Around Completions

24th May

I R comm. noeth. ring

$D(R)$ Derived category

$V \subseteq \operatorname{Spec} R$ specialization closed

: $\mathfrak{p} \subseteq \mathfrak{q}$ & $\mathfrak{p} \in V \Rightarrow \mathfrak{q} \in V$

$\equiv$ $\mathfrak{p} \in V \Rightarrow \overline{\{\mathfrak{p}\}} \subseteq V$

closure in Zariski topology.

$M \in \operatorname{Mod} R$

$\Gamma_V M := \operatorname{Ker}\left(M \longrightarrow \prod_{\mathfrak{p} \notin V} M_{\mathfrak{p}}\right)$

V-torsion submodule of M

$M \in D(R)$

$R\Gamma_V M := \Gamma_V(iM) \subseteq iM \xleftarrow{\sim} M$

Local cohomology supported on V

injective resl. of M

$$\Gamma_V D(R) := \{ M \in D(R) \mid R\Gamma_V M \xrightarrow{\sim} M \}$$

V-torsion part of $D(R)$

- $\boxed{\text{A localizing subcategory of } D(R).}$

Because $M \in \Gamma_V D(R) \iff$

$$H_*(M)_p = 0 \quad \forall\, p \notin V$$

Compact objects: Write

$$V = \bigcup_i V(p_i)$$

For example, take p_i minimal in V

$K(p_i) :=$ Koszul cx. on some finite generating set for p_i

Then $K(p_i) \in \Gamma_V D(R)$

1 $$\boxed{\begin{aligned} \Gamma_V D(R)^c &= D(R)^c \cap \Gamma_V D(R) \\ &= \mathrm{Thick}(k(\mathfrak{p}_i) \mid i) \end{aligned}}$$

Moreover

2 $$\boxed{\Gamma_V D(R) = \mathrm{Loc}(k(\mathfrak{p}_i) \mid i)}$$

This has many consequences:

Complete $R\Gamma_V M \to M$ to a Δ^{le}:

$$R\Gamma_V M \longrightarrow M \longrightarrow L_V M \longrightarrow$$

"localising" away from V

"nullification"

- Can also interpret $M \to L_V M$ as homology localization.

Example: Fix $\mathfrak{p} \in \operatorname{Spec} R$

$$Z(\mathfrak{p}) := \operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}}$$

- specialization closed

Then $R\Gamma_{Z(\mathfrak{p})} M \longrightarrow M \longrightarrow M_{\mathfrak{p}} \longrightarrow$

$$M_{\mathfrak{p}} = L_{Z(\mathfrak{p})} M$$

Exercise: Verify this.

(3) From (2) we get:

$$R\Gamma_V M \simeq R\Gamma_V R \otimes^{\mathbb{L}}_R M$$

$$L_V M \simeq L_V R \otimes^{\mathbb{L}}_R M$$

Another consequence:

(4) $\bigoplus R\Gamma_V(M_i) \longrightarrow R\Gamma_V(\bigoplus M_i)$

for any collection $\{M_i\}$

Consider adjoint pair:

$$\begin{tikzcd} \Gamma_V D(R) \arrow[r, hook, shift left] \arrow[rr, bend right, "\rho"'] & D(R) \arrow[l, shift left, "R\Gamma_V"] & {} \end{tikzcd}$$

4 implies $R\Gamma_V$ has a right adjoint:

$$L\Lambda^V M := \rho R\Gamma_V M$$

Completion along V

$$M \to L\Lambda^V M$$

Example: $V = V(I)$ $I \subseteq R$ ideal

$M \in \mathrm{Mod}\, R$

$$\Lambda^I M := \lim \left(M / I^n M \right)$$

For $M \in D(R)$ let

$$L\Lambda^I M := \Lambda^I(pM)$$

Derived Completion

Greenlees-May

projective resln. of M

One has $M \longrightarrow L\Lambda^I M$

- Can also be interpretted as a homology localisation.

For $M \in D^b(\mathrm{mod}\, R)$

$$\boxed{\begin{aligned} L\Lambda^I M &\simeq L\Lambda^I R \otimes^{\mathbb{L}}_R M \\ &\simeq \Lambda^I R \otimes_R M \end{aligned}}$$

- This is false in general

<u>Example:</u> (R, m, k) local ring

$L\Lambda^{m} E_p(k) \simeq$ a dualizing cx. for $\hat{R}$, the m-adic completion of R.

Exercise: check this.

Some tools to help understand completions:

5 Greenlees-May duality:

$$\boxed{R\mathrm{Hom}_R(R\Gamma_V M, N) \simeq R\mathrm{Hom}_R(M, L\Lambda^{V} N)}$$

Example: Say R domain complete w.r.t. some ideal $\underset{\neq 0}{I} \subseteq R$

Then $R\mathrm{Hom}_R(Q, R) = 0$

fraction field of R.

Thus $\operatorname{Ext}^*_R(Q, R) = 0$ for $R = k[\![x]\!]$.

Recall $\operatorname{Ext}^*_{\mathbb{Z}}(Q, \mathbb{Z}) \neq 0$

6 Say $V = \operatorname{supp}_R P$ P perfect cx.

Set $E = \operatorname{End}_R(P)$

- Example $V = V(I)$ and $P = K(I)$

Koszul cx. on I

$$\begin{tikzcd} D(R) \arrow[r, shift right, "\operatorname{RHom}_R(P,-)"'] & D(E^{op}) \arrow[l, shift right, "- \otimes^L_E P"'] \arrow[r, shift right, "\operatorname{RHom}_{E^{op}}(P^\#, -)"'] & D(R) \arrow[l, shift right, "\operatorname{RHom}_R(P,-)"'] \end{tikzcd}$$

$P^\# = \operatorname{RHom}_R(P, R)$

These restrict to equivalences:

$$\underbrace{\Gamma_V D(R) \equiv D(E^{op}) \equiv \Lambda^V D(R)}_{\equiv}$$

In particular

$$\begin{tikzcd} \Gamma_V D(R) \arrow[r, shift left, "L\Lambda^V(-)"] & \overset{V}{\Lambda} D(R) \arrow[l, shift left, "R\Gamma_V(-)"] \end{tikzcd} \quad \cong$$

$$R\Gamma_V M \longrightarrow M \quad \& \quad M \longrightarrow L\Lambda^V M$$

induce

$$L\Lambda^V R\Gamma_V M \xrightarrow{\simeq} M \quad \& \quad R\Gamma_V M \xrightarrow{\simeq} R\Gamma_V L\Lambda^V M$$

— x —

II Support & Cosupport $M \in D(R)$

$$\operatorname{supp}_R M := \{ \mathfrak{p} \in \operatorname{Spec} R \mid M \overset{L}{\underset{R}{\otimes}} k(\mathfrak{p}) \neq 0 \}$$

$$\Updownarrow$$

$$H_*(M \overset{L}{\underset{R}{\otimes}} k(\mathfrak{p})) \neq 0$$

$$k(\mathfrak{p}) := (R/\mathfrak{p})_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$$

$$= \text{Residue field of } R_\mathfrak{p}.$$

$$\mathrm{Cosupp}_R M := \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{RHom}_R(k(\mathfrak{p}), M) \neq 0 \}$$

<u>Exercise</u>: $\mathrm{supp}_R\, k(\mathfrak{p}) = \{\mathfrak{p}\} = \mathrm{Cosupp}_R\, M$

Support is better understood:

For $M \in D^b(\mathrm{mod}\, R)$

$$\begin{aligned} \mathrm{supp}_R M &= \mathrm{supp}_R H_*(M) \\ &= \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid H_*(M)_{\mathfrak{p}} \neq 0 \} \\ &= V(\mathrm{ann}_R H_*(M)) \end{aligned}$$

— a closed subset of $\mathrm{Spec}\, R$

Not so with Cosupport: It need not be closed.

For $M \in D^b(\mathrm{mod}\, R)$

$\mathrm{Cosupp}_R M = \mathrm{supp}_R M$ when

$R = k[\underline{x}]/I$, i.e. affine algebras.

- T. Nakamura & P. Thompson

Exercise: Say R is I adically complete then

$$\operatorname{Cosupp}_R M \subseteq V(I)$$

$\forall \quad M \in D^b(\operatorname{mod} R)$

Exercise: (R, m, k) local

$$\operatorname{Cosupp}_R E_R(k) = \operatorname{Spec} R$$

$$\operatorname{Supp}_R E_R(k) = \{m\}$$

Example: $R = k[[t]][x]$

$\operatorname{Cosupp}_R R$ is not closed.

- It is specialization closed.

Always so?

$$\boxed{\operatorname{Max}(\operatorname{Cosupp}_R M) = \operatorname{Max}(\operatorname{Supp}_R M) \quad \forall\, M \in D(R).}$$

In particular, $\operatorname{Cosupp}_R M = \emptyset \iff$
$\operatorname{Supp}_R M = \emptyset \iff$
$M = 0$.

"Everything" so far holds in great generality. Now to things special to $D(R)$

III $\forall\, M, N \in D(R)$ one has

$$\boxed{\begin{aligned} \operatorname{Supp}_R (M \overset{L}{\underset{R}{\otimes}} N) &= \operatorname{Supp}_R M \cap \operatorname{Supp}_R N \\ \operatorname{Cosupp}_R \operatorname{RHom}_R(M, N) &= \operatorname{Supp}_R M \cap \operatorname{Cosupp}_R N \end{aligned}}$$

Extremely useful

Proof: Fix $\mathfrak{p} \in \operatorname{Spec} R$

$$\operatorname{RHom}_R(k(\mathfrak{p}), \operatorname{RHom}_R(M, N))$$

$$\simeq \operatorname{RHom}_R(k(\mathfrak{p}) \otimes^{\ell}_R M, N)$$

$$\simeq \underbrace{\operatorname{RHom}_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes^{\ell}_R M, \operatorname{RHom}_R(k(\mathfrak{p}), N))}$$

$$\neq 0 \iff k(\mathfrak{p}) \otimes^{\ell}_R M \neq 0 \ \underline{\text{and}}$$

$$\operatorname{RHom}_R(k(\mathfrak{p}), N) \neq 0.$$

Thus $\mathfrak{p} \in \operatorname{Cosupp}_R \operatorname{RHom}_R(M, N)$

$$\iff \mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Cosupp}_R N.$$

The argument for $\operatorname{Supp}_R(M \otimes^{\ell}_R N)$ is equally simple $\square$

These lead to a classification of the localizing & colocalizing subcategories of $D(R)$.