Infosys-ICTS Chandrasekhar Lectures, 2024 Aug.
ICTS Program
"Quantum Information, Quantum Field Theory and Gravity"

Pseudo entropy and de Sitter Holography (Lecture 3)

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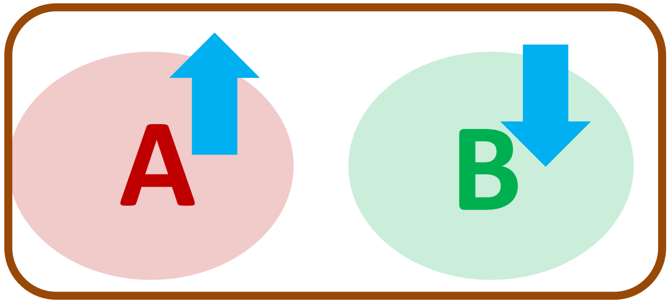
① Introduction

Quantum Entanglement (QE)



Two parts (subsystems) A and B in a total system are quantum mechanically correlated.

e.g. Bell state: $|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right]$ ➡ **Minimal Unit of Entanglement**

Pure States: Non-zero QE $\Leftrightarrow$ $|\Psi\rangle_{AB} \neq |\Psi_1\rangle_A \otimes |\Psi_2\rangle_B$.

Direct Product

The best (or only) measure of quantum entanglement for pure states is known to be **entanglement entropy (EE)**.

EE ＝ # of Bell Pairs between A and B

Entanglement entropy (EE) in "HEP"

Divide a quantum system into two subsystems A and B:

$$H_{tot} = H_A \otimes H_B \quad .$$

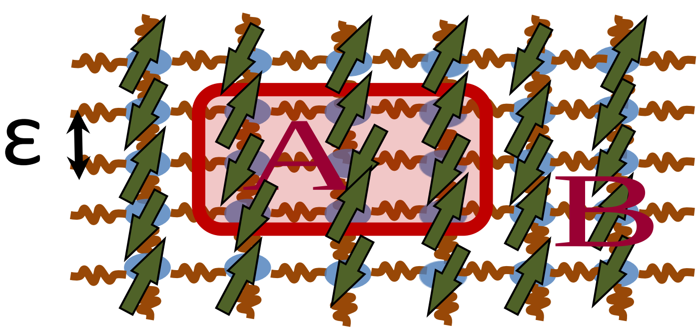
Define the reduced density matrix by $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

The entanglement entropy S_A is defined by

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \, .$$

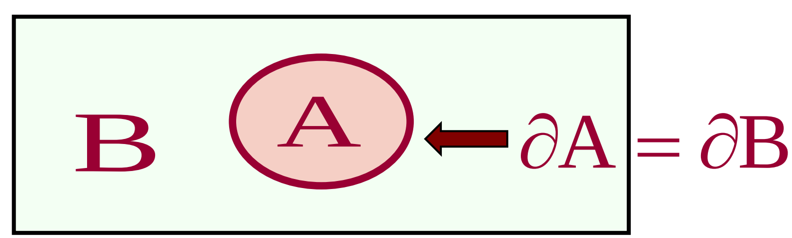
(von-Neumann entropy)

Quantum Many-body Systems

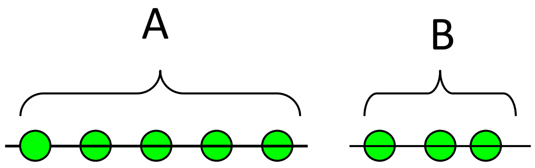


Continuum Limit ε→0

Quantum Field Theories (QFTs)

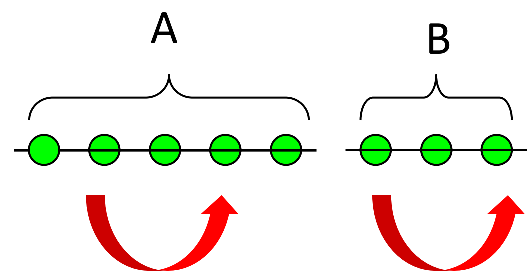


Entanglement Entropy (EE) in QI Text Book

Setup 

$\Rightarrow H_{tot} = H_A \otimes H_B$

LO (=Local Operations)



Projection measurements and unitary trfs.

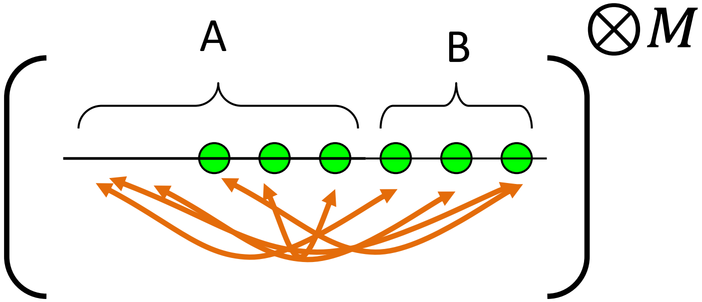
which act either A or B only.

CC (=Classical Communications between A and B)

⇒These operations are combined and called LOCC.

A basic example of LOCC: quantum teleportation

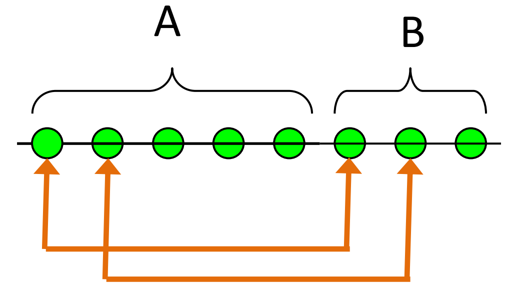
$$\left(|\Psi\rangle_{AB}\langle\Psi|\right)^{\otimes M}$$



Entangled in a very complicated way

LOCC

Distillation



N Bell pairs

$$\left(|\Psi\rangle_{AB}\langle\Psi|\right)^{\otimes M} \Rightarrow \left(|\mathrm{Bell}\rangle\langle\mathrm{Bell}|\right)^{\otimes N}$$

Well-known fact in QI:

$$\boxed{S(\rho_A) = \lim_{M\to\infty} \frac{N}{M}}$$

$$\rho_A \equiv \mathrm{Tr}_B\left[|\Psi\rangle_{AB}\langle\Psi|\right]$$

[Bennett-Bernstein-Popescu-Schumacher 95, Nielsen 98]

In this talk, we will introduce a generalization of entanglement entropy, called pseudo entropy.

Motivation 1

Generalization of entanglement entropy to post-selection processes
→ It depends on both the initial and final state.

Motivation 2

Generalization of holographic entanglement entropy to Euclidean time-dependent backgrounds → Ver. 3 HEE formula

Motivation 3

Holographic entanglement for dS/CFT ? → Need pseudo entropy

Dual CFTs are non-Hermitian !

Main References

[1] arXiv:2005.13801 [Phys.Rev.D 103 (2021) 026005]

with Yoshifumi Nakata (YITP, Kyoto), Yusuke Taki (YITP, Kyoto)

Kotaro Tamaoka (Nihon U.), Zixia Wei (Harvard U.) .

▶Original paper of pseudo entropy

[2] arXiv:2210.09457 [PRL130(2023)031601]

arXiv:2302.11695 [JHEP 05 (2023) 052]

with Kazuki Doi (YITP), Jonathan Harper (YITP),

Ali Mollabashi (IPM), Yusuke Taki (YITP).

▶pseudo entropy in dS/CFT

[3] arXiv: 2405.14237

with Kazuki Doi (YITP), Naoki Ogawa (YITP),

Kotaro Shinmyo (YITP), Yu-ki Suzuki (YITP).

▶Dual CFT states in dS/CFT

Contents

② Ver.3 of Holographic Entanglement Entropy ?

Ver. 1 Holographic EE for Static Spacetimes

[Ryu-TT 06]

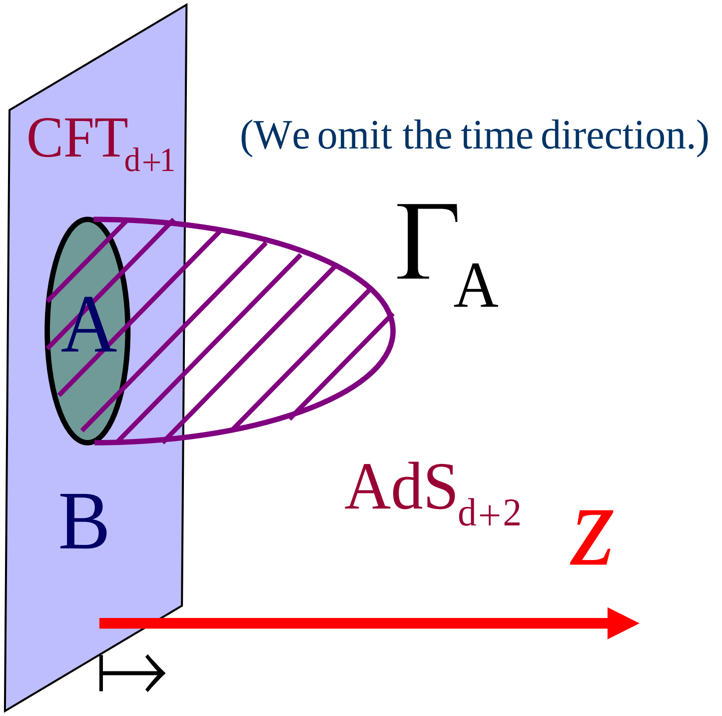
For static asymptotically AdS spacetimes:

$$S_A = \underset{\substack{\partial\Gamma_A=\partial A \\ \Gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N} \right]$$

Γ_A is the minimal area surface (codim.=2) on the time slice such that

$$\partial A = \partial\gamma_A \ \text{ and } \ A \sim \gamma_A \ .$$

homologous



$z > \varepsilon$ (UV cut off)

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

Ver. 2 Covariant Holographic Entanglement Entropy

[Hubeny-Rangamani-TT 07]

A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

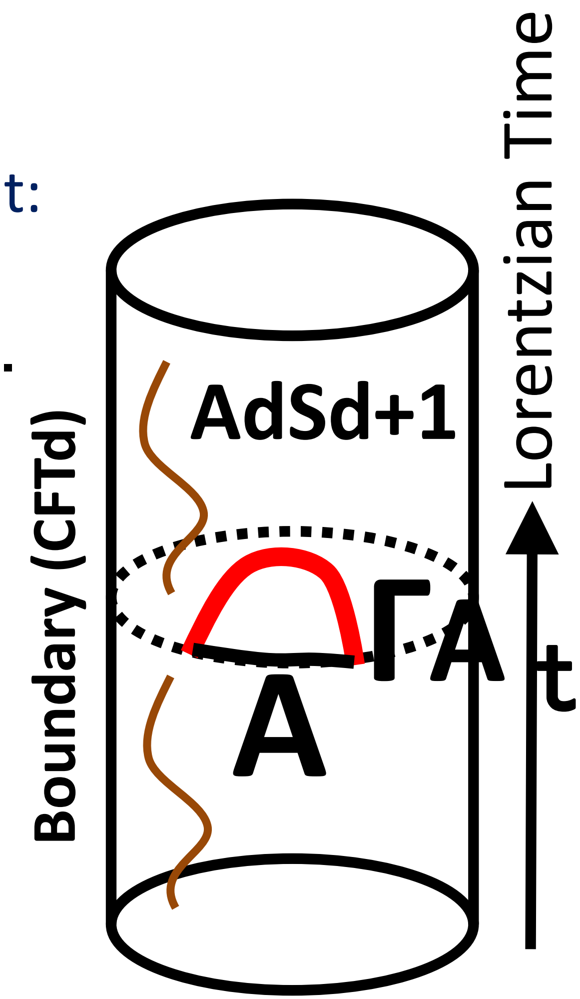
The entanglement entropy gets time-dependent:

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t).$$

This is computed by the holographic formula:

$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

$\partial A = \partial\gamma_A$ and $A \sim \gamma_A$.



Ver 3. Formula ?

Minimal areas in ***Euclidean time dependent*** asymptotically AdS spaces

= What kind of QI quantity (Entropy ?) in CFT ?

➡ **The answer is Pseudo Entropy !**

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

③ Pseudo Entropy

(3-1) Definition of Pseudo (Renyi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the *transition matrix*:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$.

and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right]$$

Pseudo Entropy

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

Renyi Pseudo Entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

(3-2) Basic Properties of Pseudo Entropy (PE)

- In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

When does PE become real ? ➡

- ◆Real valued Euclidean PI= Holographic PE
- ◆Pseudo Hermiticity [Guo-He-Zhan 2022]
- ◆Entanglement Phase Transition [Kanda-Kawamoto-Suzuki-Wei-TT 2023]

- If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state) , then
$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0.$$

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\varphi|\psi}\right)\right]^{\dagger}$.

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right)$. →"$S_A=S_B$"

- If $|\psi\rangle=|\varphi\rangle$, then $S^{(n)}\left(\tau_A^{\psi|\varphi}\right)$ = Renyi entropy.

Comment: In quantum theory, transition matrices arise when we consider *post-selection*.

$$\frac{\langle\varphi|O_A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[O_A\tau_A^{\psi|\varphi}]$$

Final state after post-selection

Initial State

This quantity is called **weak value** and is complex valued in general.

[Aharanov-Albert-Vaidman 1988,…]

Thus, **pseudo entropy =weak value of "modular operator"**:

= Area Operator

$$S\left(\tau_A^{\psi|\varphi}\right) = \frac{\langle\varphi|H_A|\psi\rangle}{\langle\varphi|\psi\rangle}.$$

H_A=-logτ_A

(3-3) Pseudo Entropy as Entanglement Distillation

Let us focus on the class E i.e. $\tau_A^{\psi|\varphi}$ and $\tau_B^{\psi|\varphi}$ are Hermitian and semi-positive definite.

Remarkably, in this case we can show a quantum information theoretical interpretation of pseudo entropy:

Claim **Pseudo Entropy** $S\left(\tau_A^{\psi|\varphi}\right)$

= # of Distillable Bell Pairs

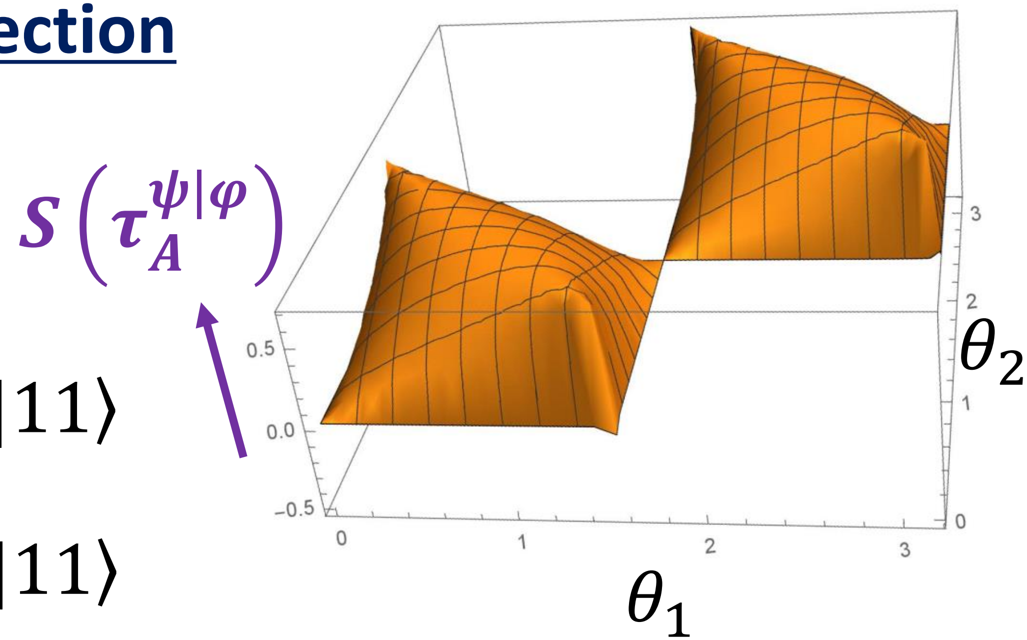
as an intermediate states

of post-selection $|\psi\rangle \rightarrow |\varphi\rangle$.

More precisely, we take asymptotic limit M→∞.

Distillation from Post-selection

In class E, we can write



$$|\psi\rangle = \cos\theta_1|00\rangle + \sin\theta_1|11\rangle$$

$$|\varphi\rangle = \cos\theta_2|00\rangle + \sin\theta_2|11\rangle$$

$$\Rightarrow \quad \tau_A^{\psi|\varphi} = \frac{\cos\theta_1\cos\theta_2|0\rangle\langle 0| + \sin\theta_1\sin\theta_2|1\rangle\langle 1|}{\cos(\theta_1-\theta_2)}$$

$$S\left(\tau_A^{\psi|\varphi}\right) = -\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)}\cdot\log\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)} - \frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}\cdot\log\frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}$$

$$(|\psi\rangle)^{\otimes M} = (\cos\theta_1|00\rangle + \sin\theta_1|11\rangle)^{\otimes M}$$
$$= \sum_{k=0}^{M}(c_1)^{M-k}\,(s_1)^k \sum_{a=1}^{{}_M C_k}|P(k),a\rangle_A|P(k),a\rangle_B$$

$$c_1 \equiv \cos\theta_1,\ s_1 \equiv \sin\theta_1$$

$k = 0$: $|P(0),1\rangle = |00\cdots0\rangle$

$k = 1$: $|P(1),1\rangle = |10\cdots0\rangle, |P(1),2\rangle = |01\cdots0\rangle, \ \cdots$

Projection to maximally entangled states with **Log[MCk]** entropy:

MCk=M!/(M-k)!k!

$$\Pi_k = \sum_{a=1}^{{}_M C_k}|P(k),a\rangle_A\langle P(k),a|$$

probability: $p_k = \langle\varphi|\Pi_k|\psi\rangle/\langle\varphi|\psi\rangle = \dfrac{(c_1c_2)^{M-k}\,(s_1s_2)^k}{(c_1c_2+s_1s_2)^M}\cdot {}_M C_k$

# of Distillable Bell pairs: $\mathbf{N} = \sum_{k=0}^{M} p_k \cdot \mathrm{Log}[{}_M C_k]$

$$\approx M\cdot S\left(\tau_A^{\psi|\varphi}\right)\ !$$

(3-4) SVD entropy [Parzygnat–Taki–Wei–TT 2023]

Motivation: Improve PE so that **(i)** it become real and non−negative and **(ii)** it has a better LOCC interpretation.

→ SVD entropy

$$S_{SVD}\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[|\tau_A^{\psi|\varphi}|\cdot \log|\tau_A^{\psi|\varphi}|\right].$$

here, $|\tau_A^{\psi|\varphi}| \equiv \sqrt{\tau_A^{\dagger\psi|\varphi}\tau_A^{\psi|\varphi}}$

- This is always non−negative and is bounded by log dim H_A.

- From quantum information theoretic viewpoint, this is the number of Bell pairs distilled from the intermediate state:

$\tau_A^{\psi|\varphi} = U \cdot \Lambda \cdot V$,

$$\frac{\langle\varphi|V^\dagger \sum_k |EPR_k\rangle\langle EPR_k| U^\dagger|\psi\rangle}{\langle\varphi|V^\dagger U^\dagger|\psi\rangle} = \sum_k p_k = 1$$

→

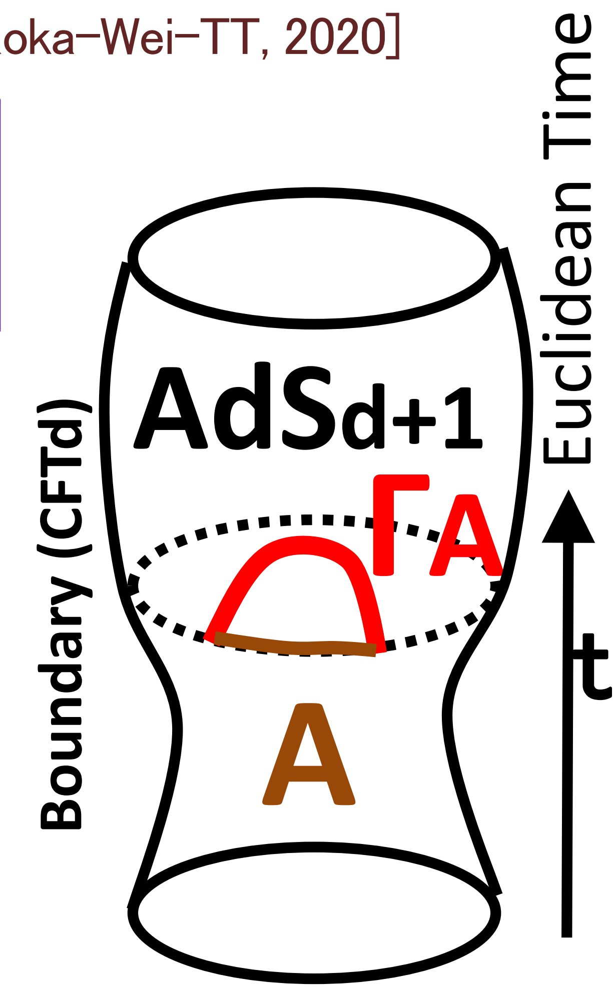
$$S_{SVD} \approx \sum_k p_k \cdot \#\text{of Bell Pairs in} |EPR_k\rangle$$

④ Holographic Pseudo Entropy

Holographic Pseudo Entropy (HPE) Formula

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$



Basic Propertie

(i) If ρ_A $is\ pure, S\left(\tau_A^{\psi|\varphi}\right)$ =0.

(ii) If ψ or φ $is\ not\ entangled,$

$S\left(\tau_A^{\psi|\varphi}\right)$ =0.

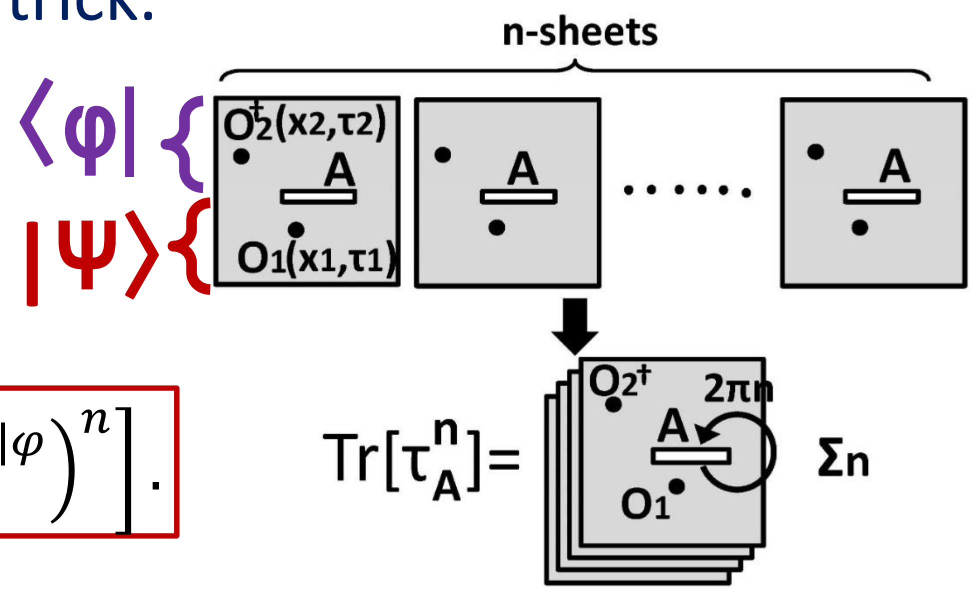
→This follows from AdS/BCFT [TT 2011]

(iii) $S\left(\tau_A^{\psi|\varphi}\right) = S\left(\tau_B^{\psi|\varphi}\right)$. **"SA=SB"**

- However, the strong subadditivity can be easily violated if we allow zigzag time slices like  .

⇒We may need to limit to straight time slices or just ignore SSA ?

- We can derive the holographic pseudo entropy formula as in [Lewkowycz-Maldacena 13] . This is because we can calculate the pseudo entropy via the standard replica trick.



$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right)=\frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

⑤ Pseudo Entropy and Quantum Phase Transitions

[Mollabashi-Shiba-Tamaoka-Wei-TT 20, 21]

(5-1) Basic Properties of Pseudo entropy in QFTs

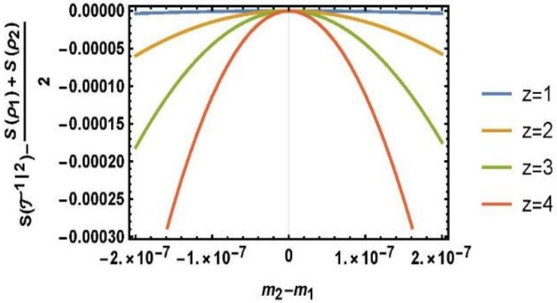
[1] Area law

$$S_A \sim \frac{\text{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2)$$

is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.



PE in a 2 dim. free scalar when we change its mass.

What happen if they belong to different phases ?
Can ΔS be positive ?

(5-2) Quantum Ising Chain with a transverse magnetic field

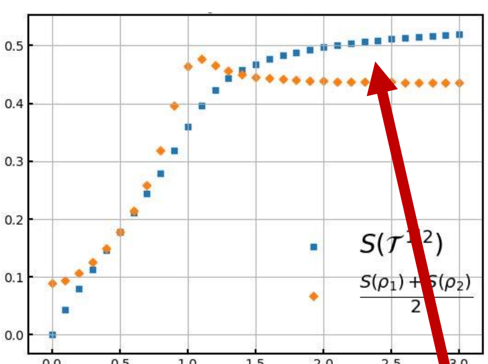
$$H = -J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=0}^{N-1} \sigma_i^x,$$

Ψ1→ vacuum of H(J1)
Ψ2→ vacuum of H(J2)
(We always set h=1)

J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

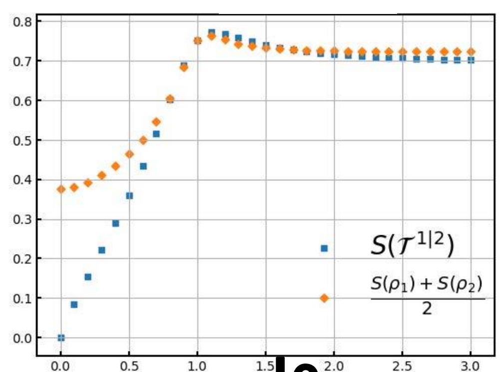
N=16, NA=8

J1=1/2



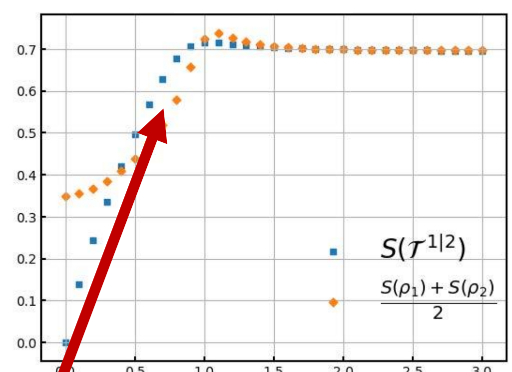
J2

J1=1



J2

J1=2

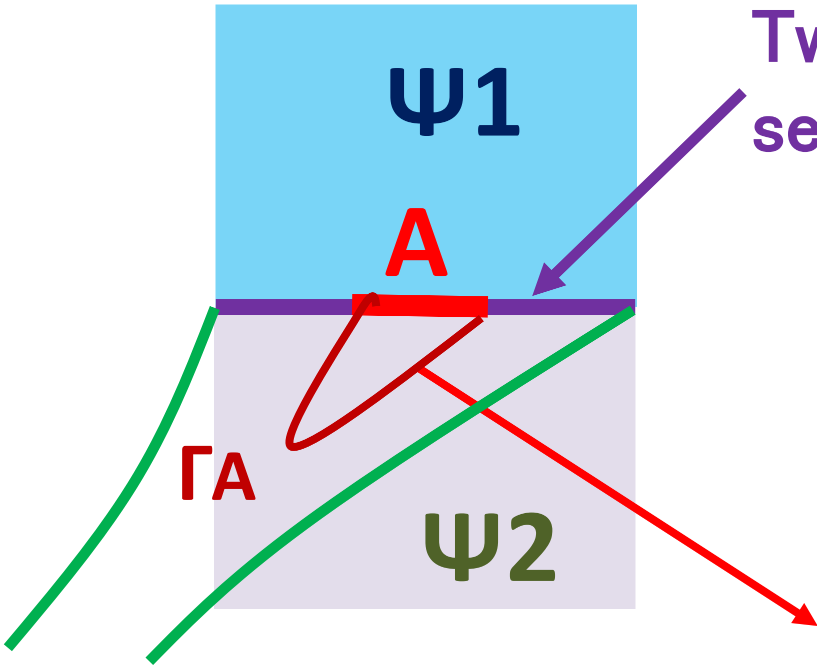


J2

We find $\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$

when Ψ1 and Ψ2 are in different phases !

Heuristic Interpretation



AdS Dual of
Gapless Interface

Two gapped phases are separated by a gapless phase.
CFT !

PE is enhanced ! $\Delta S > 0$

The gapless interface (edge state) also occurs in topological orders.

➔Topological pseudo entropy

[Nishioka-Taki-TT 2021, Caputa-Purkayastha-Saha-Sułkowski 2024]

⑥ dS Holography and Pseudo Entropy

Holographic entanglement entropy suggests that the extra dimension in AdS/CFT emerges from quantum entanglement.

However, the Universe, which we live, has been considered as de Sitter space $(\Lambda>0)$ rather than anti de Sitter space $(\Lambda<0)$.

Q. Does our universe emerges from quantum information ?

➡ **Consider holographic entanglement in dS gravity !**

Let us first remember what we know about dS holography.

A Sketch of dS/CFT [Strominger 2001, Witten 2001, Maldacena 2002,⋯.]

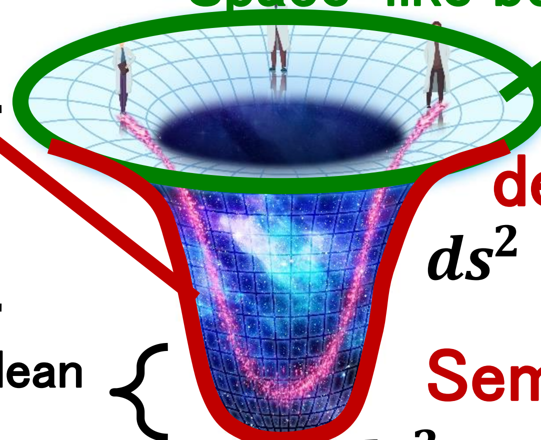
d+1 dim. Lorentzian de−Sitter spacetime

Dual

Euclidean d. dim CFT on S^d

Space−like bdy

Time



$t=\infty$

Lorentzian time

Euclidean time

de Sitter

$$ds^2 = L_{dS}^2(-dt^2 + \mathrm{Cosh}^2 t\, d\Omega^2)$$

Semi sphere

$$ds^2 = L_{dS}^2(d\theta^2 + \mathrm{Sin}^2\theta d\Omega^2)$$

$$\theta = it + \frac{\pi}{2}$$

Time emerges from Euclidean CFT !

Ψ [dS gravity]=Z [CFT]

What we expect for dS/CFT

➔Let us assume dS Einstein gravity and extract general expectations.

d+1 dim. (Lorentzian) de-Sitter $$ds^2 = L_{dS}^2(-dt^2 + \mathrm{Cosh}^2 t\, d\Omega^2)$$

⬇

S^{d+1} (Euclidean de-Sitter) $$ds^2 = L_{dS}^2(d\theta^2 + \mathrm{Sin}^2\theta d\Omega^2)$$

⬇ $L_{AdS} = iL_{dS},\ \rho = i\theta$

Euclidean AdS (H^{d+1}) $$ds^2 = L_{AdS}^2(d\rho^2 + \mathrm{Sinh}^2\rho d\Omega^2)$$

Central charge: $$c \sim \frac{L_{AdS}^{d-1}}{G_N} = i^{d-1} \cdot \frac{L_{dS}^{d-1}}{G_N}$$

We are interested in d=2 case in this talk !

➡

(i) Central charge becomes imaginary for d=even !

(ii) Central charge gets larger in classical gravity limit.

CFT dual of dS in Einstein gravity

[Hikida–Nishioka–Taki–TT, 2021]

Large c limit of SU(2)k WZW model (a 2dim. CFT)

= **Einstein Gravity on 3 dim. de Sitter** (radius L_{dS})

Level

$$k \approx -2 + \frac{4iG_N}{L_{dS}} \longrightarrow \Delta \approx iL_{dS} \cdot E_{dS}$$

Conformal dim. **Energy in dS**

Central charge

$$c = \frac{3k}{k+2} \approx i\frac{3L_{dS}}{2G_N}$$

$$Z[S^3, R_j] = |S_j^0|^2 \approx e^{\frac{\pi L_{dS}}{2G_N}\sqrt{1-8G_N E}}$$

CFT partition function — De Sitter Entropy

[This k=−2 is equivalent to k=∞ via triality in Gaberdiel–Gopakumar 2012]

This non-unitary CFT is essentially equivalent to the two Liouville CFTs at $b^{-2} \approx \pm \frac{i}{4G_N}$. [Hikida–Nishioka–Taki–TT 2022]

[→Reproduced by Verlinde–Zhang 2024 via the Double Scaled SYK]

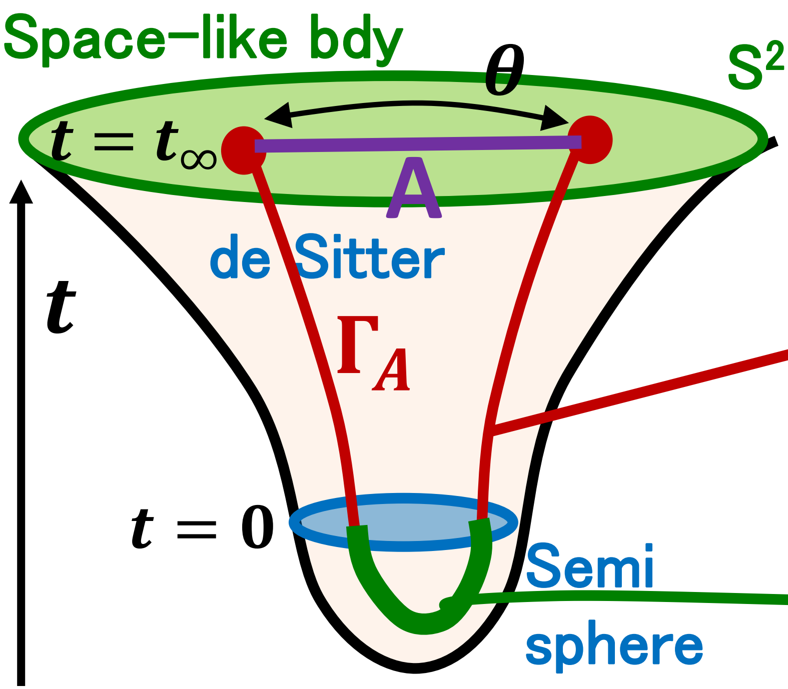
Holographic Pseudo Entropy in dS3/CFT2

[No space-like extreme surface ending on bdy →complex valued EE: Narayan, Sato 2015, Interpretation as PE: Doi-Harper-Mollabashi-Taki-TT 2022]

If we naively apply the HEE in AdS/CFT to dS/CFT, we obtain

$$S_A = \frac{\mathrm{L}(\Gamma_A)}{4G_N} = i\frac{c_{ds}}{3}\log\left(\frac{2}{\epsilon}\mathrm{Sin}\frac{\theta}{2}\right) + \underbrace{\frac{c_{dS}}{6}\pi}_{S_{dS}/2}.$$

$$ds^2 = L_{ds}^2(-dt^2 + \mathrm{Cosh}^2 t\,(d\theta^2 + Sin^2\theta d\varphi^2))$$



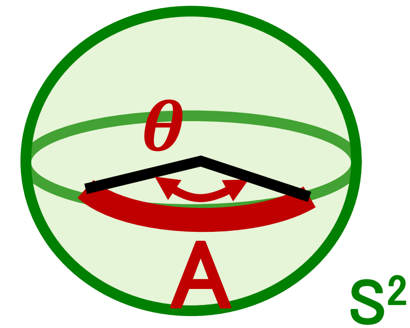
$$\mathrm{L}(\Gamma_A) = \underbrace{2it_\infty + i\log(\mathrm{Sin}^2\tfrac{\theta}{2})}\ + \pi$$

Length of time-like geodesics
➔imaginary value

Space-like geodesic (Semi circle)

This nicely reproduces the familiar 2d CFT result as follows:

$$S_A = \frac{c_{CFT}}{6}\log\left[\frac{\mathrm{Sin}^2\frac{\theta}{2}}{\tilde{\varepsilon}^2}\right],$$ by setting



$c_{CFT} = ic_{dS}$ and $\tilde{\varepsilon} = i\varepsilon = ie^{-t_\infty}$.

However, one may wonder why the EE is complex valued.

We argue it is more properly considered as the pseudo entropy.

[Doi-Harper-Mollabashi-Taki-TT 2022]

This is because the reduced density matrix ρ_A **is not Hermitian** in the CFT dual to dS, as it is not unitary.

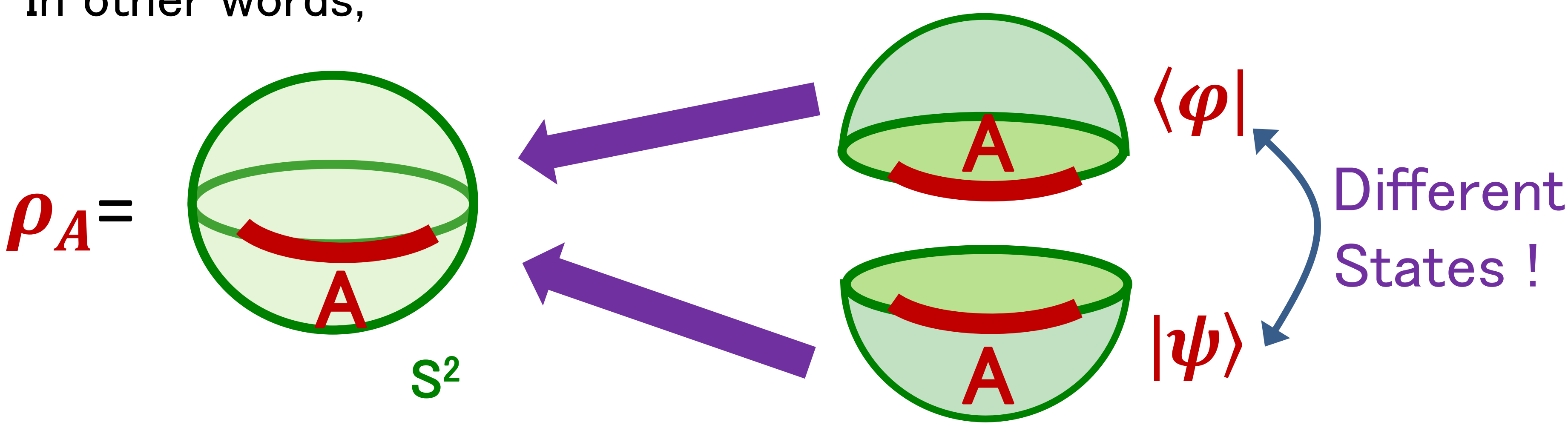
➔For the dual 2d CFT on Σ with metric $h_{ab} = e^{2\phi}\delta_{ab}$, we have

[See e.g. Boruch−Caputa−Ge−TT 2021]

$$Z_{CFT}(S^2) \approx e^{-I_{CFT}[\phi]}, \quad I_{CFT}[\phi] = i\frac{c_{ds}}{24\pi}\int d^2x[(\partial_a\phi)^2 + e^{2\phi}].$$

Complex valued ! ➔ $\rho_A \neq \rho_A^\dagger$

In other words,



Thus, the emergent time coordinate = imaginary part of PE.

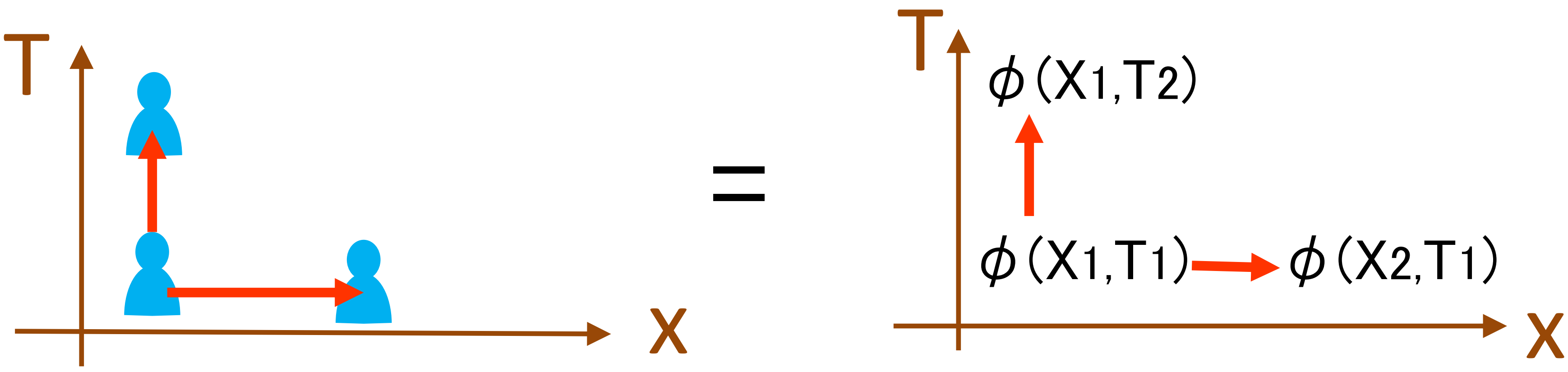
⑦ Probing dS from CFT [Doi–Ogawa–Shinmyo–Suzuki–TT 2024]

Consider an observer in 2d CFT.

➡How does the observer feel that he or she lives in AdS or dS ?

To probe the spacetime, we introduce an local excitation.

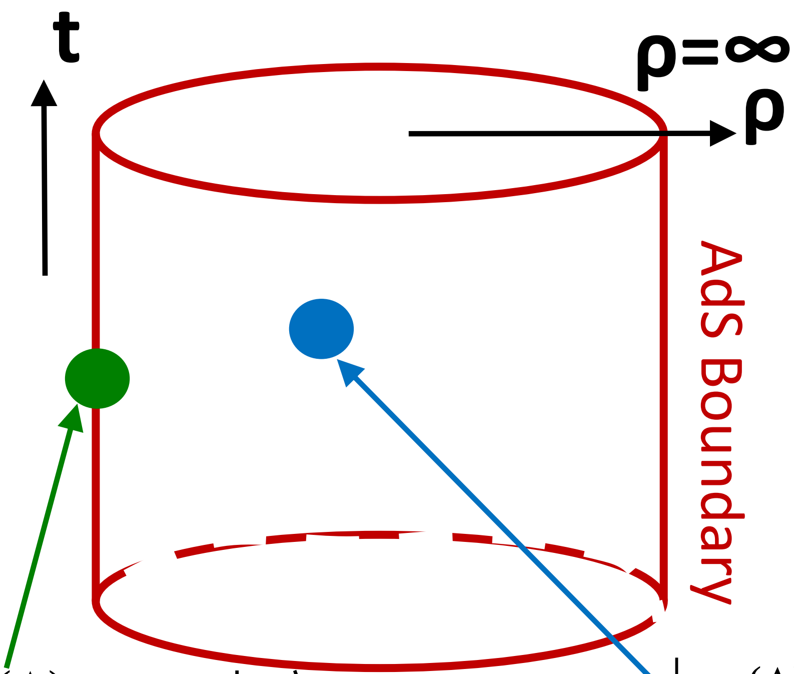
"How many directions can the observer move ?"



In flat spacetime, $|\Psi_{\mathrm{Flat}}(t,x)\rangle = e^{-iHt}e^{iPx}\hat{\phi}(0)|0\rangle$

Probing AdS from CFT

AdS metric $ds^2 = R_{AdS}^2(-\mathrm{Cosh}^2\rho dt^2 + d\rho^2 + \mathrm{Sinh}^2\rho d\varphi^2)$



$O^{(\Delta)}(t,\varphi)|0\rangle$

Primary operator in CFT

$\Delta_{\pm} = 1 \pm \sqrt{1 + M^2 R_{AdS}^2}$

$|\Psi_{AdS}^{(\Delta)}(\rho,t,\varphi)\rangle$

CFT dual of
Bulk local excitation

Geometric Symmetry of AdS3
= SO(2,2)=$SL(2,R)_L \times SL(2,R)_R$

$L_0, L_{-1}, L_1 \quad \tilde{L}_0, \tilde{L}_{-1}, \tilde{L}_1$

$[L_n, L_m] = (n-m)L_{n+m}$

Among 6 generators,
3 linear combinations
act trivially on $|\Psi_{AdS}^{(\Delta)}(\rho,t,\varphi)\rangle$

We can find quantum states in the CFT which describe localized excitations in AdS (cf. HKLL):

[Miyaji-Numasawa-Shiba-Watanabe-TT 2015]

$$\left|\Psi^{(\Delta)}_{AdS}(\rho,t,\varphi)\right\rangle = e^{-i(L_0+\tilde{L}_0)t} e^{i\varphi(L_0-\tilde{L}_0)} e^{-\frac{\rho}{2}\left(L_{-1}+\tilde{L}_{-1}-L_1-\tilde{L}_1\right)} e^{\frac{\pi i}{2}(L_0+\tilde{L}_0)} \left|I^{(\alpha)}\right\rangle.$$

CFT state dual to a localized excitation in the bulk AdS

SL(2,R) Crosscap State

SL(2,R) Ishibashi State

$$(L_n - \tilde{L}_{-n})\left|I^{(\alpha)}\right\rangle = 0, \qquad n = 0, \pm 1.$$

$$\left|I^{(\alpha)}\right\rangle = \sum_{k=0}^{\infty} c_k \left(L_{-1}\tilde{L}_{-1}\right)^k \left|\Delta\right\rangle$$

Note: $\{L_n\}$ ($|n|>1$) are not relevant as they change the CFT vacuum.

Two point function

We can show:

Geodesic distance in AdS3

$$\left\langle \Psi^{(\Delta)}_{AdS}(X) \middle| \Psi^{(\Delta)}_{AdS}(X') \right\rangle = G_{AdS}(X, X') = \frac{e^{-(\Delta-1)\cdot D_{AdS}(X,X')}}{2\sinh D_{AdS}(X, X')}$$

$$X = (\rho, t, \varphi)$$

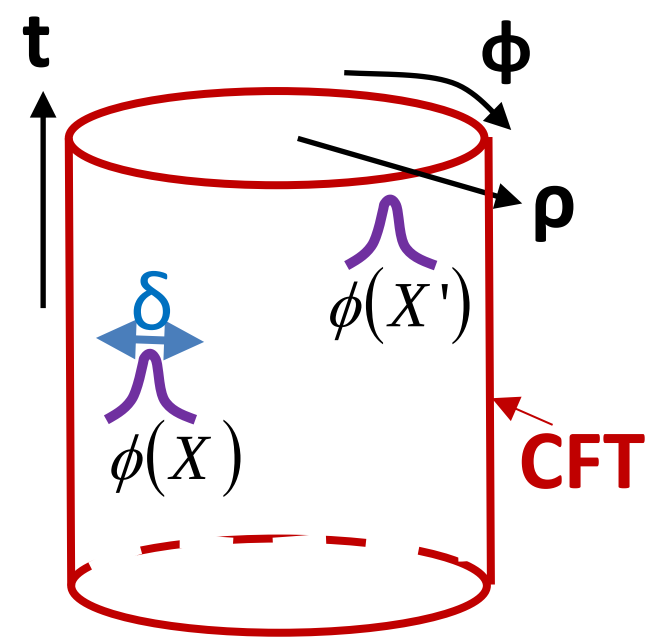
Green function of scalar field in AdS3 with M, s.t. $M^2 = \Delta(\Delta - 2)$

Information Metric (Bures Metric)

$$ds^2_{\inf} = G_{ab} dx^a dx^b = 1 - \left| \left\langle \Psi(x) \middle| \Psi(x+dx) \right\rangle \right|$$

If we smear the localized excitation up to the scale δ, we find

$$\left| \left\langle \Psi_{AdS}(x) \middle| \Psi_{AdS}(x') \right\rangle \right| \approx \frac{\delta}{\sqrt{D_{AdS}(x, x') + \delta^2}}.$$



$$\Rightarrow \quad ds^2_{\inf} = \frac{1}{\delta^2}\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2\right)$$

Probing dS from CFT

[Doi-Ogawa-Shinmyo-Suzuki-TT 2024]

First we note the "formal" relation between AdS and dS:

AdS metric

$$ds^2 = R_{AdS}^2(-\mathrm{Cosh}^2\rho dt^2 + d\rho^2 + \mathrm{Sinh}^2\rho d\varphi^2)$$

$$R_{AdS}^2 = -R_{dS}^2$$

$$t = i\tau$$

$$\rho = i\theta$$

$$(L_0)^\dagger = L_0, \quad (L_{\pm1})^\dagger = L_{\mp1}$$

dS metric

$$ds^2 = R_{dS}^2(-\mathrm{Cos}^2\theta d\tau^2 + d\theta^2 + \mathrm{Sin}^2\theta d\varphi^2)$$

$$(L_0)^\dagger = -\tilde{L}_0, \quad (L_{\pm1})^\dagger = \tilde{L}_{\pm1}$$

$$H^\dagger = -H$$

Hamiltonian is anti-Hermitian !

→Non-unitary Euclidean CFT! → Emergent Lorentzian time

Relevance of pseudo entropy

To probe the dS geometry from the dual CFT, we would like to find quantum states in the CFT which describe localized excitations in dS:

$$\left|\tilde{\Psi}_{dS}^{(\Delta_+)}(\tau,\theta,\varphi)\right\rangle = e^{(L_0+\tilde{L}_0)t} e^{i\varphi(L_0-\tilde{L}_0)} e^{i\frac{\theta}{2}\left(L_1+\tilde{L}_1-L_{-1}-\tilde{L}_{-1}\right)} e^{\frac{\pi i}{2}(L_0+\tilde{L}_0)} \left|I^{(\alpha)}\right\rangle.$$

Non-unitary evolution
→emergent time

SL(2,R) Crosscap State

Primary operator in CFT

$$O^{(\Delta_+)}(t,\varphi)$$

SL(2,R) Ishibashi State

$$(L_n - \tilde{L}_{-n})\left|I^{(\alpha)}\right\rangle = 0$$

$$\Delta_\pm = 1 \pm i\sqrt{M^2 R_{dS}^2 - 1}$$

However, the above naïve analytically continued result from AdS leads to the confusing result (due to the *unusual conjugation*):

$$\left\langle \tilde{\Psi}_{dS}^{(\Delta_+)}(\tau,\theta,\varphi) \middle| \tilde{\Psi}_{dS}^{(\Delta_+)}(\tau',\theta',\varphi') \right\rangle = 0.$$

The correct answer is found by requiring the CPT invariance state:

$$\left|\Psi_E^{(\Delta_+)}(\tau,\theta,\varphi)\right\rangle = \frac{1}{\sqrt{i}}\left|\tilde{\Psi}_{dS}^{(\Delta_+)}(\tau,\theta,\varphi)\right\rangle + CPT\cdot\left(\frac{1}{\sqrt{i}}\left|\tilde{\Psi}_{dS}^{(\Delta_+)}(\tau,\theta,\varphi)\right\rangle\right)$$

$$= \frac{1}{\sqrt{i}}\left|\tilde{\Psi}_{dS}^{(\Delta_+)}(\tau,\theta,\varphi)\right\rangle + \sqrt{i}\left|\tilde{\Psi}_{dS}^{(\Delta_-)}\underline{(\tau,\pi-\theta,\varphi+\pi)}\right\rangle.$$

Antipodal map

[cf. gauging CPT in QG: Harlow–Numasawa 2023]

Indeed, this reproduces the correct dS Green function at Euclidean vacuum

$$\left\langle\Psi_E^{(\Delta_+)}(x)\middle|\Psi_E^{(\Delta_+)}(x')\right\rangle = G_{dS}^E(x,x') = \frac{\sinh\mu(\pi - D_{dS}(x,x'))}{4\pi\sinh\pi\mu\cdot\sin D_{dS}(x,x')}.$$

After a regularization, we find that the information metric leads to dS metric:

$$ds_{\rm inf}^2 = \frac{1}{\delta^2}\left(-\cos^2\theta d\tau^2 + d\theta^2 + \sin^2\theta d\varphi^2\right).$$

⑧ Conclusions

Pseudo entropy (PE) is a generalization of entanglement entropy.

- PE depends on both the initial and final state.
- PE is in general complex valued.
- PE for `non-exotic states' measures the amount of quantum entanglement in the intermediate states.
- ΔS for two states in different phases can be positive, while ΔS in the same phase is always non-positive.
  ➡ New quantum order parameter
- In AdS/CFT, PE is equal to the minimal surface area in Euclidean time-dependent asymptotically AdS geometry.
  ➡ Emergence of space from real part of PE
- In dS/CFT, PE becomes complex valued.
  ➡ Emergence of time from imaginary part of PE
  (Non-Hermitian nature of the dual CFT)

Future directions

- Quantum information meaning of the complex values of PE ?
- Applications to cond-mat physics / statistical mechanics ?
- Implications to quantum gravity ?
- Holographic dual of SVD entropy ?
- Constraints on QFTs using PE ?

:

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Thank you very much !

Appendix A: More Results in Qubit systems

An Example of Exotic Transition Matrix

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\tau_A^{\psi|\varphi} = \frac{1}{1+e^{i\theta}}\left(|0\rangle\langle 0| + e^{i\theta}|1\rangle\langle 1|\right).$$

→ Complex conjugate pair of Eigenvalues

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\left[\frac{\cos\frac{n\theta}{2}}{2^{n-1}\cos^n\frac{\theta}{2}}\right]$$

→Only special values of θ can give positive values pseudo entropy.

Monotonicity in 2 Qubit systems

We can prove the following monotonicity under unitary transformation:

Claim Consider two states related by local unitary trf.

$$|\psi\rangle = (U_A \otimes V_B)|\varphi\rangle.$$

If $\tau_A^{\psi|\varphi}$ has non-negative eigenvalues (i.e. class B=C), then

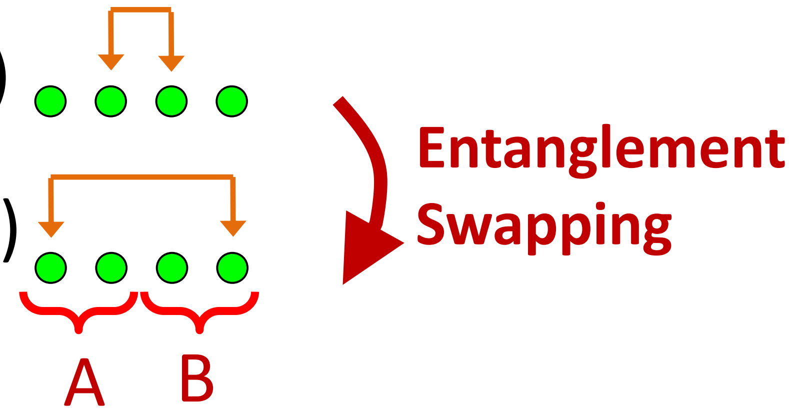
$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) \geq S^{(n)}(\mathrm{Tr}_B[|\psi\rangle\langle\psi|]) = S^{(n)}(\mathrm{Tr}_B[|\varphi\rangle\langle\varphi|]).$$

Note: However, this claim is limited to 2 qubit systems.

Decreasing Pseudo Entropy Examples

Ex1 $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0110\rangle)$

$|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1001\rangle)$



$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0$$

$$S^{(n)}(\mathrm{Tr}_B[|\psi\rangle\langle\psi|]) = S^{(n)}(\mathrm{Tr}_B[|\varphi\rangle\langle\varphi|]) = \log 2.$$

Ex2 **Thermofield States in CFTs**

$$|\psi_i\rangle = \frac{1}{\sqrt{Z(\beta_1)}}\sum_n e^{-\frac{\beta_i}{2}E_n}|n\rangle|n\rangle \quad (i = 1,2)$$

$$S_{th}\left(\frac{\beta_1+\beta_2}{2}\right) \leq \frac{1}{2}[S_{th}(\beta_1) + S_{th}(\beta_2)]$$

$$\Rightarrow S\left(\tau_A^{\psi_1|\psi_2}\right) \leq \frac{1}{2}[S(\mathrm{Tr}_B[|\psi_1\rangle\langle\psi_1|]) + S(\mathrm{Tr}_B[|\psi_2\rangle\langle\psi_2|])].$$

$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2)$ in Two Qubit System

$$|\psi_1\rangle_{AB} = \cos\theta_1|00\rangle_{AB} + \sin\theta_1|11\rangle_{AB}, \quad |\psi_2\rangle_{AB} = \cos\theta_2|00\rangle_{AB} + \sin\theta_2|11\rangle_{AB},$$

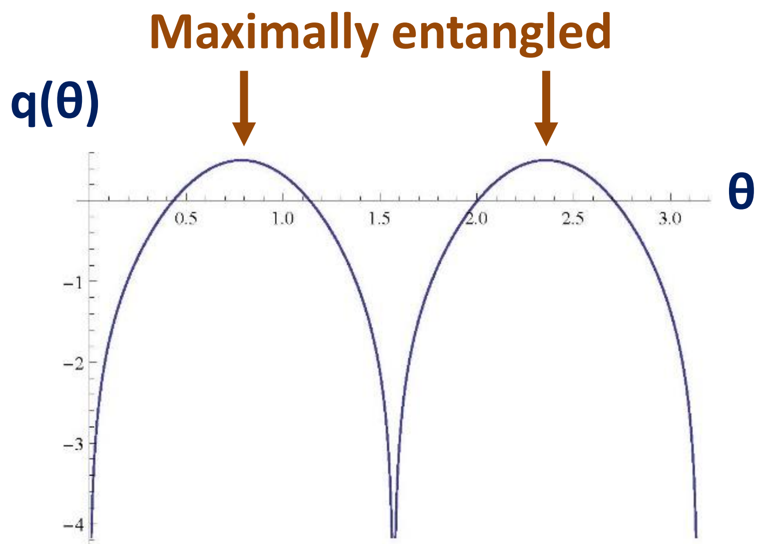
where we assume $0 \leq \theta_1, \theta_2 \leq \frac{\pi}{2}$. The pseudo entropy is computed as [16]

$$S(\tau_A^{1|2}) = -\left(\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)}\right)\cdot\log\left(\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)}\right) - \left(\frac{\sin\theta_1\sin\theta_2}{\cos(\theta_1-\theta_2)}\right)\cdot\log\left(\frac{\sin\theta_1\sin\theta_2}{\cos(\theta_1-\theta_2)}\right).$$

We are interested in a small perturbation $\theta_2 - \theta_1 = \delta \ll 1$. Then the interesting difference looks like

$$2S(\tau_A^{1|2}) - S(\rho_A^1) - S(\rho_A^2) \simeq q(\theta_1)\delta^2 + O(\delta^3),$$

where we find



Even when the two states are closed to each other, the difference is not always negative !

Appendix B: Behavior of SVD Entropy

- This entropy also shows an enhancement similar to PE for two difference states in different quantum orders.

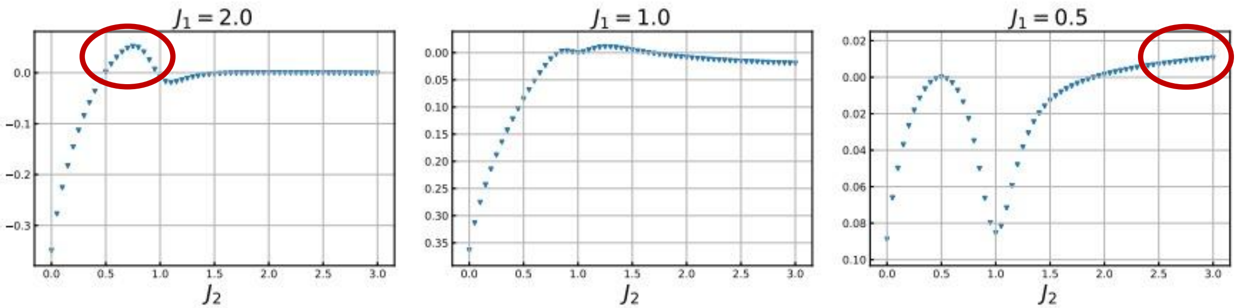
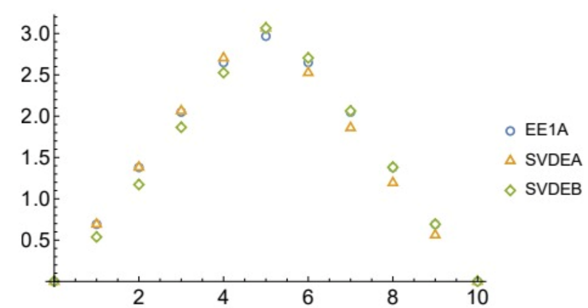


Figure 23: Plots of $S(\rho_A^{1|2}) - \left(S(\rho_A^1) + S(\rho_A^2)\right)/2$ in different cases. $S(\rho_A^{1|2}) - \left(S(\rho_A^1) + S(\rho_A^2)\right)/2 \leq 0$ when the two states are in the same quantum phase, and can be violated when the two states are in different quantum phases.

- This SVD entropy also shows the Page curve like behavior.



- However, we have $S_A \neq S_B$, as opposed to pseudo entropy !

(This suggest the gravity dual will be very complicated･･･.)

Appendix C: Free Scalar Computations

Our Free Scalar Model

When m=0, we have a Lifshitz scaling sym. $(x,t) \to (\lambda x, \lambda^z t)$.

Lifshitz scalar in 2dim.

z= dynamical exponent

$$H = \frac{1}{2} \int dx \left[\pi^2 + (\partial_x^z \phi)^2 + m^{2z} \phi^2 \right],$$

m= mass

where ϕ and π are the scalar field and its momentum.

In order to do concrete calculations, we consider its lattice regularization:

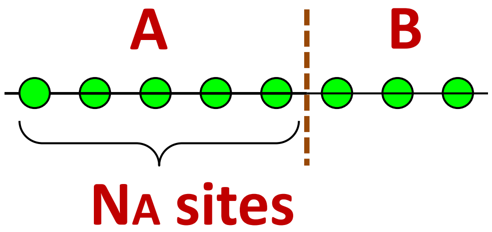
$$H = \sum_{i=1}^{N} \left[\frac{\pi_i^2}{2} + \frac{m^{2z}}{2} \phi_n^2 + \frac{1}{2} \left(\sum_{k=0}^{z} (-1)^{z+k} \binom{z}{k} \phi_{i-1+k} \right)^2 \right].$$

We set

|Ψ1> = the vacuum of H(m1,z1) and |Ψ2> = the vacuum of H(m2,z2).

Calculating Pseudo Entropy

We can calculate PE from correlation functions of ϕ, π since the model is Gaussian.



$$\Gamma = \begin{pmatrix} X & R \\ R^T & P \end{pmatrix},$$

$$X_{ij} = \mathrm{Tr}[\phi_i \phi_j \tau_A^{1|2}], \quad P_{ij} = \mathrm{Tr}[\pi_i \pi_j \tau_A^{1|2}],$$

$$R_{ij} = \frac{1}{2}\mathrm{Tr}[(\phi_i \pi_j + \pi_i \phi_j)\tau_A^{1|2}].$$

R takes complex values, though X and P are real symmetric matrices. Therefore, we consider a complexified symplectic transformation $Sp(2N_A, \mathbb{C})$ to diagonalize Γ into the form (see appendix A for more details)

$$\Gamma \to \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}, \qquad (6)$$

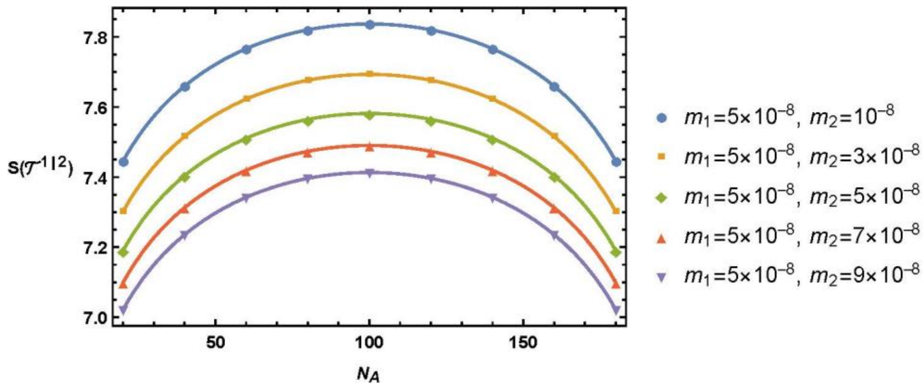
$$S(\tau_A^{1|2}) = \sum_{i=1}^{N_A} f(\nu_i)$$

$$f(x) \equiv \left(x + \tfrac{1}{2}\right)\log\left(x + \tfrac{1}{2}\right) - \left(x - \tfrac{1}{2}\right)\log\left(x - \tfrac{1}{2}\right)$$

Numerical Results

Relativistic Case: z1=z2=1

$$S\left(\tau_A^{1|2}\right) = \frac{1}{3}\log\left(\frac{L}{\pi\varepsilon}\mathrm{Sin}\left(\frac{\pi l_A}{L}\right)\right)$$
$$+\mathrm{g}(m_1, m_2, l_{A,}, L).$$



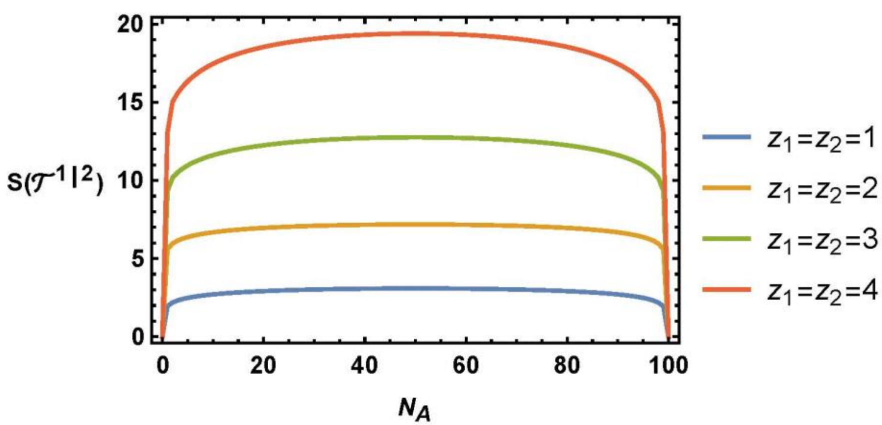
⇒ We have Area Law as in EE.

If $m_{1,2}L \sim 1$ and $m_{1,2}l \ll 1$, we can find the l dependence

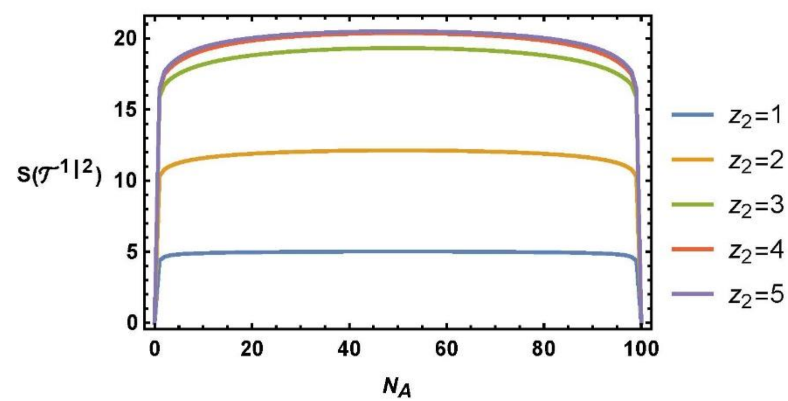
$$f(m_1, m_2, L, l) \simeq \frac{1}{2}\log\left[-\frac{m_1^2\log[m_1 l] - m_2^2\log[m_2 l]}{m_1^2 - m_2^2}\right] + f_0(m_1, m_2, L),$$

Generic Case [Note: larger z → larger EE]

$m_1 = 10^{-3}, m_2 = 10^{-5}$

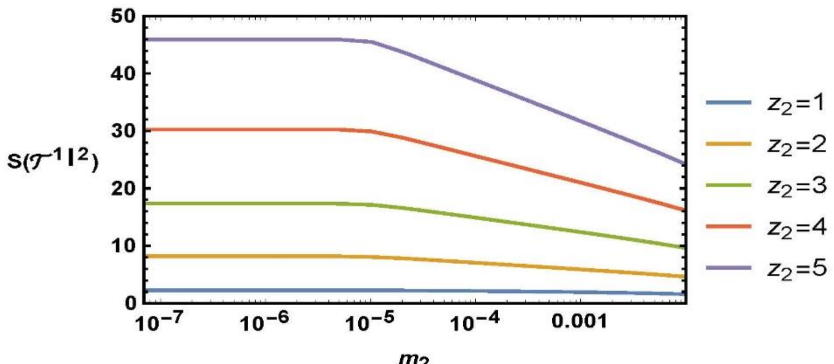


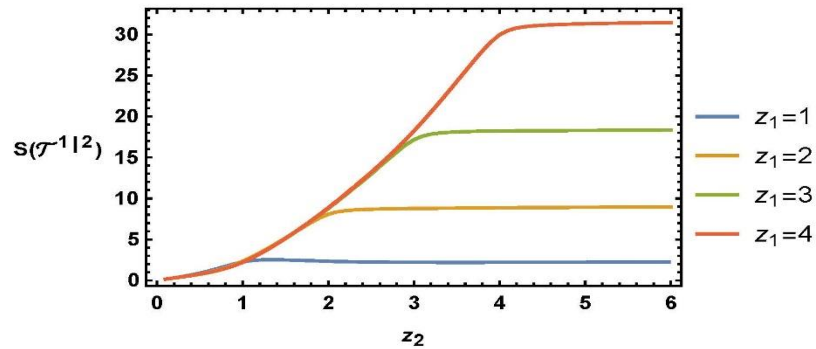
$m_1 = m_2 = 10^{-5}, z_1 = 3$



Saturation Behavior

$m_1 = 10^{-5}, \; z_1 = 1$



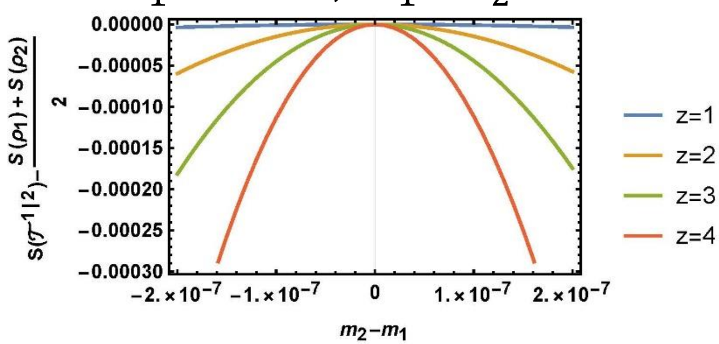


$$S\left(\tau_A^{1|2}\right) \lesssim \mathrm{Min}[S(\rho_A^1), S(\rho_A^2)]$$

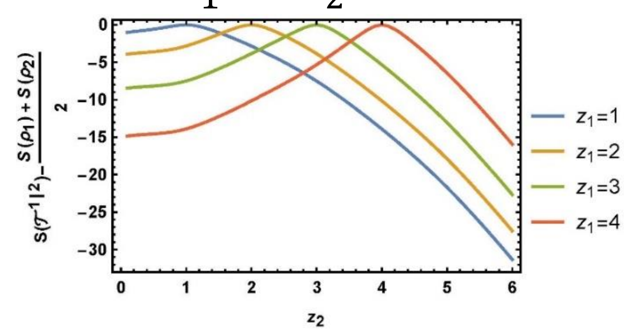
Saturation behavior

Negativity of the Difference

$m_1 = 10^{-5}, \; z_1 = z_2 = z$



$m_1 = m_2 = 10^{-5}$



$$2S\left(\tau_A^{1|2}\right) - \left(S(\rho_A^1) + S(\rho_A^2)\right) \leq 0$$

Is this inequality always true ?