Dualisable objects in local algebra

26th May

(I) R Comm. noeth. ring
$M \in D(R)$

$$\mathrm{Supp}_R M := \{ p \in \mathrm{Spec}\, R \mid k(p) \overset{L}{\underset{R}{\otimes}} M \neq 0 \}$$

$$\mathrm{Cosupp}_R M := \{ p \in \mathrm{Spec}\, R \mid \mathrm{RHom}_R(k(p), M) \neq 0 \}$$

- $\mathrm{Supp}_R M \subseteq \mathrm{Spec}\, R$ closed $\forall M \in D^b(\mathrm{mod} R)$
- Need always so for $\mathrm{Cosupp}_R M$.

<u>Example</u>: $R = k[[t]][x]$

Claim: $\mathrm{Cosupp}\, R \neq V(J)$ for any $J \subseteq R$

- Always $\mathrm{Max}(\mathrm{Supp}_R M) = \mathrm{Max}(\mathrm{Cosupp}_R M)$.

Thus $\mathrm{Max}(\mathrm{Spec}\, R) \subseteq \mathrm{Cosupp}_R R$

So if $\mathrm{Cosupp}\, R = V(J)$, then

$$J \subseteq \bigcap_{m \in \mathrm{Max}(\mathrm{Spec}\, R)} m = (0)$$

check this

[In fact, $m_i = (1 - x t^i)$, $i \geq 1$, are all maximal and

$$\bigcap_{i=1}^{\infty} m_i = (0) \,]$$

So we get $\operatorname{Cosupp} R = V(0) = \operatorname{Spec} R$.

However $\mathfrak{p} = (x) \notin \operatorname{Cosupp} R$:

$$R \twoheadrightarrow \frac{R}{(x)} = k[[t]] \hookrightarrow k(\mathfrak{p}) = k[[t]][\tfrac{1}{t}]$$

field of fractions of $k[[t]]$

Thus $\operatorname{RHom}_R(k(\mathfrak{p}), R)$

$$\simeq \operatorname{RHom}_{k[[t]]}(k(\mathfrak{p}), \operatorname{RHom}_R(k[[t]], R))$$

$$\simeq \operatorname{RHom}_{k[[t]]}(k(\mathfrak{p}), \Sigma^{-1} k[[t]])$$

$\because$ $\operatorname{RHom}_R(k[[t]], R) \simeq \Sigma^{-1} k[[t]]$

$$\simeq \Sigma^{-1} \operatorname{RHom}_{k[[t]]}(k(\mathfrak{p}), k[[t]])$$

$$\simeq 0$$

(Compute using $0 \to R \xrightarrow{x} R \to 0$, $R \xrightarrow{\sim} k[[t]]$).

$\because$ $k[[t]]$ is (t)-adically complete.

So $\mathfrak{p} \notin \operatorname{Cosupp} R$.

In fact, the argument shows $\overline{\operatorname{Cosupp} R}$ is not contained in any proper closed set.

II Stratification (Neeman)

$$\left\{ \begin{array}{c} \text{Localising} \\ \text{subcategories} \\ \text{of } D(R) \end{array} \right\} \overset{\operatorname{supp}_R(-)}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Subsets of} \\ \operatorname{Spec} R \end{array} \right\}$$

bijection.

$$\mathcal{L} \longmapsto \bigcup_{M \in \mathcal{L}} \operatorname{supp}_R M$$

$$\{ M \in D(R) \mid \operatorname{supp}_R M \subseteq \{\mathfrak{p}\} \} \longleftarrow \mathfrak{p} \in \operatorname{Spec} R$$

$$\Updownarrow$$

$$\Gamma_{V(\mathfrak{p})}(M_{\mathfrak{p}}) \simeq M$$ i.e. $H_*(M)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.

$$\Updownarrow$$

$$M \in \Gamma_{\mathfrak{p}} D(R)$$

So $\Gamma_{\mathfrak{p}} D(R)$ are the minimal localising subcategories in $D(R)$

– the "atoms" of $D(R)$.

Can we "look" deeper into $\Gamma_{\mathfrak{p}} D(R)$?

What we know:

- $\Gamma_p D(R) \subseteq D(R)$
  Δ^{lalen} subcategory, and closed under $\oplus$ (in $D(R)$).
- It is compactly generated:

$$\boxed{\Gamma_p D(R)^c = D(R_p)^c \cap \Gamma_{V(p)} D(R_p)}$$

Caveat: $\Gamma_p D(R)^c$ only compact in $D(R_p)$, not in $D(R)$.

In fact, let $K :=$ Koszul cx. on some finite generating set for $pR_p \subseteq R_p$.

Then $\Gamma_p D(R)^c = \operatorname{Thick}(K)$

And $\Gamma_p D(R) = \operatorname{Loc}(K)$

More structure: $\Gamma_{\mathfrak{p}} D(R)$ inherits a $\otimes$ from $D(R)$:

$$M, N \in \Gamma_{\mathfrak{p}} D(R) \Rightarrow M \overset{\ell}{\underset{R}{\otimes}} N \in \Gamma_{\mathfrak{p}} D(R)$$

[Recall: $\Gamma_{\mathfrak{p}} M \simeq \Gamma_{\mathfrak{p}} R \overset{\ell}{\underset{R}{\otimes}} M \quad \forall M$]

Moreover $M \overset{\ell}{\underset{R}{\otimes}} \Gamma_{\mathfrak{p}} R \simeq M$. So

$$\boxed{\Gamma_{\mathfrak{p}} D(R) \text{ is a tt-category with unit } \underline{\Gamma_{\mathfrak{p}} R}.}$$

N.B. $\Gamma_{\mathfrak{p}} R$ not compact in $\Gamma_{\mathfrak{p}} D(R)$ unless $\mathfrak{p} \in \operatorname{Min}(R)$.

[$\because$ $\operatorname{length}_{R_{\mathfrak{p}}} C < \infty \ \ \forall\, C \in \Gamma_{\mathfrak{p}} D(R)^c$]

– This turns out to be key.

III $(\mathcal{T}, \otimes, 1\!\!1)$ tt-Category

unit of $-\otimes-$.

- assume it has internal Homs

$F(-,-)$ so

$$\mathrm{Hom}_{\mathcal{T}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{T}}(X, F(Y,Z)).$$

$\forall\ X, Y, Z.$

Example: ① $(D(R), \overset{\mathbb{L}}{\otimes}, R)$ and

$$F(-,-) = R\mathrm{Hom}_R(-,-)$$

② $(\Gamma_{\mathfrak{p}} D(R), -\overset{\mathbb{L}}{\otimes}_{R_{\mathfrak{p}}}-, \Gamma_{\mathfrak{p}} R)$ and

$$F(-,-) = \Gamma_{\mathfrak{p}} R\mathrm{Hom}_R(-,-)$$

—x—

$D \in \mathcal{T}$ is dualisable if

$$F(D, 1\!\!1) \otimes X \xrightarrow{\cong} F(D, X)$$

Natural map. $\forall\ X \in \mathcal{T}$

Recall $C \in \mathcal{T}$ is compact means

$$\bigoplus \operatorname{Hom}_{\mathcal{T}}(C, X_i) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{T}}(C, \oplus X_i)$$

$$\forall \ \{X_i\}$$

- In general, neither property implies the other.

Set $\mathcal{T}^d :=$ Dualizable objects in $\mathcal{T}$

$$\mathbb{1} \in \mathcal{T}^d$$

- This is a thick subcategory, closed under $\otimes$: $D, D' \in \mathcal{T}^d \Rightarrow D \otimes D' \in \mathcal{T}^d$

So $\mathcal{T}^d$ is a tt-category in its own right.

$\mathcal{T}^c :=$ Compact objects in $\mathcal{T}$

- thick subcategory
- May not be $\otimes$-closed.

However: $\boxed{D \in \mathcal{T}^d \text{ and } C \in \mathcal{T}^c \text{; then } D \otimes C \in \mathcal{T}^c}$

So $\mathcal{T}^d$ "acts" on $\mathcal{T}^c$.

- In particular $\mathbb{1} \in \mathcal{T}^c \iff \mathcal{T}^d \subseteq \mathcal{T}^c$

But this doesn't always hold.

Example: ① $\mathcal{T} = D(R)$

$$D(R)^c = \mathrm{Thick}(R) = D(R)^d$$

(Check!)

② What about $\mathcal{T} = \Gamma_{\mathfrak{p}} D(R)$?

$$\Gamma_{\mathfrak{p}} R \in \mathcal{T}^d, \text{ so}$$

unit

$$\underset{\substack{\| \\ \mathcal{T}^c}}{\mathrm{Thick}(k)} \subseteq \mathrm{Thick}(\Gamma_{\mathfrak{p}} R) \subseteq \mathcal{T}^d$$

Benson, —, Krause, Pevtsova (2023):

For $X \in \Gamma_{\mathfrak{p}} D(R)$, the following conditions are equivalent:

1. $X \in \Gamma_{\mathfrak{p}} D(R)^{d}$
2. $\operatorname{rank}_{k(\mathfrak{p})} H_*^{\ell}(k(\mathfrak{p}) \otimes_R X) < \infty$
3. $X \in \operatorname{Thick}(\Gamma_{\mathfrak{p}} R)$.

Appears in: Local dualisable objects in local algebra; arxiv. 2023.

- ① ⇔ ② ⇐ ③ true fairly generally.

  ① ⇒ ③ special to $D(R)$

  (and also holds in $\operatorname{StMod} kG$, for instance, but not in $Ho(Sp)$).

Comment on ① $\Leftrightarrow$ ② :

Say $(R, \mathfrak{m}, k)$ local, and $M \in D(R)$ is s.t. $\operatorname{rank}_k (k \otimes^{\mathbb{L}}_R M) < \infty$

- When $M \in D^b(\operatorname{mod} R)$, this holds

$\Leftrightarrow$ $M \in \operatorname{Thick}(R)$ i.e. M perfect.

- For general $M \in D(R)$, the thm. above yields that $\Gamma_{\mathfrak{m}} M \in \operatorname{Thick}(\Gamma_{\mathfrak{m}} R)$, and conversely.

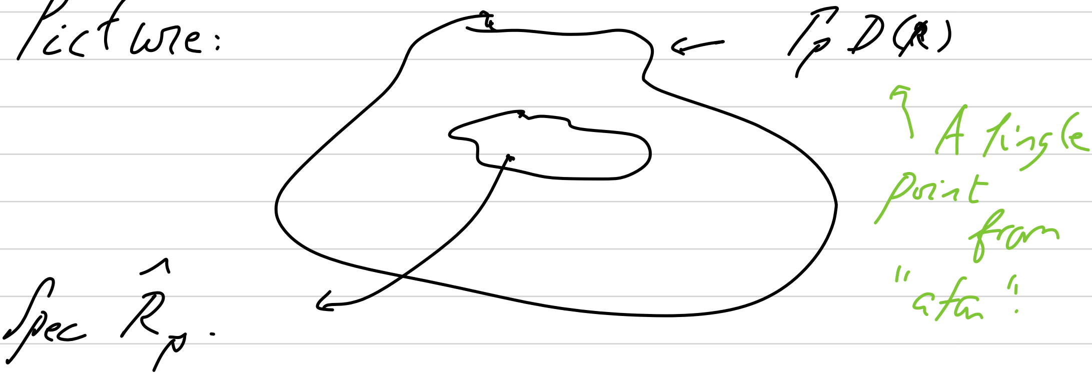
A Corollary (using Greenlees–May)

$$\boxed{\Gamma_{\mathfrak{p}} D(R)^{d} \equiv \operatorname{Thick}(\hat{R}_{\mathfrak{p}}) \subseteq D(\hat{R}_{\mathfrak{p}})}$$

$\hat{R}_{\mathfrak{p}}$ = Completion of $R_{\mathfrak{p}}$ at its maximal ideal.

Consequently $\boxed{\operatorname{Spec} \prod_p D(R)^{d} = \operatorname{Spec}(\hat{R}_p).}$

Picture:



Rmk: $\operatorname{Spec}(\hat{R}_p)$ can be quite diff. from. $\operatorname{Spec}(R_p)$.

- Passage from $\prod_p D(R)^c$ to $\prod_p D(R)^d$ is a "completion".