

Around Completions

24th May

①

① R comm. noeth. ring

$D(R)$ Derived category

$V \subseteq \text{Spec } R$ specialization closed

$$: \mathfrak{p} \subseteq \mathfrak{q} \text{ \& } \mathfrak{p} \in V \Rightarrow \mathfrak{q} \in V$$

$$\equiv \mathfrak{p} \in V \Rightarrow \overline{\{\mathfrak{p}\}} \subseteq V$$

↑ closure in Zariski topology.

$M \in \text{Mod } R$

$$T_V M := \text{Ker} (M \rightarrow \prod_{\mathfrak{p} \notin V} M_{\mathfrak{p}})$$

V -torsion submodule of M

$M \in D(R)$

$$R\Gamma_V M := T_V (i^* M) \subseteq i^* M \xleftarrow{\sim} M$$

Local cohomology
supported on V

↑ injective res. of M

(2)

$$\Gamma_V D(R) := \{ M \in D(R) \mid R\Gamma_V M \simeq M \}$$

\uparrow
 V-torsion part of $D(R)$

- A localizing subcategory of $D(R)$.

Because $M \in \Gamma_V D(R) \Leftrightarrow$

$$H_*^p(M) = 0 \quad \forall p \notin V$$

Compact objects: Write

$$V = \bigcup_i V(\mathcal{P}_i)$$

For example, take \mathcal{P}_i minimal in V

$K(\mathcal{P}_i) :=$ Koszul cx. on some finite generating set for \mathcal{P}_i

Then $K(\mathcal{P}_i) \in \Gamma_V D(R)$

$$\begin{aligned} \textcircled{1} \quad T_{\vee}^c D(R) &= D(R)^c \cap T_{\vee}^c D(R) \\ &= \text{Thick}(K(\mathbb{Z})/i) \end{aligned}$$

Moreover

$$\textcircled{2} \quad T_{\vee}^c D(R) = \text{Loc}(K(\mathbb{Z})/i)$$

This has many consequences:

Complete $\text{RP}_{\vee}^c M \rightarrow M$ to a Δ^{le} :

$$\text{RP}_{\vee}^c M \rightarrow M \rightarrow L_{\vee} M \rightarrow$$

} "localising" away from \vee
"nullification"

- Can also interpret $M \rightarrow L_{\vee} M$ as homology localization.

(4)

Example: Fix $p \in \text{Spec } R$

$$Z(p) := \text{Spec } R \setminus \text{Spec } R_p$$

- specialization closed

$$\begin{array}{ccccccc} \text{Then} & \mathcal{R}T_{Z(p)} M & \longrightarrow & M & \longrightarrow & M_p & \longrightarrow \\ & & & & & \parallel & \\ & & & & & \mathcal{L}_{Z(p)} M & \end{array}$$

Exercise: Verify this.

(3) From (2) we get:

$$\mathcal{R}T_V M \simeq \mathcal{R}T_V R \oplus_R^{\mathcal{L}} M$$

$$\mathcal{L}_V M \simeq \mathcal{L}_V R \oplus_R^{\mathcal{L}} M$$

Another consequence:

$$(4) \quad \oplus \mathcal{R}T_V(\pi_i) \longrightarrow \mathcal{R}T_V(\oplus M_i)$$

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for any collection $\{M_i\}$

Consider adjoint pair:

$$\begin{array}{ccc} \prod_v D(R) & \xrightleftharpoons{\quad} & D(R) \\ & \searrow_{R\Gamma_v} & \\ & \xrightarrow{\rho} & \end{array}$$

④ implies $R\Gamma_v$ has a right adjoint:

$$L\Lambda^v M := \rho R\Gamma_v M$$

↑
Completion along v

$$M \rightarrow L\Lambda^v M$$

Example: $v = V(I)$ $I \subseteq R$
ideal

$M \in \text{Mod } R$

$$\Lambda^I M := \varprojlim (M/I^n M)$$

For $M \in D(R)$ set

$$L\Lambda^I M := \Lambda^I(pM)$$

Derived Completion
Greenlees-May

projective
resln. of M

One has $M \rightarrow L\Lambda^V M$

- Can also be interpreted as a homology localization.

For $M \in D^b(\text{mod } R)$

$$\begin{aligned}
L\Lambda^I M &\simeq L\Lambda^I R \otimes_R^L M \\
&\simeq \Lambda^I R \otimes_R M
\end{aligned}$$

- This is false in general

Example: (R, \mathfrak{m}, k) local ring

⑦

$L\hat{A}^m E_P(R) \cong$ a dualizing c.x. for \hat{R} , the m -adic completion of R .

Exercise: check this.

Some tools to help understand completion:

⑤ Greenlees-May duality:

$$R\text{Hom}_R(R\Gamma_V M, N) \simeq R\text{Hom}_R(M, L\hat{A}^V N)$$

Example: Say R domain complete
w.v.t. some ideal $I \subseteq R$
 \mathfrak{h}_0

Then $R\text{Hom}_R(Q, R) = 0$
 $R \uparrow$

fraction field of R .

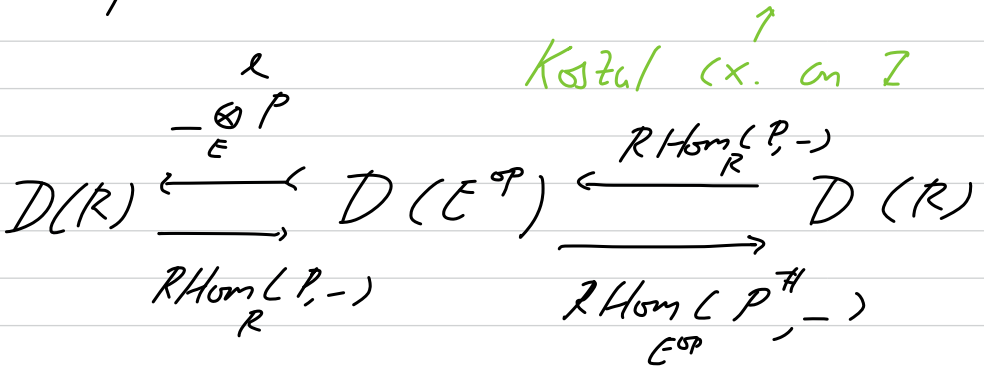
Thus $\text{Ext}_R^*(Q, R) = 0$ for $R: k[[x]] \rightarrow D$.

Recall $\text{Ext}_R^*(Q, R) \neq 0$

⑥ Say $V = \text{supp}_R P$ P perfect (x).

Set $E = \text{End}_R(P)$

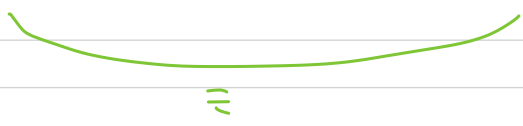
- Example $V = V(I)$ and $P = k(I)$



$$P^{\#} = \text{RHom}_R(P, R)$$

These restrict to equivalences:

$$P^{\#}_V D(R) \cong D(E^{\text{op}}) \cong \Lambda^V D(R)$$



In particular

$$\begin{array}{ccc} \Gamma_{\vee} D(R) & \xrightarrow{\cong} & \Lambda D(R) \\ & & \uparrow \\ & & \mathbb{R}P_{\vee}(-) \end{array}$$

$$\mathbb{R}P_{\vee} M \rightarrow M \quad \& \quad M \rightarrow \Lambda^{\vee} M$$

and also

$$\Lambda^{\vee} \mathbb{R}P_{\vee} M \xrightarrow{\cong} M \quad \& \quad \mathbb{R}P_{\vee} M \xrightarrow{\cong} \mathbb{R}P_{\vee} \Lambda^{\vee} M$$

- x -

(II) Support & Cosupport $M \in D(R)$

$$\text{Supp}_R M := \{ \mathfrak{p} \in \text{Spec } R \mid M \otimes_R^L k(\mathfrak{p}) \neq 0 \}$$

$$\text{Cosupp}_R M := \{ \mathfrak{p} \in \text{Spec } R \mid H^i(M \otimes_R^L k(\mathfrak{p})) \neq 0 \}$$

$$k(\mathfrak{p}) := (R/\mathfrak{p})_{\mathfrak{p}} = R_{\mathfrak{p}} / \mathfrak{p}R_{\mathfrak{p}}$$

= Residue field of $R_{\mathfrak{p}}$.

$$\text{Cosupp}_R M := \{ \mathfrak{p} \in \text{Spec } R \mid R\text{Hom}_R(k(\mathfrak{p}), M) \neq 0 \}$$

Exercise: $\text{supp}_R k(\mathfrak{p}) = \{ \mathfrak{p} \} = \text{Cosupp}_R M$

Support is better understood:

For $M \in D^b(\text{mod } R)$

$$\begin{aligned} \text{supp}_R M &= \text{supp}_R H_* (M) \\ &= \{ \mathfrak{p} \in \text{Spec } R \mid H_* (M)_{\mathfrak{p}} \neq 0 \} \\ &= V(\text{Ann}_R H_* (M)) \end{aligned}$$

- a closed subset of $\text{Spec } R$

Not so with Cosupp : *it need not be closed.*

For $M \in D^b(\text{mod } R)$

$\text{Cosupp}_R M = \text{supp}_R M$ when

$R = k[x] / I$, i.e. affine algebras.

(11)

- T. Nakamura & P. Thompson

Exercise: Say R is \mathcal{I} -adically complete

Then

$$\text{Cosupp}_R M \subseteq V(\mathcal{I})$$

$$\nleftrightarrow M \in \mathcal{D}^b(\text{mod } R)$$

Exercise: (R, \mathfrak{m}, k) local

$$\text{Cosupp}_R E_R(k) = \text{Spec } R$$

$$\text{Supp}_R E_R(k) = \{\mathfrak{m}\}$$

Example: $R = k[[t]][z]$

$\text{Cosupp}_R R$ is not closed.

- It is specialization closed.

Always so?

$$\text{Max}(\text{Cosupp}_R M) = \text{Max}(\text{Supp}_R M) \\ \forall M \in D(R).$$

In particular, $\text{Cosupp}_R M = \emptyset \Leftrightarrow$
 $\text{Supp}_R M = \emptyset \Leftrightarrow$
 $M = 0.$

"Everything" so far holds in great generality. Now to things special to $D(R)$

III $\forall M, N \in D(R)$ one has

$$\text{Supp}_R (M \otimes_R N) = \text{Supp}_R M \cap \text{Supp}_R N \\ \text{Cosupp}_R \text{RHom}_R(M, N) = \text{Supp}_R M \cap \text{Cosupp}_R N$$

Extremely useful

Proof: Fix $p \in \text{Spec } R$

$$R\text{Hom}_R(k(p), R\text{Hom}_R(M, N))$$

$$\cong R\text{Hom}_R(k(p) \otimes_R^L M, N)$$

$$\cong R\text{Hom}_{k(p)}(k(p) \otimes_R^L M, R\text{Hom}_R(k(p), N))$$

$$\neq 0 \iff k(p) \otimes_R^L M \neq 0 \text{ and}$$

$$R\text{Hom}_R(k(p), N) \neq 0.$$

Thus $p \in \text{Supp}_R R\text{Hom}_R(M, N)$

$$\iff p \in \text{Supp}_R M \cap \text{Supp}_R N.$$

The argument for $\text{Supp}_R(M \otimes_R^L N)$ is equally simple □

These lead to a classification of the localizing & colocalizing subcategories of $\mathcal{D}(R)$.