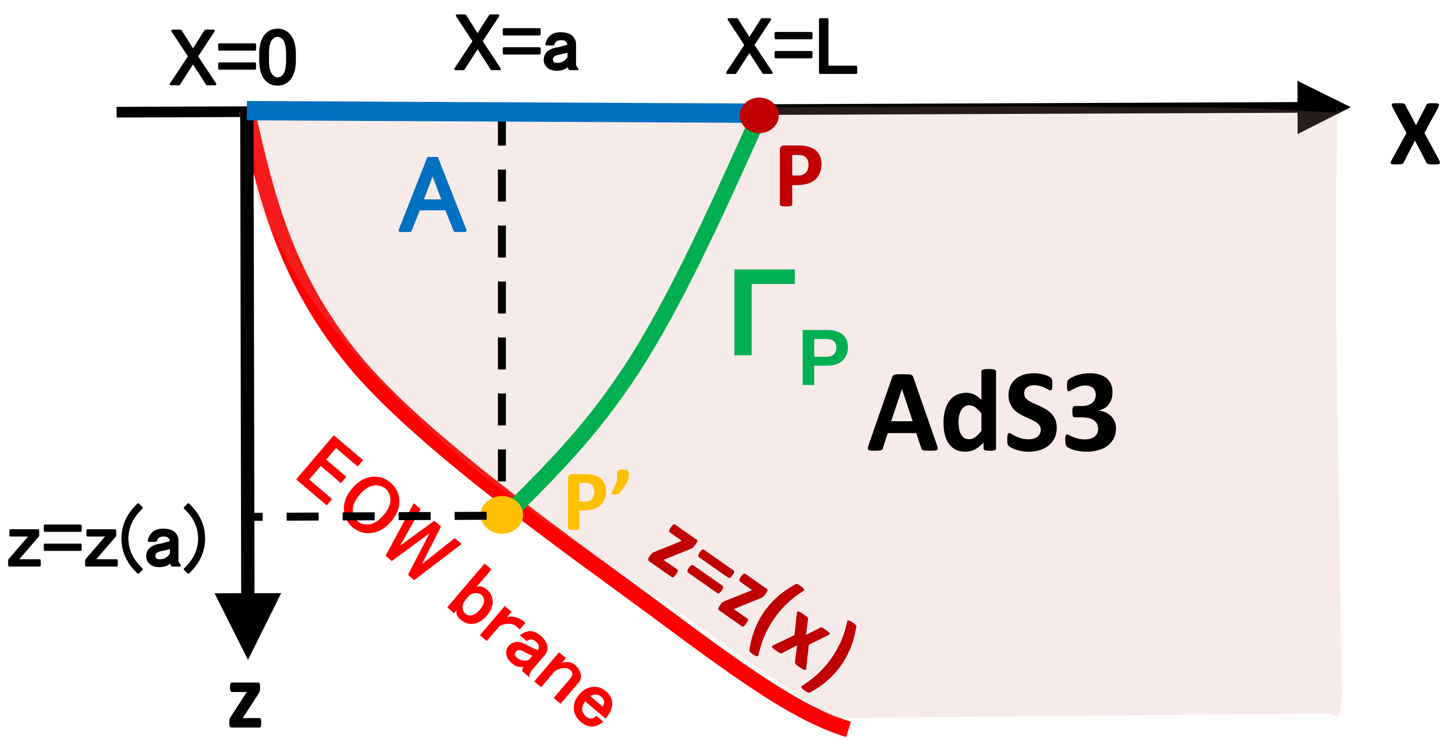
We can prove SSA as in the case without the EOW branes.

[SSA in Static AdS/CFT: Headrick-TT 2007]

Similarly we can derive the MMI. [MMI in AdS/CFT: Hayden-Headrick-Maloney 2011]

(4-2) SSA in Lorentzian setups of AdS/BCFT

Generally, a 2d CFT on a half space (x>0) with a bdy RG flow is dual to a generic shaped EOW brane in AdS3.



Below we evaluate the HEE: $S_{dis}(L) = \frac{c}{6}|\Gamma_P|$ as a function of L.

For a generic profile of EOW brane: z=z(x), we can relate the boundary point x=L and the intersection of the minimal surface and EOW brane x=a by

$$L = a - \frac{z(a)}{\dot{z}(a)} + z(a)\sqrt{1+\frac{1}{\dot{z}(a)^2}}\ .$$

The geodesic length reads $|\Gamma_P| = \log\left[\frac{2z(a)\sqrt{1+\dot{z}(a)^2}}{\epsilon\left(\sqrt{1+\dot{z}(a)^2}+1\right)}\right].$

Finally we obtain $$\frac{6}{c}\cdot L\frac{dS_{dis}(L)}{dL} - 1 = \frac{a\dot{z}(a) - z(a)}{z(a)\sqrt{1+\dot{z}(a)^2}}.$$

The SSA is satisfied if this is non-negative.

↑

We can guarantee this by assuming the null energy condition on the EOW brane.

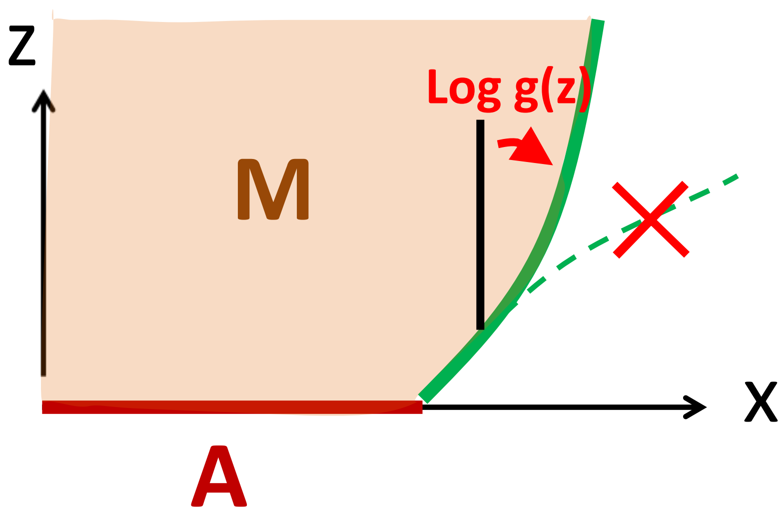
$$T^Q_{ab}N^aN^b \propto -\ddot{z}(a) \geq 0$$

[SSA in Lorentzian AdS/CFT: Wall 2012]

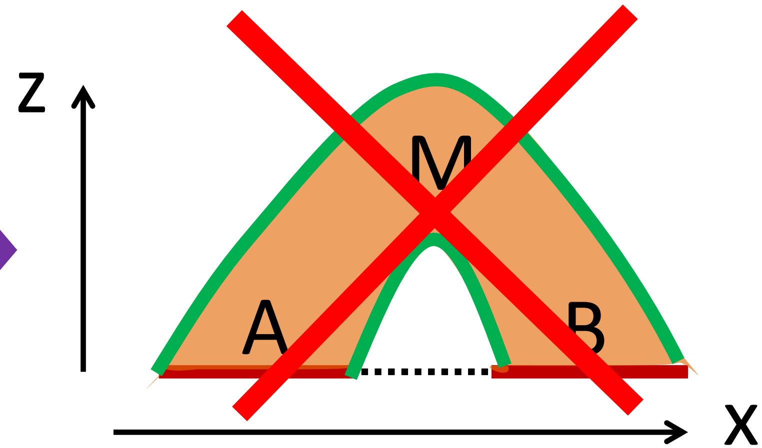
g-theorem and wormhole

Null Energy Condition

$$T^{Q}_{ab} N^a N^b \geq 0$$



"Topological Censorship with Bdy"
No traversable wormhole
in classical gravity



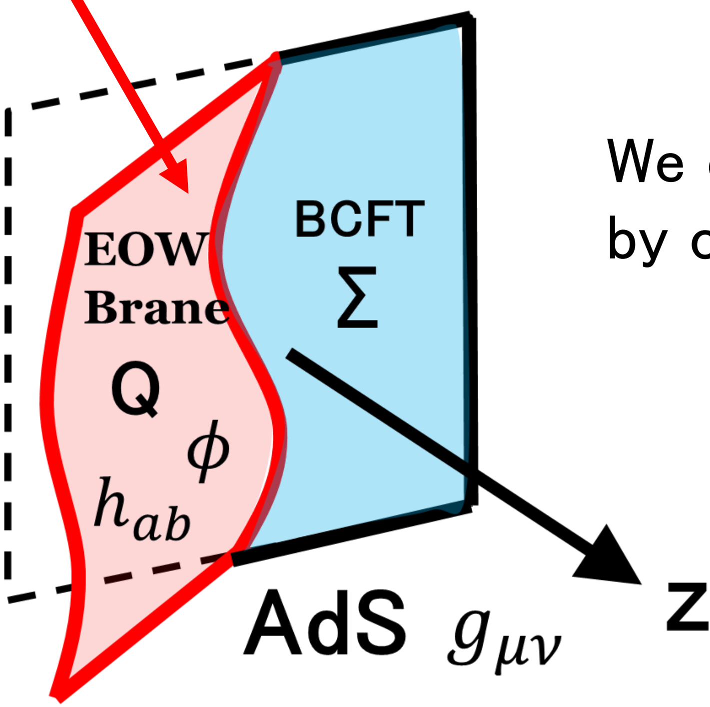
[cf. top. censorship without boundary
Galloway-Schleich-Witt-Woolgar 99]

⑤ AdS/BCFT with boundary localized scalar

[Kanda–Sato–Suzuki–Wei–TT 2023]

(5-1) Localized scalar model

$$I_{\mathrm{brane}} = -\frac{1}{8\pi G_N}\int_Q d^dx\sqrt{h}(K - h^{ab}\partial_a\phi\partial_b\phi - V(\phi)),$$



We can design the shape of Q by choosing the potential V(ϕ).

Bdy condition: $K_{ab} - Kh_{ab} = -\left(h^{cd}\partial_c\phi\partial_d\phi + V(\phi)\right)h_{ab} + 2\partial_a\phi\partial_b\phi.$

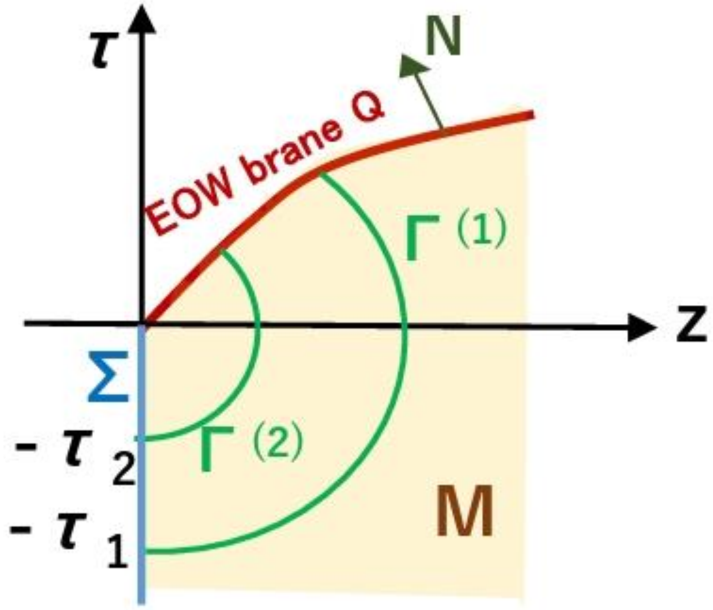
(5-2) Gravity dual of Bdy RG flow

Boundary EOMs lead to

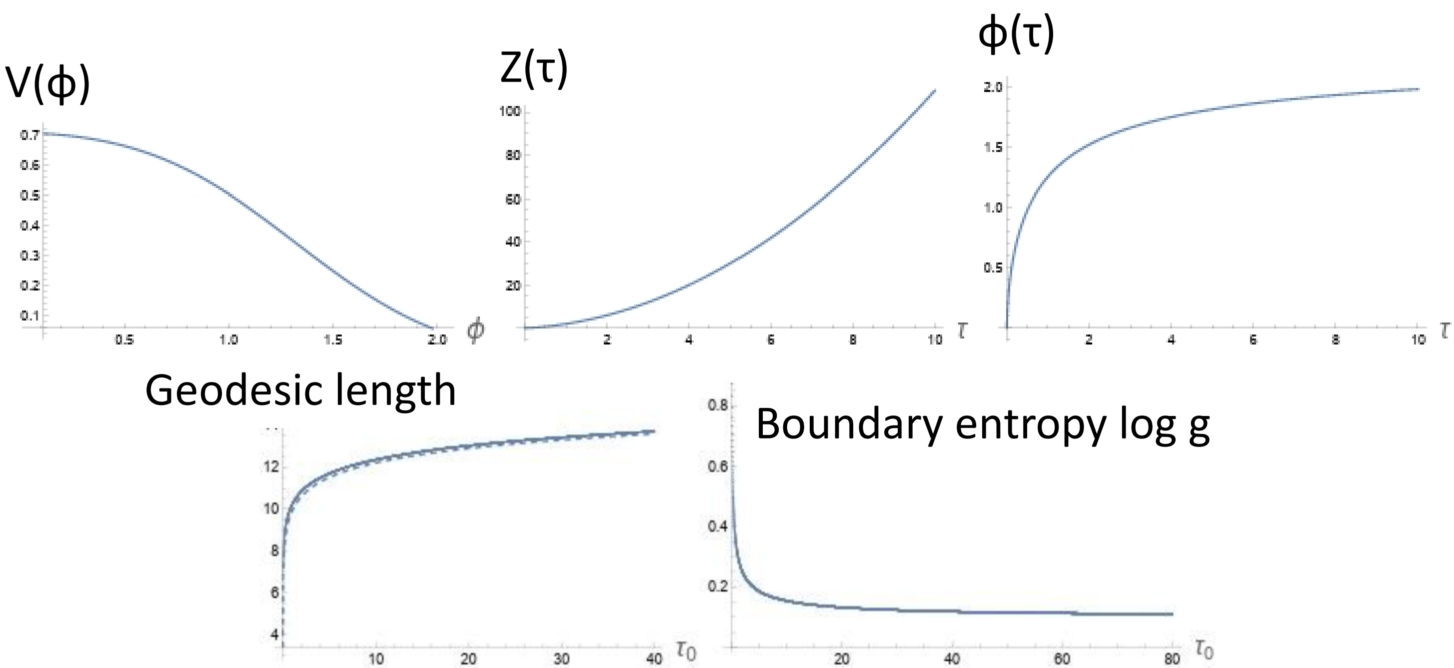
$$\dot{\phi}^2 = \frac{\ddot{z}}{2z\sqrt{1+\dot{z}^2}} \geqq 0$$

(Null energy condition)

$$V(\phi) = \frac{z^2\dot{\phi}^2}{1+\dot{z}^2} + \frac{1}{\sqrt{1+\dot{z}^2}}.$$

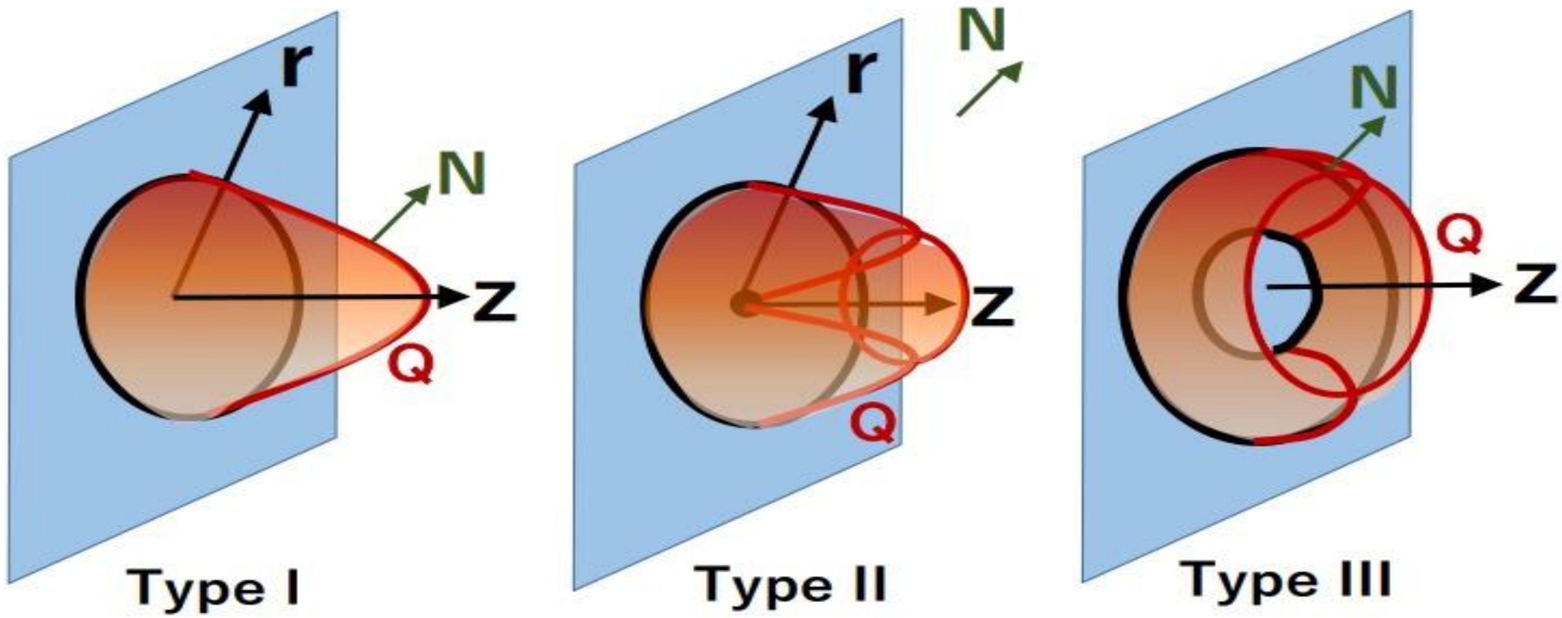


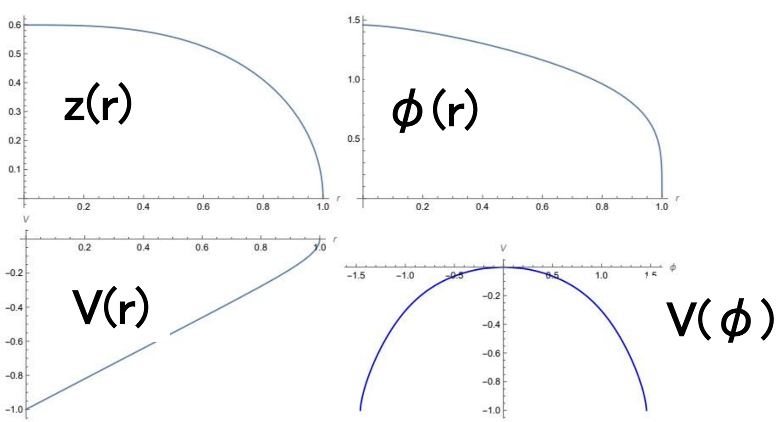
Plots for $z(\tau) = \tau + \tau^2$

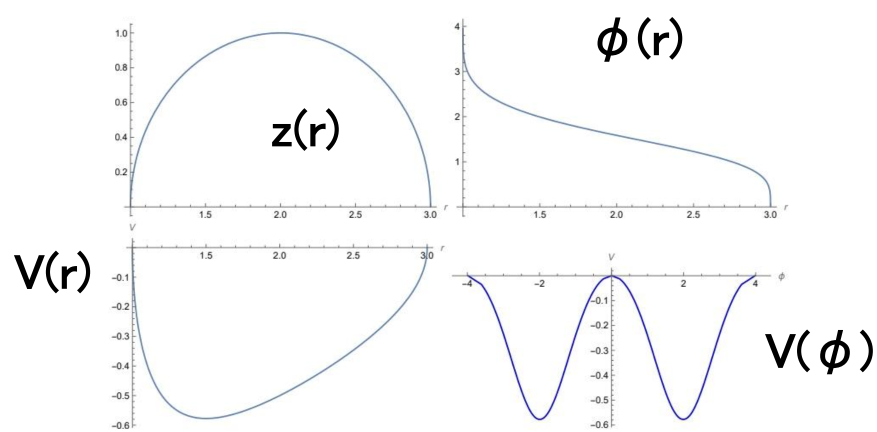


Another Example: we can also find rotationally invariant solutions.

→ Boundary RG flows on a disk and annulus.



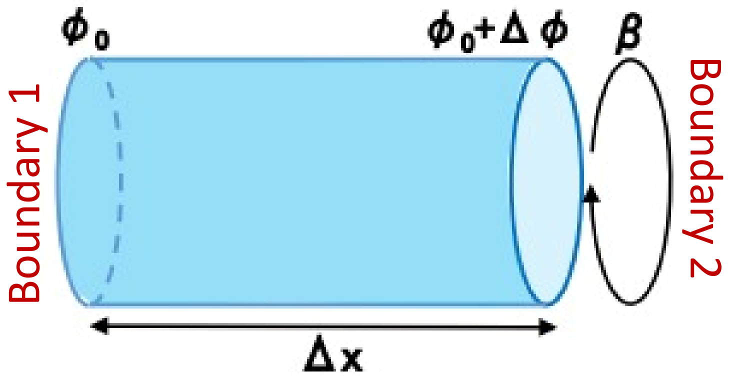




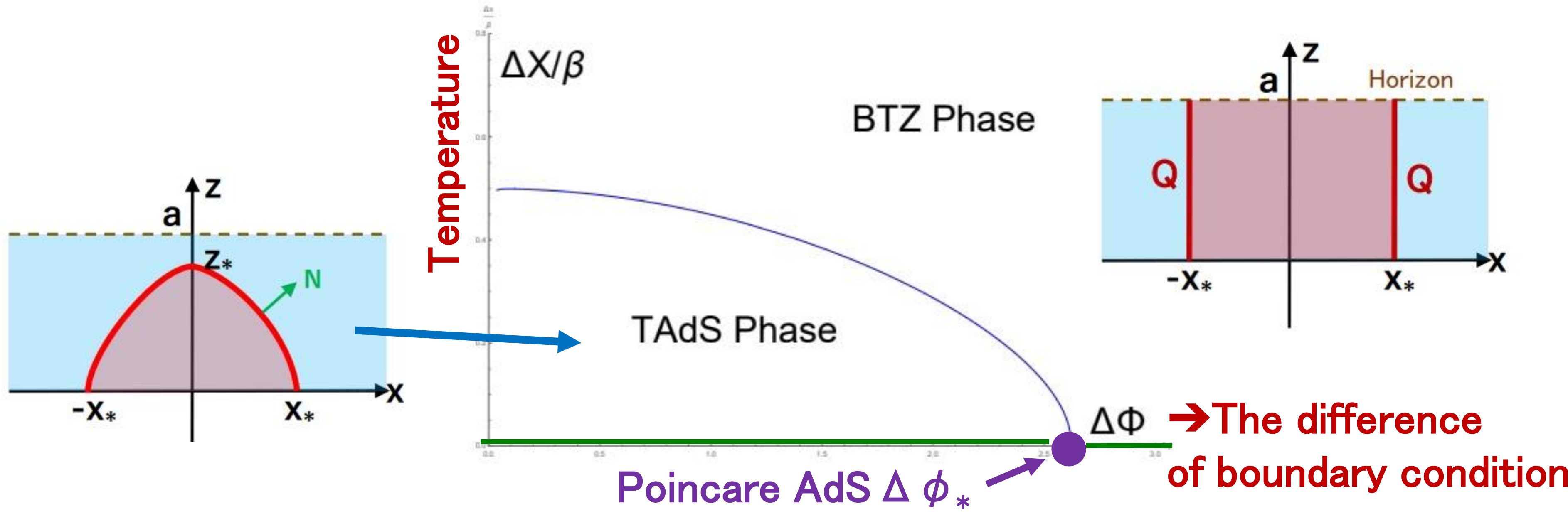
(5-3) Entanglement Phase Transition

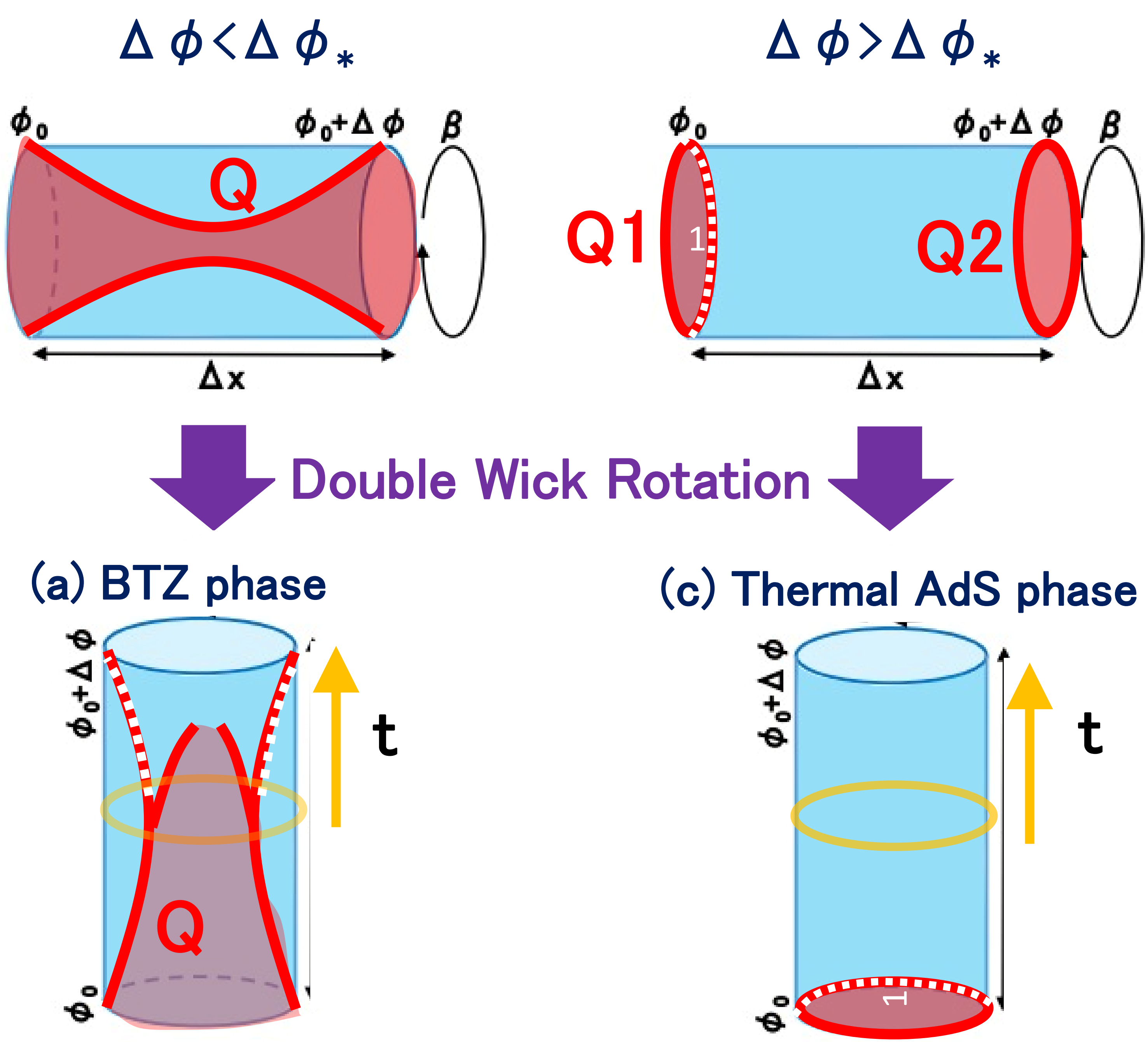
[Kanda-Sato-Suzuki-Wei-TT 2023] [Kanda-Kawamoto-Suzuki-Tasuki-Wei-TT 2023]

Consider the AdS3/BCFT2
with V(ϕ)=0
for a 2d CFT on a cylinder.
("Boundary Janus solutions")



Phase diagram based on the free energy

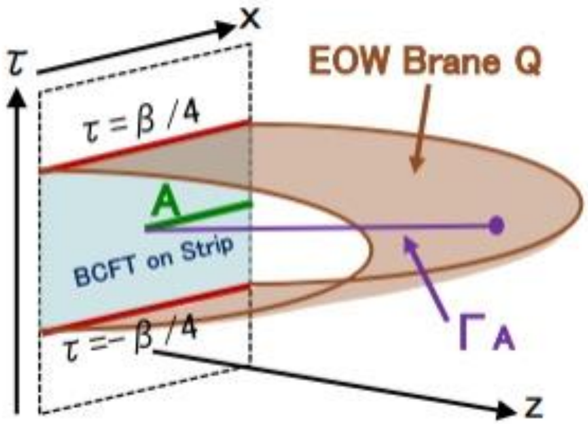




Δ φ<Δ φ*
Δ φ>Δ φ*
φ0
φ0+Δ φ
β
Q
Δx
Q1
Q2
1
Double Wick Rotation
(a) BTZ phase
(c) Thermal AdS phase
t

Double Wick Rotation and Time Evolution

We take β→∞ limit.



A= a half space

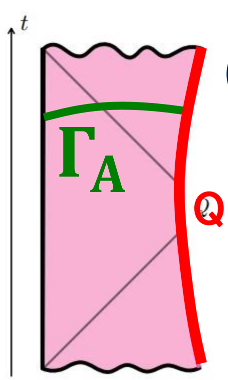
τ =it

Imaginary valued scalar field on EOW brane ➔ Non-unitary

$$\rho(t) = e^{-\left(\frac{\beta}{4}+it\right)H} \left| B(\varphi_0 + \Delta\varphi) \right\rangle \left\langle B(\varphi_0) \right| e^{-\left(\frac{\beta}{4}-it\right)H}$$

|B>=Boundary state ~ Direct product state

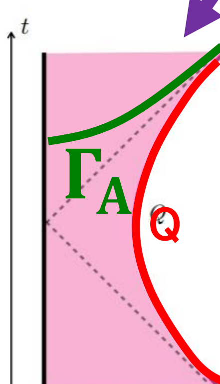
$\Delta\phi < \Delta\phi_*$ $\Delta\phi = \Delta\phi_*$ $\Delta\phi > \Delta\phi_*$



(a) BTZ phase

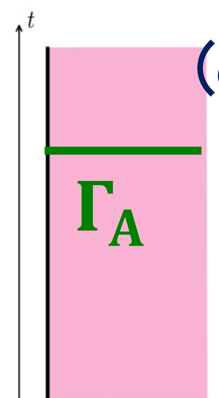
$$S_A \approx \frac{c}{6} \cdot \frac{\pi}{2\alpha} \cdot t$$

[Δ ϕ=0→Hartman-Maldacena 2013]



(b) Poincare AdS phase

$$S_A \approx \frac{c}{3} \cdot \log t.$$



(c) Thermal AdS phase

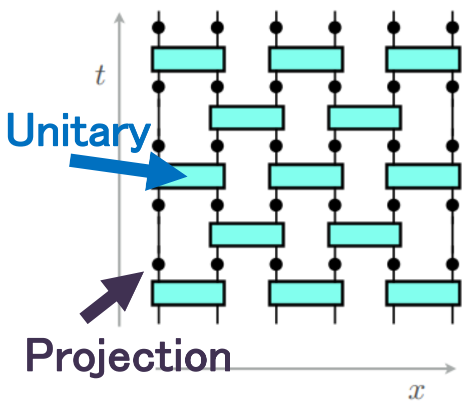
$$S_A \approx \text{const.}$$

This result looks very analogous to

Entanglement Phase Transition

(Measurement Induced Phase Transition)

[Skinner–Ruhman–Nahum,
Li–Chen–Fisher 2018]

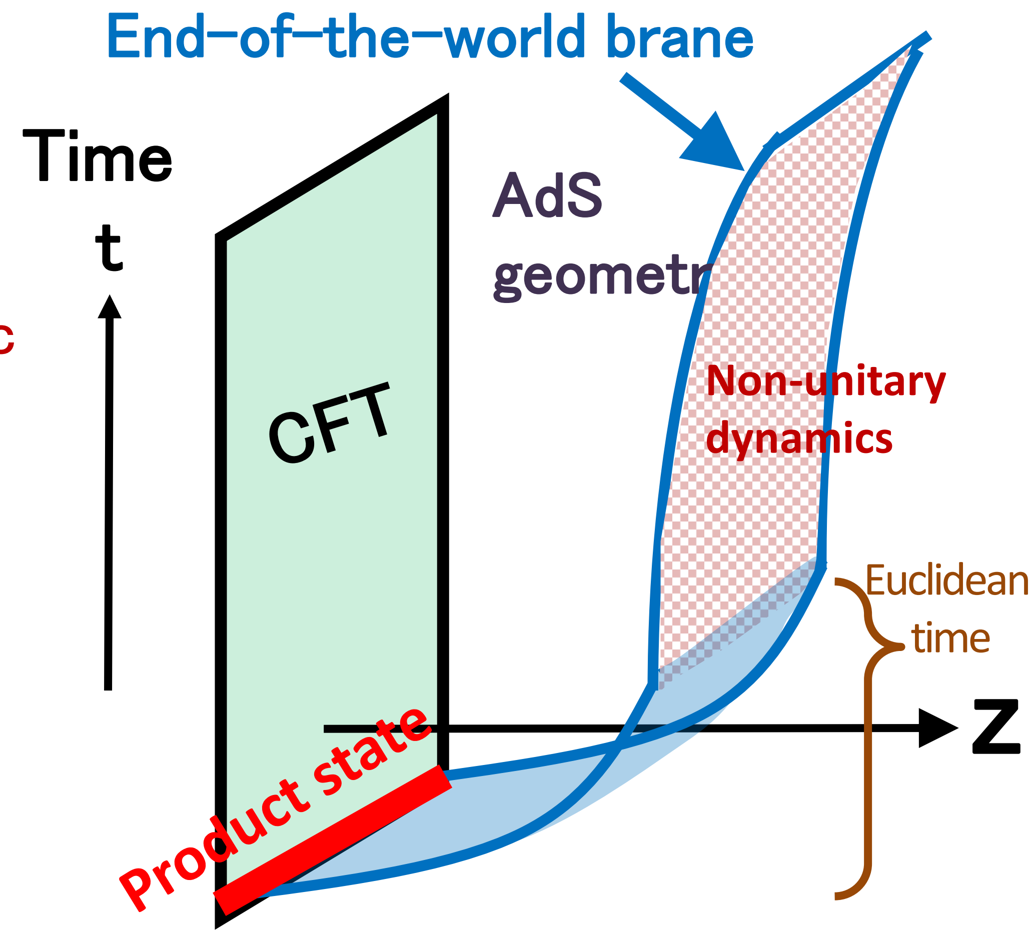


$$(i)\quad p < p_* : \quad S_A \propto t,$$

$$(ii)\quad p = p_* : \quad S_A \propto \log t,$$

$$(iii)\quad p > p_* : \quad S_A = \text{finite},$$

Holographic
Model ?



[For other holographic approaches refer to Antonini–Bentsen–Cao–Harper –Jian–Swingle, 2022, Goto–Nozaki–Tamaoka, –Tan 2022]

However, note that this entropy SA computed in the AdS/BCFT should be regarded as the pseudo entropy instead of EE.

Pseudo Entropy $$S(\mathcal{T}_A^{\psi|\varphi}) = -\mathrm{Tr}\left[\mathcal{T}_A^{\psi|\varphi}\log\mathcal{T}_A^{\psi|\varphi}\right]$$

Transition matrix
(Non-Hermitian in general)

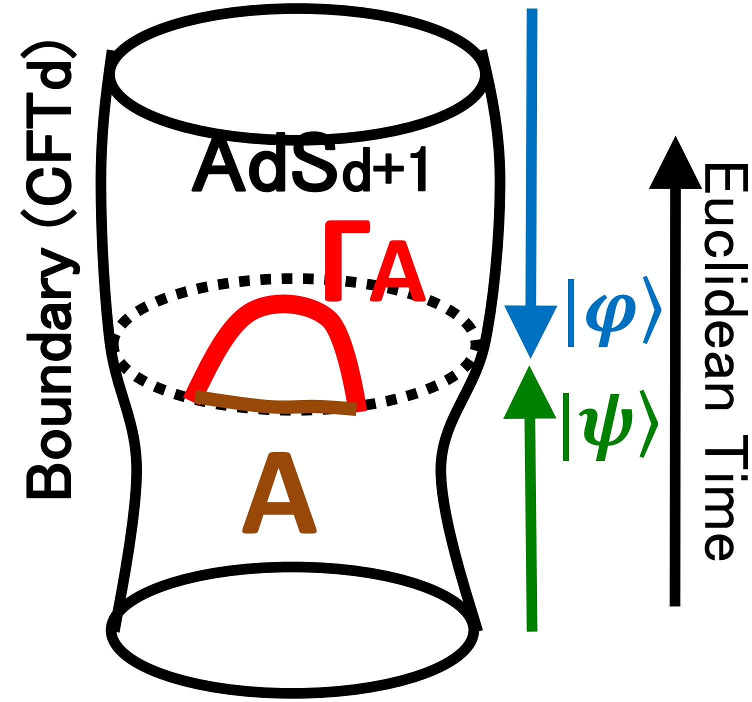
Initial state → $|\psi\rangle$; Final state → $\langle\varphi|$

$$\mathcal{T}^{\psi|\varphi} := \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} \quad \left(\mathcal{T}_A^{\psi|\varphi} := \mathrm{Tr}_{\bar{A}}\mathcal{T}^{\psi|\varphi}\right)$$

Holographic pseudo entropy

$$S(\mathcal{T}_A^{\psi|\varphi}) = \min_{\Gamma_A}\frac{\mathrm{Area}(\Gamma_A)}{4G_N}$$

➡ For pseudo entropy, refer to lecture 3



⑥ Conclusions

- SSA provides a new geometric proof of g-theorem under boundary RGs for 2d CFTs.
- In the holographic analysis of AdS/BCFT, SSA is automatically satisfied in static backgrounds.
- In Lorentzian boosted setups of AdS/BCFT, SSA is satisfied if the null energy condition is imposed on the EOW brane.
- We constructed a class of explicit gravity duals of boundary RG flow by adding a scalar field localized on the EOW brane.
- When two boundary conditions are different in a BCFT on a cylinder, its gravity dual predicts entanglement phase transition.

Future directions

- Higher (co-)dimensional generalizations ?
- SSA in Time-dependent cases ?
- SSA in AQFT and c/g theorem ?

Thank you very much ！