# Chromatic tower of $D(R)$

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ICTS

- inspired by eponymous paper by Neeman.

(I) $\mathcal{T}$ = any triangulated category with all coproducts

e.g. $\mathcal{T} = D(R)$  $R$ comm. noeth. ring

or $D(\text{q-Coh}\, X)$, $\text{StMod}\, kG$, $\text{Ho}(Sp)$, ...

- A subcategory $\mathcal{S} \subseteq \mathcal{T}$ is thick if:

full ✓ and $0 \in \mathcal{S}$

- $\mathcal{S}$ is triangulated:

$$X \to Y \to Z \to \Sigma X$$

Any two of $X, Y, Z$ in $\mathcal{S}$ $\Rightarrow$ all three in $\mathcal{S}$

- $\mathcal{S}$ is closed under retracts

$$X \oplus Y \in \mathcal{S} \Rightarrow X, Y \in \mathcal{S}$$

- $\mathcal{S}$ closed under finite sums.

Example: $F : \mathcal{T} \to \mathcal{U}$ exact functor

$\text{Ker}(F)$ is a thick subcategory

If $S$ is closed under all coproducts, then it is <u>localizing</u>.

Fix $X \in T$

$\mathrm{Thick}(X) :=$ Smallest thick subcategory of $T$ containing $X$

$\mathrm{Loc}(X) :=$ Smallest localizing subcategory containing $X$.

<u>Examples</u>: $T = D(R)$

(1) $\mathrm{Thick}(R) =$ Perfect complexes

$\mathrm{Loc}(R) = D(R)$

(2) $I \subseteq R$ ideal. Then

$$\mathrm{Loc}(R/I) = \{ M \in D(R) \mid R\Gamma_{V(I)} M \xrightarrow{\sim} M \}$$

$$= \Gamma_{V(I)} H_i(M) = H_i(M) \ \forall i$$

$$= \mathrm{Supp}_R H(M) \subseteq V(I)$$

$\mathrm{Thick}(R/I)$ has no such simple description in general

<u>Exercise</u>: Say $R/I$ is regular (e.g. $I \subseteq R$ maximal)

Then $\mathrm{Thick}(R/I) = \{ M \in D^b(\mathrm{mod}\, R) \mid R\Gamma_{V(I)} M \xrightarrow{\sim} M \}$

Basic question: Given $Y \in \mathcal{T}$, is

$$Y \in \mathrm{Thick}(X)\,? \qquad Y \overset{?}{\in} \mathrm{Loc}(X)\,?$$

Why care: Properties of $X$ are inherited by objects in $\mathrm{Thick}(X)$.

Example: $\mathcal{T} = D(R)$. Say $X$ is perfect, then any $Y \in \mathrm{Thick}(X)$ is perfect.

[ But $X$ projective $\not\Rightarrow$ $Y$ projective ]

— x —

Sometimes $Y \in \mathrm{Loc}(X) \Rightarrow Y \in \mathrm{Thick}(X)$

( Converse is clear)

An object $C \in \mathcal{T}$ is small (or compact) if for any set $\{X_i\}$ in $\mathcal{T}$, the natural map

$$\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(C, X_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(C, \bigoplus_i X_i)$$

is bijective.

$$\mathcal{T}^c := \text{Compact objects in } \mathcal{T}$$

- Thick subcategory.

Example: ① $D(R)^c$ = Perfect Complexes

② $(\mathrm{St\text{-}Mod}\, A)^c = \mathrm{st\text{-}mod}\, A$

$A$ any Artin, Gorenstein algebra

③ $\mathrm{Ho}(\mathrm{Sp})^c$ = Finite Spectra

Thm: $\boxed{\mathcal{C}, \mathcal{D} \text{ compact and } \mathcal{D} \in \mathrm{Loc}(\mathcal{C}) \Rightarrow \mathcal{D} \in \mathrm{Thick}(\mathcal{C}).}$

Neeman?

II From now on $\mathcal{T} = D(R)$

$\mathfrak{p} \in \operatorname{Spec} R$ set $k(\mathfrak{p}) := \left( R_{\mathfrak{p}} / \mathfrak{p} R_{\mathfrak{p}} \right)$

- Residue field of $R_{\mathfrak{p}}$.

For $M \in D(R)$ set

$$\operatorname{supp}_R M := \{ \mathfrak{p} \in \operatorname{Spec} R \mid M \overset{\mathbb{L}}{\underset{R}{\otimes}} k(\mathfrak{p}) \neq 0 \}$$

("small") Support of $M$

Notable features:

1. $\mathrm{supp}_R M = \emptyset \iff M = 0$. In fact

$$\boxed{\mathrm{Loc}(M) = \mathrm{Loc}(M \otimes^{\mathbb{L}}_R k(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\, R)}$$

"M can be built out of its strata $\{M \otimes^{\mathbb{L}}_R k(\mathfrak{p})\}$"

Local-to-global principle for $D(R)$.

2. $N \in \mathrm{Loc}(M) \Rightarrow \mathrm{supp}_R N \subseteq \mathrm{supp}_R M$

3. $\mathrm{Supp}_R(\oplus M_i) = \bigcup_i \mathrm{supp}_R(M_i)$

4. $\mathrm{supp}_R(M \otimes^{\mathbb{L}}_R N) = \mathrm{supp}_R M \cap \mathrm{supp}_R N$

5. $\mathrm{supp}_R\, k(\mathfrak{p}) = \{\mathfrak{p}\}$

— x —

Hopkins: $F, G$ perfect cx'es with $\mathrm{supp}_R F \subseteq \mathrm{supp}_R G$. Then $F \in \mathrm{Thick}(G)$.

"Proof" Set $V = \mathrm{supp}_R G$. Then

$$\begin{array}{l} R\Gamma_V F \xrightarrow{\sim} F \quad (\because \mathrm{supp}_R F \subseteq V) \\ \wr\wr \\ \mathrm{Cell}_G F \in \mathrm{Loc}(G) \end{array}$$

Thus $F \in \mathrm{Loc}(G)$

$\Rightarrow F \in \mathrm{Thick}(G)$ ( $\because$ $F, G$ perfect) $\square$

What is hidden?

Application: Say $R \longrightarrow S$, $N \in D^b(\mathrm{mod}\, S)$

finite map of rings

Wellknown: . $S$ perfect $/R$ + $N$ perfect $/S$

$\Rightarrow N$ perfect $/R$.

(i.e. $\mathrm{projdim}_R S < \infty$ + $\mathrm{projdim}_S N < \infty \Rightarrow \mathrm{projdim}_R N < \infty$).

. Clearly $S$ + $N$ perfect $/R$ $\not\Rightarrow$ $N$ perfect $/S$

(Example?)

Thm: $\boxed{N \text{ perfect} / R \text{ and over } S \Rightarrow S \text{ perfect} / R \text{ on } \mathrm{Supp}_S N.}$

(i.e. $\mathrm{prodim}_{R_{\mathfrak{q} \cap R}} S_{\mathfrak{q}} < \infty$ $\forall$ $\mathfrak{q} \in \mathrm{Supp}_S N$)

Sketch of proof: May reduce to

$R \longrightarrow S$ local map of local rings

$0 \neq N \in D^b(\mathrm{mod}\, S)$

with $N$ perfect / $S'$ and $R$.

Let $\underline{s} = s_1, \dots, s_n$ be a generating set for $m_S$, the maximal ideal of $S$

$K := K(\underline{s}; S)$ the Koszul cx. on $\underline{s}$.

Then $\operatorname{supp}_S K = \{m_S\} \subseteq \operatorname{supp}_S N$

non-empty closed subset of Spec $S'$

Thus $K \in \operatorname{Thick}_{S'}(N)$ ($\because$ Hopkins)

So $N$ perfect / $R$ $\Rightarrow$ $K$ perfect / $R$.

Now $K = K(s_1, \dots, s_n) = K(s_1; \overbrace{K(s_2, \dots, s_n)}^{X})$

Koszul cx. on $s_1$ with coefficient in $K(s_2, \dots, s_n)$

**Lemma:** Say $X \in D^b(\operatorname{mod} S)$ and $s \in m_S$. If $K(s; X)$ is perfect / $R$, then so is $X$.

- Follows from

$$0 \to X \to K(s; X) \to \Sigma X \to 0$$

Applying $\mathrm{Tor}^R(k, -)$ $k$ the residue field of $R$

yields:

$$\longrightarrow \mathrm{Tor}_i^R(k, X) \xrightarrow{s} \mathrm{Tor}_i^R(k, X) \longrightarrow \mathrm{Tor}_i^R(k, K(s; X)) \rightarrow$$

$$\uparrow \qquad\qquad\qquad\qquad \quad //$$

So onto for $i \gg 0$. $\quad 0 \;\forall\, i \gg 0$

$X \in D^b(\mathrm{mod}\, S) \Rightarrow \mathrm{Tor}_i^R(k, X)$ f.g. $S$-module

so $\mathrm{Tor}_i^R(k, X) = 0 \quad \forall\, i \gg 0$

i.e. $X$ perfect $/R$. $\square$

Repeated application of this lemma yields:

$K(\underline{s})$ perfect $/R$ $\Rightarrow$ $S$ perfect $/R$. $\square$

See: Finiteness in the derived category of local rings, by DGI.

Neeman: $\boxed{\text{For any } M, N \in D(R), \text{ if } \mathrm{Supp}_R M \subseteq \mathrm{Supp}_R N \text{ then } M \in \mathrm{Loc}(N).}$

- Implies Hopkins (explain):

  $F, G$ perfect with $\mathrm{Supp}_R F \subseteq \mathrm{Supp}_R G$

  $\Rightarrow \quad F \in \mathrm{Loc}(G) \quad$ (Neeman)

  $\Rightarrow \quad F \in \mathrm{Thick}(G) \quad (\because F, G$ compact).

- Really a reformulation of the local-to-global principle: $\quad \mathrm{Supp}_R M \subseteq \mathrm{Supp}_R N$

  $\Leftrightarrow \quad \mathrm{Loc}(M \otimes^{L}_{R} k(p)) \subseteq \mathrm{Loc}(N \otimes^{L}_{R} k(p))$

  $(\because k(p)$ field$)$

  $\Rightarrow \quad \mathrm{Loc}(M) \subseteq \mathrm{Loc}(N) \qquad \square$

## III Revisiting Dwyer/Greenlees:

Fix $G \in D(R)$ perfect.

- Can assume $G$ is a bounded cx. of f.g. projectives.

$E = \operatorname{End}_R(G)$ dg $R$-algebra

For any $M \in D(R)$

$\operatorname{RHom}_R(G, M) \simeq \operatorname{Hom}_R(G, M)$ is a dg $E^{op}$-module.

One has

$$\begin{tikzcd} D(R) \arrow[r, shift right, "\operatorname{RHom}_R(G,-)"'] & D(E^{op}) \arrow[l, shift right, "- \otimes^{\mathbf{L}}_E G"'] \end{tikzcd}$$

Then

$$\operatorname{RHom}_R(G, M) \otimes^{\mathbf{L}}_E G \longrightarrow M$$
$$\cong$$
$$\operatorname{Cell}_G(M)$$

Implies $- \otimes^{\mathbf{L}}_E G : D(E^{op}) \xrightarrow{\equiv} \operatorname{Loc}(G) = R\Gamma_V D(R)$

$V = \operatorname{supp}_R G$

# "Derived Morita theory"

There is more: Set $\mathcal{G}^{\#} = \mathrm{Hom}_R(\mathcal{G}, R)$

$\mathcal{E}^{op}$-module.

Then $$\begin{tikzcd} D(R) \arrow[r, shift left, "R\mathrm{Hom}_R(\mathcal{G},-)"] & D(\mathcal{E}^{op}) \arrow[l, shift left, "R\mathrm{Hom}_{\mathcal{E}}(\mathcal{G}^{\#},-)"] \end{tikzcd}$$ adjoint pair

$$M \longrightarrow R\mathrm{Hom}_{\mathcal{E}}(\mathcal{G}^{\#}, R\mathrm{Hom}_R(\mathcal{G}, M))$$

$$M \cong R\mathrm{Hom}_R(R, M) \xrightarrow[R\mathrm{Hom}_R(\mathcal{G},-)]{} R\mathrm{Hom}_{\mathcal{E}}(R\mathrm{Hom}_R(\mathcal{G}, R), R\mathrm{Hom}_R(\mathcal{G}, M))$$

- This map is derived completion w.r.t $V = \mathrm{supp}_R\, \mathcal{G}$

We get

$$\begin{tikzcd} D(R) \arrow[r, "R\mathrm{Hom}_R(\mathcal{G},-)" description] & D(\mathcal{E}^{op}) \arrow[l, bend right, "- \otimes^{L}_{\mathcal{E}} \mathcal{G}"'] \arrow[l, bend left, "R\mathrm{Hom}_{\mathcal{E}}(\mathcal{G}^{\#},-)"] \end{tikzcd}$$

Then

$$\begin{tikzcd} R\Gamma_V D(R) \arrow[r, shift left, "\cong"] & L\Lambda^V D(R) \arrow[l, shift left] \end{tikzcd}$$

"Greenlees-May".

Back to the example: $R = \mathbb{Z}$, $G = \mathbb{Z}/p \simeq 0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$

$E = \mathrm{End}_{\mathbb{Z}}(0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0)$

Generated as a $\mathbb{Z}$-module by:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] \arrow[dl, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$f_{-1}$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal, shift right] \arrow[d, "0", shift left] & \mathbb{Z} \arrow[r] \arrow[d, equal, shift right] \arrow[d, "0", shift left] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$f_{00}$ $\quad$ $f_{01}$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[dr, equal] & \mathbb{Z} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$f_{1}$

$[\partial, f_{-1}] = p(f_{00} + f_{01})$ $\qquad$ $[\partial, f_{00}] = p f_{-1}$ $\qquad$ $[\partial, f_{-1}] = 0$

$[\partial, f_{01}] = -p f_{-1}$

$\therefore\ H^0(E) = \mathbb{Z}[f_{00} + f_{01}]$ and $p \cdot [f_{00} + f_{01}] = 0$

$H^1(E) = \mathbb{Z}[f_{-1}]$ and $p \cdot [f_{-1}] = 0$.

To compute $- \overset{\ell}{\underset{E}{\otimes}} G$ one would have to resolve $G$ as an $E$-module.

N.B. $G \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ is a $\mathbb{Z}$-morphism, but not $E$-linear.