Hard probes – Lecture notes

Jean-Paul Blaizot*

Institut de Physique Théorique, CEA-Saclay, 91191 Gif-sur-Yvette Cedex, France.

(Dated: March 24, 2026)

These are unpolished notes that are meant to complement the lectures, by providing explicitly worked out exercises. These may be updated. Please point out any possible typos/errors.

*Electronic address: `jean-paul.blaizot@ipht.fr`

I. THE INFLUENCE FUNCTIONAL IN THE ABELIAN CASE

We perform the explicit calculation of the influence functional for the case of a single "abelian"' $Q\bar{Q}$ pair immersed in an electromagnetic plasma of light charged particles and photons. By abelian quarks we mean electrically charged particles (with charge $+g$ for the quark and $-g$ for the antiquark). In Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, the Hamiltonian of the system reads

$$\begin{aligned} H &= \frac{1}{2M}\left(\mathbf{p}^2 + \bar{\mathbf{p}}^2\right) + \int \mathrm{d}\boldsymbol{x}\ \psi^\dagger(\boldsymbol{x})\ \left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i} + m\gamma_0\right)\ \psi(\boldsymbol{x}) \\ &+ \frac{1}{2}\iint \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\ \rho_{\mathrm{tot}}(\boldsymbol{x})K(\boldsymbol{x}-\boldsymbol{y})\rho_{\mathrm{tot}}(\boldsymbol{y}), \end{aligned} \tag{1}$$

where $\alpha^i = \gamma_0\gamma^i$ is a Dirac matrix, and $\rho_{\mathrm{tot}} = \rho + \rho_\psi$ is the total charge density, with

$$\rho(\boldsymbol{x}) = g\ \left[\delta(\boldsymbol{x}-\hat{\boldsymbol{q}}) - \delta(\boldsymbol{x}-\hat{\bar{\boldsymbol{q}}})\right], \tag{2}$$

the charge density of the heavy quark- antiquark pair, and

$$\rho_\psi(\boldsymbol{x}) = g\,\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}) \tag{3}$$

the density of the charged plasma particles. The plasma is supposed to be electrically neutral, that is, it contains the same number of light particles and antiparticles. It is useful to rewrite the Hamiltonian (1) by separating its various contributions as follows

$$H = H_Q + H_1 + H_{\mathrm{pl}}, \tag{4}$$

with H_Q describing the dynamics of the heavy quarks in the absence of the plasma,

$$H_Q = \frac{1}{2M}\left(\mathbf{p}^2 + \bar{\mathbf{p}}^2\right) + \frac{1}{2}\iint \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\rho(\boldsymbol{x})K(\boldsymbol{x}-\boldsymbol{y})\rho(\boldsymbol{y}), \tag{5}$$

H_1 the Hamiltonian coupling the heavy quarks to the plasma charged particles

$$H_1 = \iint \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\rho(\boldsymbol{x})K(\boldsymbol{x}-\boldsymbol{y})\rho_\psi(\boldsymbol{y}), \tag{6}$$

and H_{pl} the Hamiltonian of the plasma in the absence of the heavy quarks

$$H_{\mathrm{pl}} = \int \mathrm{d}\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{i} + m\gamma_0\right)\psi(\boldsymbol{x}) + \frac{1}{2}\iint \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\rho_\psi(\boldsymbol{x})K(\boldsymbol{x}-\boldsymbol{y})\rho_\psi(\boldsymbol{y}). \tag{7}$$

We represent the heavy particles in first quantization (they are non relativistic particles whose number is conserved), while the light particles are represented by the fermion fields $\psi(\boldsymbol{x})$ and $\psi^\dagger(\boldsymbol{x})$. Note that the interaction term in Eq. (5) contains contributions of self interactions. Such terms will not contribute in the final equations. The last term in Eq. (1) is the total Coulomb energy, with

$$K(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \qquad -\boldsymbol{\nabla}_x^2 K(x-y) = \delta(x-y). \tag{8}$$

We shall first consider the case where the heavy particles interact only with an external potential $A_0(\boldsymbol{x})$, with a Hamiltonian $H_1 = g\int_{\boldsymbol{x}} \rho(\boldsymbol{x}) A_0(\boldsymbol{x})$. In this case

$$P(\boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i) = \left|\langle \boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i\rangle\right|^2 . \tag{9}$$

and the probability amplitude $\langle \boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i\rangle$ is given by a Feynman path integral

$$\langle \boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i\rangle \;=\; \int_{\boldsymbol{Q}_i}^{\boldsymbol{Q}_f} D\boldsymbol{Q}\ \mathrm{e}^{i(S_0[\boldsymbol{Q}]+S_1[\boldsymbol{Q},A_0])}, \tag{10}$$

where the paths $\boldsymbol{Q}(t)$ satisfy $\boldsymbol{Q}(t_i) = \boldsymbol{Q}_i$ and $\boldsymbol{Q}(t_f) = \boldsymbol{Q}_f$. Here $\boldsymbol{Q} = \{\boldsymbol{q}, \bar{\boldsymbol{q}}\}$ denotes collectively the coordinate $\boldsymbol{q}$ of the quark, and that $\bar{\boldsymbol{q}}$ of the antiquark. The actions S_0 and S_1 are given by

$$S_0[\boldsymbol{Q}] \;=\; \frac{M}{2}\int_{t_i}^{t_f} \mathrm{d}t\left(\dot{\boldsymbol{q}}^2 + \dot{\bar{\boldsymbol{q}}}^2\right), \tag{11}$$

$$\begin{aligned} S_1[\boldsymbol{Q}, A_0] \;&=\; -\int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, \rho(x) A_0(x), \\ &=\; -g\int_{t_i}^{t_f} \mathrm{d}t\, \left[A_0(\boldsymbol{q}(t)) - A_0(\bar{\boldsymbol{q}}(t))\right]. \end{aligned} \tag{12}$$

where we have set $x = (t, \boldsymbol{x})$.

The probability (9) can be represented by a very similar formula, by using the closed-time path formalism. We introduce a contour in the complex time plane, as illustrated in Fig. 1, and consider the coordinates $\{\boldsymbol{q}, \bar{\boldsymbol{q}}\}$ as functions of the complex time $t_{\mathcal{C}}$ running along the contour. Alternatively, we may keep time real, but duplicate the coordinates, denoting by $\boldsymbol{Q}_1 = \{\boldsymbol{q}_1, \bar{\boldsymbol{q}}_1\}$ and $\boldsymbol{Q}_2 = \{\boldsymbol{q}_2, \bar{\boldsymbol{q}}_2\}$ the coordinates of the heavy particles living respectively on the upper branch ($\mathcal{C}_1$, corresponding to the amplitude) and the lower branch ($\mathcal{C}_2$, corresponding to the complex conjugate amplitude) of the contour. We can then write

$$P(\boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i) = \int_{\mathcal{C}} D\boldsymbol{Q}\ \mathrm{e}^{i(S_0[\boldsymbol{Q}]+S_1[\boldsymbol{Q},A_0])}, \tag{13}$$

where the actions S_0 and S_1 are given by the formulae (11, 12) in which the time integrations are replaced by integrations along the Schwinger-Keldysh contour. Thus, for instance, S_0 is given by

$$S_0[\boldsymbol{Q}] = \frac{M}{2}\int_{\mathcal{C}} \mathrm{d}t^c \left(\dot{\boldsymbol{q}}^2 + \dot{\bar{\boldsymbol{q}}}^2\right), \tag{14}$$

where the time coordinate t^c runs along the contour $\mathcal{C}$, that is, from t_i to t_f slightly above the real time axis, and returns from t_f to t_i slightly below it. Thus,

$$\int_{\mathcal{C}} \mathrm{d}t^c\, \dot{\boldsymbol{q}}^2 = \int_{t_i+i\eta}^{t_f+i\eta} \mathrm{d}t^c\, \dot{\boldsymbol{q}}^2 + \int_{t_f-i\eta}^{t_i-i\eta} \mathrm{d}t^c\, \dot{\boldsymbol{q}}^2 = \int_{t_i}^{t_f} \mathrm{d}t\, (\dot{\boldsymbol{q}}_1^2 - \dot{\boldsymbol{q}}_2^2), \tag{15}$$

where in the last step we have duplicated the coordinate $\boldsymbol{q}(t)$ as discussed above. The last expression appears naturally in the action when one multiplies the probability amplitude by its complex conjugate in order to build the probability (13), with $\boldsymbol{q}_1(t)$ labelling the path in the amplitude and $\boldsymbol{q}_2(t)$ the path in the complex conjugate amplitude.

It is straightforward to extend the formula (13) to include the interactions among the heavy particles and with the light plasma constituents. We have (see e.g. [2] or [4])

$$P(\boldsymbol{Q}_f, t_f|\boldsymbol{Q}_i, t_i) = \int_{\mathcal{C}} D\boldsymbol{Q} \int_{\mathcal{C}} D(\bar{\psi}, \psi)\; \mathrm{e}^{\mathrm{i}\, S[\boldsymbol{Q},\psi,\bar{\psi}]}, \tag{16}$$

where the contour now includes a vertical piece, $\mathcal{C}_3$ corresponding to the thermal average of the plasma degrees of freedom at the initial time (i.e., the trace over the equilibrium density matrix of the plasma). Accordingly, the fermionic fields in Eq. (16) obey anti-periodic boundary conditions on $\mathcal{C}_3$, $\psi(0,\boldsymbol{x}) = -\psi(-\mathrm{i}\beta,\boldsymbol{x})$, $\overline{\psi}(0,\boldsymbol{x}) = -\overline{\psi}(-\mathrm{i}\beta,\boldsymbol{x})$. The action $S[\boldsymbol{Q},\psi,\bar{\psi}]$ is given by

$$S[\boldsymbol{Q},\psi,\bar{\psi}] = S_0[\boldsymbol{Q}] + \int_{\mathcal{C}} \mathrm{d}^4x\; \bar{\psi}(x)(\; \mathrm{i}\gamma^\mu\partial_\mu - m\;)\psi(x) \tag{17}$$

$$-\frac{1}{2}\iint_{\mathcal{C}} \mathrm{d}^4x\, \mathrm{d}^4y\; \rho_{\rm tot}(x)K(x-y)\rho_{\rm tot}(y)\,, \tag{18}$$

where $K(x-y) = \delta(t_x - t_y)K(\boldsymbol{x}-\boldsymbol{y})$ represents the (instantaneous) Coulomb interaction, and $\rho_{\rm tot}$ is the total charge density. It is important to stress that the heavy particles do not

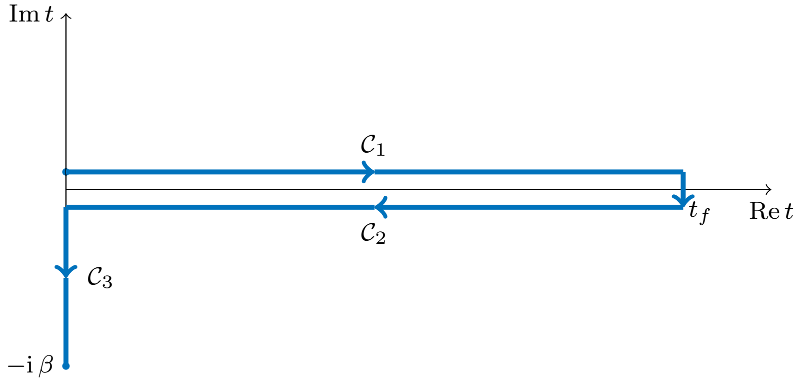

FIG. 1: The Keldysh contour $\mathcal{C}$, with its different branches.

take part in the thermal average, and consequently they do not propagate along the imaginary time sector of the Keldysh contour[5]. We may take $\rho(t = -i\tau, \boldsymbol{x}) = 0$, with $0 < \tau \le \beta$.

The next step consists in eliminating the light fermion field in favor of a Coulomb potential A_0. To this end, we use the formal identity[6]:

$$\exp\left[-\frac{\mathrm{i}}{2}\,\rho_{\rm tot}\cdot K\cdot\rho_{\rm tot}\right] = \mathcal{N}\int_{\mathcal{C}} DA_0\;\exp\left[\frac{\mathrm{i}}{2}\,A_0\cdot K^{-1}\cdot A_0 - \mathrm{i}\,A_0\cdot\rho_{\rm tot}\right], \tag{19}$$

(20)

where $\mathcal{N} \sim (\det[\nabla^2])^{\frac{1}{2}}$ is a normalization constant, $K^{-1}(\boldsymbol{x}-\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\nabla}^2_{\boldsymbol{y}}$, and we use a matrix notation to simplify the formulae, e.g.,

$$\rho\cdot K\cdot\rho = \iint_{\mathcal{C}} \mathrm{d}^4x\,\mathrm{d}^4y\,\rho(x)K(x,y)\rho(y). \tag{21}$$

When using this identity, it is important to remember that part of the A_0 potential is the Coulomb field created by the heavy particles. We shall call $A_0^{\rm cl}$ this contribution, and write accordingly $A_0 = A_0^{\rm cl} + \tilde{A}_0$. By definition, we have

$$-\boldsymbol{\nabla}^2 A_0^{\rm cl}(\boldsymbol{x}) = \rho(\boldsymbol{x}), \qquad A_0^{\rm cl}(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{y}\,K(\boldsymbol{x}-\boldsymbol{y})\rho(\boldsymbol{y}). \tag{22}$$

The integration over A_0 is then truly an integration over the field $\tilde{A}_0$, and this field satisfies the imaginary time (KMS) periodic boundary condition $\tilde{A}_0(0,\boldsymbol{x}) = \tilde{A}_0(-\mathrm{i}\,\beta,\boldsymbol{x})$. We could take this explicitly into account by performing a shift of integration variables. It is more convenient not to do so, provided that we remember that A_0^{cl} plays the role of a constant in the functional integral, in particular the result of such integration will depend on A_0^{cl}.

Using the identity above, and remembering that $\rho_{\mathrm{tot}} = \rho + g\,\psi^\dagger\psi$, we can perform the Gaussian integrals over the light fermion fields

$$\int D(\bar{\psi},\psi)\ \exp\left[\mathrm{i}\int_{\mathcal{C}}\mathrm{d}x\ \bar{\psi}(x)(\mathrm{i}\gamma^\mu\partial_\mu - m - g\gamma^0 A_0(x))\psi(x)\right] \tag{23}$$

$$= \exp\left[\mathrm{Tr}\ \ln\left[\mathrm{i}\gamma^\mu\partial_\mu - m - g\gamma^0 A_0\right]\right]. \tag{24}$$

The probability (16) takes then the form

$$P(\boldsymbol{Q}_f,t_f|\boldsymbol{Q}_i,t_i) = \int_{\mathcal{C}} D\boldsymbol{Q}\int_{\mathcal{C}} DA_0\ \mathrm{e}^{\mathrm{i}\,(S_0[\boldsymbol{Q}]+S_1[\boldsymbol{Q},A_0]+S_2[\boldsymbol{Q},A_0])}, \tag{25}$$

where S_1 is given by Eq. (12) and

$$S_2[\boldsymbol{Q},A_0] = -\frac{1}{2}\int_{\mathcal{C}}\mathrm{d}x\,\big(\,A_0(x)\nabla^2 A_0(x)\big) - \mathrm{i}\ \mathrm{Tr}\ \ln\left[\ \mathrm{i}\gamma^\mu\partial_\mu - m - g\gamma^0 A_0(x)\ \right]. \tag{26}$$

The dependence of S_2 on $\boldsymbol{Q}$ arises from the dependence of A_0^{cl} on the positions of the heavy particles.

It is convenient to rewrite the integral over A_0 as the exponential of an effective action, the so-called Feynman-Vernon (FV) influence functional [3]:

$$\mathrm{e}^{\mathrm{i}\Phi[\boldsymbol{Q},A_0^{\mathrm{cl}}]} = \int \boldsymbol{D}A_0\,\mathrm{e}^{-\mathrm{i}\int_{\mathcal{C}}\mathrm{d}^4x\,\rho(x)A_0(x)}\,\mathrm{e}^{\mathrm{i}S_2[A_0]}. \tag{27}$$

The exponential of the FV functional is the thermal average over the A_0 fluctuations of the exponential factor that contains the linear interaction ρA_0 between the heavy particles, with charge density ρ, and the total Coulomb field A_0. This particular structure is a consequence of the fact that the heavy quark is linearly coupled to the total field A_0. So far, no approximation has been made (within the present Abelian context). We shall now introduce the main approximation of the whole approach, that consists in neglecting the non linear self-interactions of the A_0 field.

The action S_2 contains a non-local term describing the coupling between light quarks and gluons, as well as the classical Coulomb interaction between the heavy particles. Its expansion in powers of A_0 gives rise to induced effective couplings to all orders in the coupling constant g. In order to be able to compute the influence functional we need to introduce some approximations. We do so by retaining only the terms up to quadratic order in the coupling g, or equivalently in the field A_0. This is certainly an excellent approximation at truly weak coupling, like in electromagnetic plasmas. In the case of QCD, the validity of this weak coupling approximation may require further investigation. The main virtue of this approximation is to make the path integral over A_0 calculable, since it becomes Gaussian. The influence functional $\Phi[\boldsymbol{Q}]$ becomes

$$\Phi[\boldsymbol{Q}] = \frac{1}{2} \iint_{\mathcal{C}} \mathrm{d}^4x \mathrm{d}^4y \; \rho(x) \Delta_{\mathcal{C}}(x-y) \rho(y) \; , \tag{28}$$

where

$$\Delta_{\mathcal{C}}(x-y) = \mathrm{i} \langle \, T_{\mathcal{C}} \, [A_0(x) A_0(y) \,] \rangle \tag{29}$$

is the longitudinal gluon propagator defined on the contour, and obeying the KMS conditions. Its inverse involves the one-loop longitudinal photon self-energy $\Pi_{00}^{\mathcal{C}}$, also defined on the contour,

$$-\Delta_{\mathcal{C}}^{-1}(x-y) = \delta_{\mathcal{C}}(t_x^c - t_y^c) \; K^{-1}(\boldsymbol{x}-\boldsymbol{y}) + \Pi_{00}^{\mathcal{C}}(x-y), \tag{30}$$

where $\delta_{\mathcal{C}}(x-y) = \delta_{\mathcal{C}}(t_x^c - t_y^c) \, \delta(\boldsymbol{x}-\boldsymbol{y})$.

It is convenient to write the influence functional using the duplicated fields, that is, we make the substitution

$$\begin{aligned} \boldsymbol{Q}(t^c) \; &\rightarrow \; (\boldsymbol{Q}_1(t), \boldsymbol{Q}_2(t)) \\ A_0(t^c, \boldsymbol{x}) \; &\rightarrow \; (A_{0,1}(t,\boldsymbol{x}), A_{0,2}(t,\boldsymbol{x})), \end{aligned} \tag{31}$$

where t^c denotes the curvilinear abscissa parametrizing the Keldysh contour, while $t \in [t_i, t_f]$ denotes the physical time. The integration over the physical time is always from t_i to t_f. With this notation Δ becomes a matrix,

$$\Delta_{ab}(t_x - t_y) = \Delta(t_x^c - t_y^c) \quad \text{with} \quad t_x^c \in \mathcal{C}_a \, , \; t_y^c \in \mathcal{C}_b \, , \quad a,b = 1,2. \tag{32}$$

We have, explicitly,

$$
\begin{aligned}
\Delta_{11}(x,y) &= \mathrm{i}\langle\, T_c\,[A_{0,1}(x)A_{0,1}(y)\,]\rangle = \mathrm{i}\langle\, T\,[A_0(x)A_0(y)\,]\rangle = \Delta(x,y),\\
\Delta_{21}(x,y) &= \mathrm{i}\langle\, T_c\,[A_{0,2}(x)A_{0,1}(y)\,]\rangle = \mathrm{i}\langle\, A_0(x)A_0(y)\,\rangle = \mathrm{i}\Delta^{>}(x,y),\\
\Delta_{12}(x,y) &= \mathrm{i}\langle\, T_c\,[A_{0,1}(x)A_{0,2}(y)\,]\rangle = \mathrm{i}\langle\, A_0(y)A_0(x)\,\rangle = \mathrm{i}\Delta^{<}(x,y),\\
\Delta_{22}(x-y) &= \mathrm{i}\langle\, T_c\,[A_{0,2}(x)A_{0,2}(y)\,]\rangle = \mathrm{i}\langle\, \tilde{T}\,[A_0(x)A_0(y)\,]\rangle = \tilde{\Delta}(x,y).
\end{aligned}
\tag{33}
$$

where T and $\widetilde{T}$ denote respectively the time ordering and anti ordering, and $\langle\cdots\rangle$ is the thermal average. Using this notation, the phase in Eq. (28) becomes

$$
\Phi[\boldsymbol{Q}] = \frac{1}{2}\int_{t_i}^{t_f}\mathrm{d}t_x\int_{t_i}^{t_f}\mathrm{d}t_y\int\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\;(-1)^{a+b}\rho_a(t_x,\boldsymbol{x})\;\Delta_{ab}(t_x-t_y,\boldsymbol{x}-\boldsymbol{y})\;\rho_b(t_y,\boldsymbol{y}).
\tag{34}
$$

Note that we integrate all times the same way, i.e., on the real time axis from t_i to t_f, so that the off diagonal terms pick up a minus sign. Note also that in the right hand sides of Eqs. (33), we have introduced the notation Δ (without subscripts) to denote the time-ordered real time propagator. Other useful relations are $\Delta = \Delta^R + \mathrm{i}\Delta^<$, $\tilde{\Delta} = -\Delta^A + \mathrm{i}\Delta^<$, where Δ^R and Δ^A denote respectively the retarded and the advanced propagators.

At this point the probability (13) is written as the following path integral

$$
P(\boldsymbol{Q}_f,t_f|\boldsymbol{Q}_i,t_i) = \int_{\mathcal{C}} D\boldsymbol{Q}\;\mathrm{e}^{iS_0[\boldsymbol{Q}]}\,\mathrm{e}^{i\Phi[\boldsymbol{Q}]},
\tag{35}
$$

with $\Phi[\boldsymbol{Q}]$ given by Eq. (34) above. The plasma degrees of freedom have been eliminated, the plasma properties entering the calculation of $\Phi[\boldsymbol{Q}]$ solely through the contour propagator $\Delta_{ab}(t_x-t_y,\boldsymbol{x}-\boldsymbol{y})$ that plays the role of an effective interaction between the heavy quarks. The problem of calculating the probability $P(\boldsymbol{Q}_f,t_f|\boldsymbol{Q}_i,t_i)$ has been reduced to that of calculating an "ordinary" Feynman path integral. This remains however a difficult task.

II. CORRELATORS/PROPAGATORS OF THE HARMONIC OSCILLATOR

1. *A single oscillator (1)*

As a warm up, we consider the environment to be constituted by a single harmonic oscillator with frequency Ω, i.e. with hamiltonian $H = \Omega\, a^\dagger a$. We shall then derive explicitly the various correlators, and verify the general relations that they satisfy.

We recall that the time dependence of the creation and annihilation operators is given by

$$a(t) = \mathrm{e}^{iHt} a\, \mathrm{e}^{-iHt} = \mathrm{e}^{-i\Omega t}, \qquad a^\dagger(t) = \mathrm{e}^{iHt} a^\dagger \mathrm{e}^{-iHt} = \mathrm{e}^{i\Omega t}. \tag{36}$$

We have

$$G^>(t-t') = \langle a(t) a^\dagger(t') \rangle = \mathrm{e}^{-i\Omega(t-t')} \langle a a^\dagger \rangle = \mathrm{e}^{-i\Omega(t-t')}(1+n_\Omega). \tag{37}$$

We have used the commutation relation $[a, a^\dagger] = 1$ and set

$$n_\Omega = \langle a^\dagger a \rangle = \frac{1}{\mathrm{e}^{\beta\Omega} - 1}, \tag{38}$$

assuming the oscillator to be in equilibrium at temperature $T = 1/\beta$. Similarly, we have

$$\begin{aligned} G^<(t-t') &= \langle a^\dagger(t')\, a(t) \rangle = \mathrm{e}^{-i\Omega(t-t')}\, n_\Omega, \\ G^>(t-t') &= \langle a(t)\, a^\dagger(t') \rangle = \mathrm{e}^{-i\Omega(t-t')}\, (1+n_\Omega), \end{aligned} \tag{39}$$

The time ordered propagator is given by (observe the factor i)

$$G(t-t') = i\theta(t-t') G^>(t-t') + i\theta(t'-t) G^>(t-t'), \tag{40}$$

and the retarded propagator by

$$\begin{aligned} G^R(t-t') &= i\theta(t-t')\, [G^>(t-t') - G^<(t-t')] \\ &= i\theta(t-t') \left\langle \left[a(t), a^\dagger(t')\right] \right\rangle = i\theta(t-t')\, \mathrm{e}^{-i\Omega(t-t')}. \end{aligned} \tag{41}$$

Note that the commutator being a c-number, the result is independent of which state is involved in the expectation value (e.g., the result is the same at any temperature). One also defines an advanced propagator

$$G^A(t-t') = -i\theta(t'-t)\, [G^>(t-t') - G^<(t-t')] = -i\theta(t'-t) \mathrm{e}^{-i\Omega(t-t')}. \tag{42}$$

We also define the spectral function

$$\rho(t-t') = \left\langle \left[a(t), a^\dagger(t')\right] \right\rangle = \mathrm{e}^{-i\Omega(t-t')}, \tag{43}$$

as well as the statistical correlator

$$F(t-t') = \left\langle \left\{a(t), a^\dagger(t')\right\} \right\rangle = \mathrm{e}^{-i\Omega(t-t')}(1+2n_\Omega) \tag{44}$$

As was the case for the retarded and the advanced propagators, the spectral function is independent of the temperature. Note that

$$G^R(t-t') - G^A(t-t') = i\mathrm{e}^{-i\Omega(t-t')} = i\rho(t-t'). \tag{45}$$

Finally, one easily establishes the (general) relation between the time-ordered and the retarded propagator

$$G(t) = G^R(t) + iG^<(t). \tag{46}$$

By taking a Fourier transform, one obtains

$$\begin{aligned}
\rho(\omega) &= \textstyle\int \mathrm{d}t\, \mathrm{e}^{i\omega t}\rho(t) = 2\pi\delta(\omega-\Omega),\\
F(\omega) &= 2\pi\delta(\omega-\Omega)(1+2n_\Omega),\\
G^>(\omega) &= 2\pi\delta(\omega-\Omega)(1+n_\Omega),\\
G^<(\omega) &= 2\pi\delta(\omega-\Omega)n_\Omega.
\end{aligned} \tag{47}$$

One then verifies the general relation (KMS)

$$G^>(\omega) = \mathrm{e}^{\beta\omega}G^<(\omega). \tag{48}$$

The Fourier transform of the retarded function is given by

$$\begin{aligned}
G^R(\omega) &= i\int \mathrm{d}t\, \mathrm{e}^{i\omega t}\theta(t)\mathrm{e}^{-i\Omega t} = i\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\mathrm{e}^{-i\Omega t}\mathrm{e}^{-\eta t}\\
&= \frac{1}{\Omega-\omega-i\eta}.
\end{aligned} \tag{49}$$

The retarded function has a pole at $\omega = \Omega - i\eta$, and is (quite generally) analytic in the upper plane of complex ω. The Fourier transform of G^A is defined by

$$\begin{aligned}
G^A(\omega) &= -i\int \mathrm{d}t\mathrm{e}^{i\omega t}\theta(-t)\mathrm{e}^{-i\Omega t} = -i\int_{-\infty}^0 \mathrm{d}t\mathrm{e}^{i\omega t}\mathrm{e}^{-i\Omega t}\mathrm{e}^{\eta t}\\
&= \frac{1}{\Omega-\omega+i\eta} = G^{R*}(\omega) = G(z\to\omega-i\eta).
\end{aligned} \tag{50}$$

We have

$$G^R(\omega) - G^A(\omega) = 2\pi i\delta(\omega-\Omega) = i\rho(\omega). \tag{51}$$

Thus $G^R - G^A$ is purely imaginary. Note also that the spectral function is twice the imaginary part of the retarded Green's function.

The Fourier transform of the time-ordered propagator can be deduced from that of the retarded propagator by using the relation (46) above. We get

$$\begin{aligned} G(\omega) &= G^R(\omega) + G^<(\omega) = \frac{1}{\Omega - \omega - i\eta} + 2\pi i\, \delta(\omega - \Omega) n_\Omega \\ &= P\left(\frac{1}{\Omega - \omega}\right) + i\pi\, \delta(\omega - \Omega)\,(1 + 2n_\Omega). \end{aligned} \tag{52}$$

The last line exhibits the real and imaginary parts of the time-ordered propagator.

We also consider the analytic propagator

$$G(z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{\rho(\omega')}{\omega' - z}. \tag{53}$$

The Matsubara propagator is given by

$$G(z = i\omega_n) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{\rho(\omega')}{\omega' - i\omega_n} \tag{54}$$

The imaginary time propagator is obtained as

$$\begin{aligned} G(\tau) &= \frac{1}{\beta} \sum_n G(i\omega_n) \mathrm{e}^{-i\omega_n \tau} = \frac{1}{\beta} \sum_n \mathrm{e}^{-i\omega_n \tau} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega'}{2\pi} \frac{\rho(\omega')}{\omega' - i\omega_n} \\ &= \mathrm{e}^{-\Omega\tau} \left[\theta(\tau)(1 + n_\Omega) + \theta(-\tau) n_\Omega.\right] \end{aligned} \tag{55}$$

The retarded propagator is obtained by the analytic continuation

$$G^R(\omega) = G(z \to \omega + i\eta). \tag{56}$$

The time-ordered propagator is given by

$$G(\omega) = \theta(\omega) G(z \to \omega + i\eta) + \theta(-\omega) G(z \to \omega - i\eta). \tag{57}$$

Thus, the time-ordered propagator identifies with the retarded propagator for positive ω and with the advanced propagator for negative ω.

2. The bath as a single oscillator (2)

We now reexpress the previous results in terms of corelators involving not the creation operators but rather the coordinate of the oscillator (getting closer to scalar field theory). We set

$$\varphi(t) = \sqrt{\frac{\hbar}{2M\Omega}} \left(a\, \mathrm{e}^{-i\Omega t} + a^\dagger \mathrm{e}^{i\Omega t}\right). \tag{58}$$

The commutator and anticommutator functions are given by

$$\begin{aligned}\rho_\varphi(t-t') = \langle[\varphi(t),\varphi(t')]\rangle = -i\tfrac{\hbar}{M\Omega}\sin\Omega(t-t'),\\ F_\varphi(t-t') = \langle\{\varphi(t),\varphi(t')\}\rangle = \tfrac{\hbar}{M\Omega}(1+2n_\Omega)\cos\Omega(t-t').\end{aligned} \tag{59}$$

The fact that ρ is purely imaginary while F is real follows from the hermiticity of φ which entails

$$\langle\varphi(t)\varphi(t')\rangle^* = \langle\varphi(t')\varphi(t)\rangle. \tag{60}$$

The retarded and advanced functions are then given by

$$\begin{aligned}G^R_\varphi(t,t') &= \frac{\hbar}{M\Omega}\theta(t-t')\sin\Omega(t-t')\\ G^A_\varphi(t,t') &= -\frac{\hbar}{M\Omega}\theta(t'-t)\sin\Omega(t-t')\end{aligned} \tag{61}$$

and $G^R_\varphi(t-t') - G^A_\varphi(t-t') = i\rho_\varphi(t-t')$, as expected. Note that

$$G^A_\varphi(t-t') = \frac{\hbar}{M\Omega}\theta(t'-t)\sin\Omega(t'-t) = G^R_\varphi(t'-t). \tag{62}$$

These propagators are related to those of the previous subsection. For instance,

$$\begin{aligned}G^R_\varphi(t) &= \frac{\hbar}{M\Omega}\theta(t)\sin\Omega t = \frac{\hbar}{M\Omega}\theta(t)\frac{\mathrm{e}^{i\Omega t}-\mathrm{e}^{-i\Omega t}}{2i}\\ &= \left[G^R_\Omega(t) - G^R_{-\Omega}(t)\right],\end{aligned} \tag{63}$$

where $G^R_\Omega(t)$ is the retarded propagator for the creation/annihilation operators. We have also

$$\begin{aligned}G^>_\varphi(t-t') &= \langle\varphi(t)\varphi(t')\rangle\\ &= \frac{\hbar}{2M\Omega}\left\{(1+2n_\Omega)\cos\Omega(t-t') - i\sin\Omega(t-t')\right\},\end{aligned} \tag{64}$$

and

$$\begin{aligned}G^<_\varphi(t-t') &= \langle\varphi(t')\varphi(t)\rangle\\ &= \frac{\hbar}{2M\Omega}\left\{(1+2n_\Omega)\cos\Omega(t-t') + i\sin\Omega(t-t')\right\},\end{aligned} \tag{65}$$

A simple calculation shows then that the time-ordered propagator is given by

$$G_\varphi(t-t') = i\frac{\hbar}{2M\Omega}\left\{(1+2n_\Omega)\cos\Omega|t-t'| - i\sin\Omega|t-t'|\right\}. \tag{66}$$

Taking the Fourier transforms, one gets

$$\rho_\varphi(\omega) = \frac{\pi\hbar}{M\Omega}\left[\delta(\omega-\Omega) - \delta(\omega+\Omega)\right]. \tag{67}$$

Note that $\rho_\varphi(\omega)$ is an odd function of ω

$$\rho_\varphi(-\omega) = -\rho_\varphi(\omega). \tag{68}$$

Note also that

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\rho_\varphi(\omega) = 0, \qquad \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\omega\rho_\varphi(\omega) = 1. \tag{69}$$

The anticommutator function is an even function of ω given by

$$F_\varphi(\omega) = \frac{\hbar}{M\Omega}(1+2n_\Omega)\left[\delta(\omega-\Omega) + \delta(\omega+\Omega)\right]. \tag{70}$$

After Fourier transform one gets

$$G_\varphi^R(\omega) = \frac{\hbar}{2M\Omega}\left(\frac{1}{\Omega-\omega-i\eta} + \frac{1}{\Omega+\omega+i\eta}\right) \tag{71}$$

$$G_\varphi^>(\omega) = \frac{\pi\hbar}{M\Omega}\left\{(1+n_\Omega)\delta(\omega-\Omega) + n_\Omega\delta(\omega+\Omega)\right\} \tag{72}$$

$$G_\varphi^<(\omega) = \frac{\pi\hbar}{M\Omega}\left\{n_\Omega\delta(\omega-\Omega) + (1+n_\Omega)\delta(\omega+\Omega)\right\} \tag{73}$$

Note that $G_\varphi^>(\omega) - G_\varphi^<(\omega) = \rho(\omega)$ as it should. One also verifies that

$$\begin{aligned} G_\varphi(\omega) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\frac{\rho_\varphi(\omega')}{\omega'-\omega-i\eta} + i\rho_\varphi(\omega)n(\omega) \\ &= P\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\frac{\rho_\varphi(\omega')}{\omega'-\omega} + \frac{i}{2}\rho_\varphi(\omega)(1+2n(\omega)). \end{aligned} \tag{74}$$

In particular, for $\omega = 0$ we get

$$G_\varphi(\omega=0) = P\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{2\pi}\frac{\rho_\varphi(\omega')}{\omega'} + iT\lim_{\omega\to 0}\frac{\rho_\varphi(\omega)}{\omega}. \tag{75}$$

This is a general formula. In the present case, there is no spectral weight at $\omega = 0$, hence no imaginary part.

3. The influence functional for a particle coupled to an oscillator

We consider the system described by the hamiltonian

$$H_S = \frac{P^2}{2M} + V(X). \tag{76}$$

The bath is consituted by a single oscillator of mass m and frequency ω. That is, the action of the bath reads

$$S_B = \int \mathrm{d}t \sum_k \frac{m}{2}\left[\dot{\varphi}^2 - \omega^2\varphi^2\right] = -\int \mathrm{d}t\, \frac{m}{2}\left[\varphi(\partial_t^2 + \omega^2)\varphi\right]. \tag{77}$$

We assume the bath oscillator to be linearly coupled to the oscillator, that is

$$H_1 = -g\varphi X. \tag{78}$$

The influence functional can then be obtained by treating H_1 as a perturbation. We get

$$\mathrm{e}^{i\Phi[X]} = \mathrm{Tr}_B\left[\mathcal{D}_B \mathcal{T}_\mathcal{C} \exp\left\{\frac{i}{\hbar} g \int_\mathcal{C} \mathrm{d}t \varphi(\mathrm{t})\mathrm{X}(\mathrm{t})\right\}\right]. \tag{79}$$

For definiteness, one can assume that the oscillator is in contact with a reservoir with which it can exchange energy and which maintains it in thermal equilibrium at temperature T. Then the expression above is a thermal average. Since the action for the field φ is quadratic, the calculation can be done using Wick's theorem. Alternatively, we can calculate Φ as a path integral. Either way, one gets

$$\mathrm{e}^{i\Phi[X]} = \exp\left\{\frac{i}{2\hbar^2}\frac{g^2}{m}\int_\mathcal{C} \mathrm{d}t\mathrm{d}t'\, X(t)\cdot\Delta_\mathcal{C}(t,t')\cdot X(t')\right\} \tag{80}$$

where $\Delta_\mathcal{C}(t,t') = i\langle T_\mathcal{C}\varphi(t)\varphi(t')\rangle$.

III. INFINITE MASS LIMIT

When the heavy quark mass is infinite simple results can be obtained analytically. We recall here some useful results concerning the single particle propagator. The free propagator obeys the Schrödinger equation

$$i\partial_t G_0^>(t,\boldsymbol{r}) = \left(M - \frac{\nabla^2}{2M}\right) G_0^>(t,\boldsymbol{r}), \tag{81}$$

whose solution, corresponding to the initial condition $G_0^>(t=0,\boldsymbol{r})=\delta(\boldsymbol{r})$ reads

$$G_0^>(t,\boldsymbol{r}) = \mathrm{e}^{-iMt}\left(\frac{M}{2\pi it}\right)^{3/2}\mathrm{e}^{iM\boldsymbol{r}^2/2t}. \tag{82}$$

In the infinite mass limit, this propagator remains proportional to $\delta(\boldsymbol{r})$. In the presence of the background field, $G^>(t,\boldsymbol{r})$ is given by the following path integral

$$G^>(t,\boldsymbol{r}) = \int_0^{\boldsymbol{r}}\mathcal{D}z\,\exp\left[i\int_0^t \mathrm{d}t'\,\frac{1}{2}M\dot{z}^2\right]\mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\,A_0^a(u,\boldsymbol{r}(u))t^a\right\} \tag{83}$$

Recall that $U(t,t_0) = U_0(t,t_0)U_I(t,t_0)$, so that the second factor may be interpreted as $U_I(t,0)$. Indeed going to the interaction representation for the system, we get $\boldsymbol{r}\to\boldsymbol{r}_I(t)$, with

$$\hat{\boldsymbol{r}}_I(t) = \mathrm{e}^{iH_Qt}\hat{\boldsymbol{r}}\mathrm{e}^{-iH_Qt}, \qquad H_Q = \frac{\boldsymbol{p}^2}{2M}. \tag{84}$$

An alternative expression in terms of the evolution operator is

$$U(t,t_0) = U_0(t,t_0)\mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\int_{\boldsymbol{r}} A_0^a(u,\boldsymbol{r})\rho_a^I(u,\boldsymbol{r})\right\}. \tag{85}$$

This can be used to calculate the S-matrix element

$$\langle\boldsymbol{r}_2,m|U(t,t_0)|\boldsymbol{r}_1,n\rangle. \tag{86}$$

The difficulty of the calculation is that U_0 is diagonal in the momentum representation, while the interaction part has a simple expression in coordinate space.

The propagator of a single heavy quark is the average S-matrix element where we ask the final state of the plasma to be identical to the initial state, that is

$$\sum_m \frac{\mathrm{e}^{-E_m^B/T}}{Z_B}\langle\boldsymbol{r}_2,m|U(t,t_0)|\boldsymbol{r}_1,m\rangle. \tag{87}$$

When the plasma is described by a quadratic hamiltonian, we can apply Wick's theorem to calculate the average over the field of the plasma. In the case of QCD, things are more complicated. One can get insight by considering the infinite mass limit.

A. Single particle propagator in the infinite mass limit

Consider then the infinite mass limit. In this case, the trajectory of the heavy quark is trivial, and we get simply

$$\sum_m \frac{\mathrm{e}^{-E_m^B/T}}{Z_B}\langle \boldsymbol{r}_2, m|U(t,t_0)|\boldsymbol{r}_1, m\rangle = \delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\langle \mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\, A_0^a(u)t^a\right\}\rangle_0, \tag{88}$$

where the subscript 0 indicates that the expectation value is taken in the plasma equilibrium state. The expectation value is often expressed as the exponential of the so-called "influence functional" (not only in the infinite mass limit):

$$\mathrm{e}^{i\Phi} = \langle \mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\, A_0^a(u)t^a\right\}\rangle_0. \tag{89}$$

This is also the average of a Wilson line. To calculate it, we may expand perturbatively

$$\begin{aligned}\mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\, A_0^a(u)t^a\right\} &\simeq 1 + ig\int_0^t \mathrm{d}u\, A_0^a(u)\, t^a \\ &\quad -\frac{g^2}{2}\int_0^t \mathrm{d}u_1\int_0^t \mathrm{d}u_2\, \mathrm{T}\left[A_0^a(u_1)\, t^a_{u_1}\, A_0^b(u_2)\, t^b_{u_2}\right] + \cdots\end{aligned} \tag{90}$$

where the subscripts u_1 and u_2 on the color matrices is here to recall the ordering with time. Taking the average over the plasma, one obtains

$$\langle \mathrm{T}\left[A_0^a(u_1)\, t^a_{u_1}\, A_0^b(u_2)\, t^b_{u_2}\right]\rangle = \delta^{ab}\langle \mathrm{T}\left[A_0^a(u_1)A_0^a(u_2)\right]\rangle \mathrm{T}\left[t^a_{u_1}\, t^a_{u_2}\right]. \tag{91}$$

The correlator $\langle \mathrm{T}\left[A_0^a(u_1)A_0^a(u_2)\right]\rangle$ is independent of the color index a (the plasma is color neutral). One can then perform freely the sum over a of the color matrices, and get $t^a_{u_1}\, t^a_{u_2} = C_F$, independently of the order of the color matrices.

One then obtains

$$\langle \mathrm{Texp}\left\{ig\int_0^t \mathrm{d}u\, A_0^a(u)t^a\right\}\rangle_0 \simeq 1 + \frac{ig^2 C_F}{2}\int_0^t \mathrm{d}u_1\int_0^t \mathrm{d}u_2\, \Delta(u_1-u_2) + \cdots \tag{92}$$

with (no sum over a here)

$$\Delta(u_1,u_2) = i\langle \mathrm{T}\left[A_0^a(u_1)\, A_0^a(u_2)\right]\rangle. \tag{93}$$

If the propagating parton is a gluon instead of a quark, the only change in Eq. (92) is the color factor which becomes $C_A = N_c$ instead of C_F.

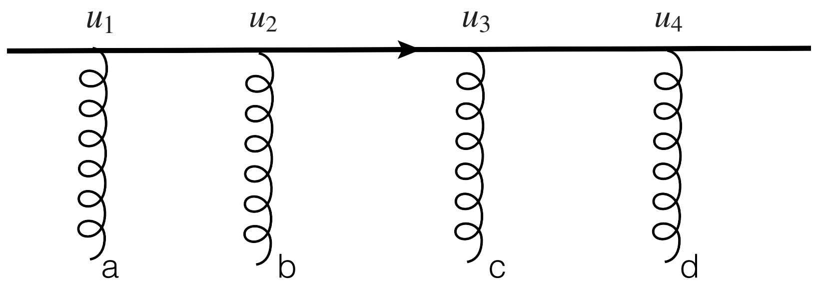

FIG. 2: The Wilson line with four insertions of the color field.

In QED, the all order result is just the exponential of the second order result. In that case, the influence functional reads

$$\Phi = \frac{g^2 C_F}{2} \int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2 \, \Delta(u_1 - u_2). \tag{94}$$

This is a priori not the case in QCD, because of the non commutation of the color matrices. The problem shows up at order g^4. The corresponding contribution reads

$$\begin{aligned} &\frac{g^4}{4!} \int_0^t \prod_{i=1}^4 \mathrm{d}u_i \langle \mathrm{T} \left[A_0^a(u_1) t_{u_1}^a \, A_0^b(u_2) t_{u_2}^b \, A_0^c(u_3) t_{u_3}^c \, A_0^d(u_4) t_{u_4}^d \right] \rangle \\ &= \frac{g^4}{4!} \int_0^t \prod_{i=1}^4 \mathrm{d}u_i \langle \mathrm{T} \left[A_0^a(u_1) A_0^b(u_2) A_0^c(u_3) A_0^d(u_4) \right] \rangle \mathrm{T} \left[t_{u_1}^a \, t_{u_2}^b \, t_{u_3}^c \, t_{u_4}^d \right]. \end{aligned} \tag{95}$$

Application of Wick's theorem (assuming Gaussian average) yields

$$\begin{aligned} &\langle \mathrm{T} \left[A_0^a(u_1) A_0^b(u_2) A_0^c(u_3) A_0^d(u_4) \right] \rangle = \langle \mathrm{T} A_0^a(u_1) A_0^b(u_2) \rangle \langle \mathrm{T} A_0^c(u_3) A_0^d(u_4) \rangle \\ &+ \langle \mathrm{T} A_0^a(u_1) A_0^c(u_3) \rangle \langle \mathrm{T} A_0^b(u_2) A_0^d(u_4) \rangle + \langle \mathrm{T} A_0^a(u_1) A_0^d(u_4) \rangle \langle \mathrm{T} A_0^b(u_2) A_0^c(u_3) \rangle \\ &= -\delta^{ab} \delta^{cd} \Delta(u_{12}) \Delta(u_{34}) - \delta^{ac} \delta^{bd} \Delta(u_{13}) \Delta(u_{24}) - \delta^{ad} \delta^{bc} \Delta(u_{14}) \Delta(u_{23}). \end{aligned} \tag{96}$$

We need to evaluate three color factors:

$$\left[t_{u_1}^a \, t_{u_2}^b \, t_{u_3}^c \, t_{u_4}^d \right] \delta^{ab} \delta^{cd} = C_F^2 = \left[t_{u_1}^a \, t_{u_2}^b \, t_{u_3}^c \, t_{u_4}^d \right] \delta^{ad} \delta^{bc}, \tag{97}$$

and

$$\left[t_{u_1}^a \, t_{u_2}^b \, t_{u_3}^c \, t_{u_4}^d \right] \delta^{ac} \delta^{bd} = \left[t_{u_1}^a \, t_{u_2}^b \, t_{u_3}^a \, t_{u_4}^b \right] \tag{98}$$

These color factors enter the calculation of the three diagrams displayed in Fig. 3 and which take care of all the topological configurations associated with the various time ordering.

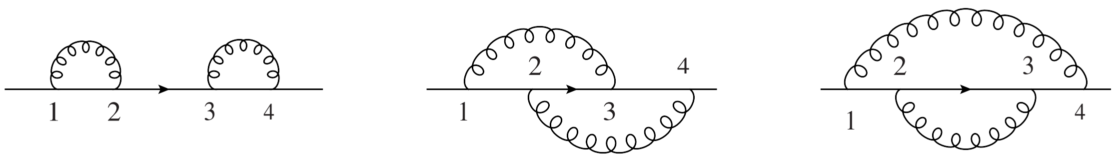

FIG. 3: These three diagrams illustrate the three color structures that enter the calculation of the HQ propagator at order g^4. While the first diagram appears in the exponential of the second order result, this is not the case of the other diagrams. We may however argue that the last two diagrams become subleading in the large time limit, as they are then growing as t, while the first diagram grows as t^2.

B. Single particle density matrix in the infinite mass limit

Let us move now to the case of the density matrix. In the infinite mass limit, this is of the form

$$\langle \boldsymbol{r}_1\alpha_1|\rho_A|\boldsymbol{r}_2\alpha_2\rangle = \delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\langle\alpha_1|\rho_A|\alpha_2\rangle. \tag{99}$$

We obtain, limiting the expansion to second order,

$$\begin{aligned}\rho_A(t) &= \rho_A(t_0)+\frac{ig^2C_F}{2}\int_0^t \mathrm{d}u_1\int_0^t \mathrm{d}u_2[\Delta(u_{12})+\tilde{\Delta}(u_{12})]\rho_A(t_0)\\ &+ g^2\int_0^t \mathrm{d}u_1\int_0^t \mathrm{d}u_2\; t^a_{u_1}\rho_A(t_0)t^a_{u_2}\Delta^<(u_{12}),\end{aligned} \tag{100}$$

where, e.g., $\Delta(u_{12})$ stands for $\Delta(u_{12},\boldsymbol{r}_1-\boldsymbol{r}_2=0)$, and similarly for the other functions. If we assume that the initial density matrix is a color singlet, $\rho_A(t_0)$ commutes with t^a and we get

$$\rho_A(t) = \rho_A(t_0) + \frac{ig^2C_F}{2}\int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2[\Delta(u_{12}) + \tilde{\Delta}(u_{12})]\rho_A(t_0) + g^2C_F \int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2\, \rho_A(t_0)\Delta^<(u_{12}). \tag{101}$$

That is, the density matrix at time t is also a color singlet, $\langle\alpha_1|\rho_A(t)|\alpha_2\rangle \propto \delta_{\alpha_1\alpha_2}$. Thus, in the infinite mass limit, and to second order in g,

$$\rho_A(t) = \rho_A(t_0)\left(1 + \frac{g^2C_F}{2}\int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2[i\Delta(u_{12}) + i\tilde{\Delta}(u_{12}) + 2\Delta^<(u_{12})]\right), \tag{102}$$

where the correlators Δ are evaluated at coincident points. But this second order correction vanishes because of the relations

$$i\Delta(u_{12}) + i\tilde{\Delta}(u_{12}) + \Delta^<(u_{12}) + \Delta^>(u_{12}) = 0, \qquad \Delta^<(u_{12}) = \Delta^>(u_{21}). \tag{103}$$

This is as it should be. In the infinite mass limit, the density matrix is diagonal, the diagonal element being the density. Since the interaction cannot change the density of the heavy quark it is natural to observe the explicit cancellation of the second order effects.

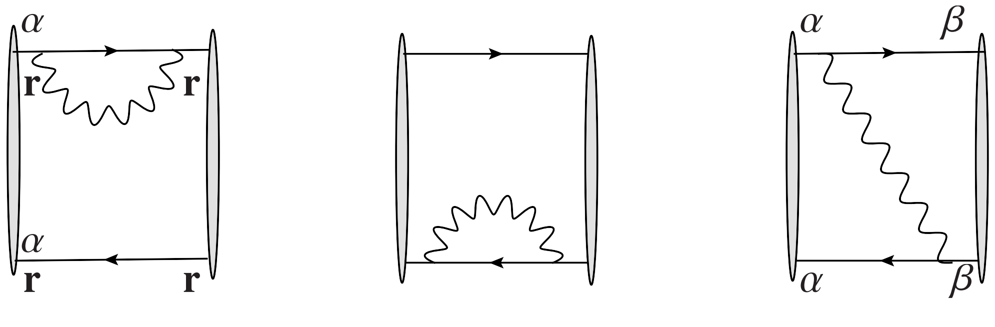

FIG. 4: The three diagrams which contribute to the second order correction to the density matrix, according to Eq. (102), and whose sum vanishes in the infinite mass limit. The first two diagrams correspond to the contribution proportional to Δ and $\tilde{\Delta}$, respectively, while the last one corresponds to the contribution proportional to $2\Delta^<$.

C. Two particle propagator in the infinite mass limit (QED)

We consider a pair of (abelian) quark and antiquark, located at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. The total charge density is

$$\rho(\boldsymbol{x}) = \delta(\boldsymbol{x} - \hat{\boldsymbol{r}}) - \delta(\boldsymbol{x} - \hat{\boldsymbol{r}}). \tag{104}$$

The matrix element of the evolution operator reads

$$\begin{aligned}\langle \boldsymbol{r}_1 \boldsymbol{r}_2 | U(t,t_0) | \boldsymbol{r}_3, \boldsymbol{r}_4 \rangle &= \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\delta(r_2 - \boldsymbol{r}_4) \\ &\times \mathrm{Texp}\left\{ ig \int_0^t \mathrm{d}u \, [A_0(u, \boldsymbol{r}_1)) - A_0(u, \boldsymbol{r}_2)] \right\}.\end{aligned} \tag{105}$$

Expanding to second order one gets for the coefficient of $-g^2/2$,

$$\begin{aligned}&\int_{u_1,u_2} \mathrm{T}\left\{ [A_0(u_1, \boldsymbol{r}_1)) - A_0(u_1, \boldsymbol{r}_2)] \, [A_0(u_2, \boldsymbol{r}_1)) - A_0(u_2, \boldsymbol{r}_2)] \right\} \\ &= \int_{u_1,u_2} \mathrm{T}[A_0(u_1, \boldsymbol{r}_1)) A_0(u_2, \boldsymbol{r}_1))] + \int_{u_1,u_2} \mathrm{T}[A_0(u_1, \boldsymbol{r}_2) A_0(u_2, \boldsymbol{r}_2)] \\ &- \int_{u_1,u_2} \mathrm{T}[A_0(u_1, \boldsymbol{r}_1)) A_0(u_2, \boldsymbol{r}_2)] - \int_{u_1,u_2} \mathrm{T}[A_0(u_1, \boldsymbol{r}_2) A_0(u_2, \boldsymbol{r}_1))].\end{aligned} \tag{106}$$

Note that only T-products enter this calculation.

By taking the average over the plasma degrees of freedom, one obtains

$$\begin{aligned}&= -i \int_{u_1,u_2} \Delta(u_1, \boldsymbol{r}_1; u_2, \boldsymbol{r}_1) - i \int_{u_1,u_2} \Delta(u_1, \boldsymbol{r}_2; u_2, \boldsymbol{r}_2) \\ &+ i \int_{u_1,u_2} \Delta(u_1, \boldsymbol{r}_1; u_2, \boldsymbol{r}_2) + i \int_{u_1,u_2} \Delta(u_1, \boldsymbol{r}_2; u_2, \boldsymbol{r}_1) \\ &= -i \int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2 \; (2\Delta(u_{12}, 0) - \Delta(u_{12}, \boldsymbol{r}_{12}) - \Delta(u_{12}, \boldsymbol{r}_{21})) \\ &= 2i \int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2 \; (\Delta(u_{12}, |\boldsymbol{r}_{12}|) - \Delta(u_{12}, 0))\end{aligned} \tag{107}$$

One then obtains

$$\begin{aligned}\sum_m \frac{\mathrm{e}^{-E_m^B/T}}{Z_B} \langle \boldsymbol{r}_1 \boldsymbol{r}_2; m | U(t,t_0) | \boldsymbol{r}_3, \boldsymbol{r}_4; m \rangle &= \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\delta(r_2 - \boldsymbol{r}_4) \\ &\times \langle \mathrm{Texp}\left\{ ig \int_0^t \mathrm{d}u \, [A_0(u, \boldsymbol{r}_1)) - A_0(u, \boldsymbol{r}_2)] \right\} \rangle \\ &= \delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\delta(r_2 - \boldsymbol{r}_4) \mathrm{e}^{i\Phi}\end{aligned} \tag{108}$$

with

$$\Phi = -g^2 \int_0^t \mathrm{d}u_1 \int_0^t \mathrm{d}u_2 \, [\Delta(u_{12}, |\boldsymbol{r}_{12}|) - \Delta(u_{12}, 0)] \, . \tag{109}$$

This is the QED influence functional obtained earlier (Eq. (34)).

IV. LOW FREQUENCY EXPANSION

The low frequency approximation reads

$$\begin{aligned} \Delta(\tau) &= \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega\tau} \left[\Delta(\omega = 0) + \omega\Delta'(\omega = 0)\right] \\ &\simeq \delta(\tau)\Delta(\omega = 0) + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}\delta(\tau)\Delta'(\omega = 0). \end{aligned} \tag{110}$$

We assume that the same expansion holds for the two components $\Delta^>$ and $\Delta^<$ as well. Note that although we have not indicated explicitly the spatial coordinates, the correlators are invariant under the permutation of the coordinate of their end points. Thus for instance

$$\Delta^>(-\tau, \boldsymbol{x}) = \Delta^<(\tau, -\boldsymbol{x}) = \Delta^<(\tau, \boldsymbol{x}). \tag{111}$$

We have the following results

$$\int_0^\infty \mathrm{d}\tau\tau\Delta(\tau) = i \int_0^\infty \mathrm{d}\tau\tau\Delta^>(\tau), \qquad \int_0^\infty \mathrm{d}\tau\tau\tilde{\Delta}(\tau) = i \int_0^\infty \mathrm{d}\tau\tau\Delta^<(\tau). \tag{112}$$

as well as

$$\begin{aligned} &\int_{-\infty}^\infty \mathrm{d}\tau \, \Delta(\tau) = \Delta(0) = \Delta^R(0) + i\Delta^<(0), \\ &\int_{-\infty}^\infty \mathrm{d}\tau \, \tilde{\Delta}(\tau) = \tilde{\Delta}(0) = -\Delta^A(0) + i\Delta^<(0), \\ &\int_{-\infty}^\infty \mathrm{d}\tau \, \Delta^>(\tau) = \int_{-\infty}^\infty \mathrm{d}\tau \, \Delta^<(\tau) = \Delta^>(0) = \Delta^<(0), \\ &\int_0^\infty \mathrm{d}\tau\tau \, \Delta^>(\tau) = i \int_0^\infty \mathrm{d}\tau\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\delta(\tau) \, \Delta'^>(0) = -\frac{i}{2}\Delta'^>(0). \end{aligned} \tag{113}$$

Note that in the right hand sides of these equations, $\Delta(0)$ stands for $\Delta(\omega = 0)$. I have used the fact that $\delta(-\tau) = \delta(\tau)$ so that

$$\int_0^\infty \mathrm{d}\tau\delta(\tau) = \frac{1}{2}, \qquad \int_0^\infty \mathrm{d}\tau\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\delta(\tau) = -\frac{1}{2}. \tag{114}$$

We have also

$$\int_0^\infty d\tau \Delta^>(\tau, \boldsymbol{x})) = \frac{1}{2}\int_0^\infty d\tau \Delta^>(\tau, \boldsymbol{x}) + \frac{1}{2}\int_{-\infty}^0 d\tau \Delta^<(\tau, \boldsymbol{x}) \\ = -\frac{i}{2}\int_{-\infty}^\infty d\tau \Delta(\tau, \boldsymbol{x}) = -\frac{i}{2}\Delta_R(0, \boldsymbol{x}) + \frac{1}{2}\Delta^<(0, \boldsymbol{x}), \tag{115}$$

where we have assumed that $\Delta^<(\tau, \boldsymbol{x}) = \Delta^<(\tau, -\boldsymbol{x})$. Similarly,

$$\int_0^\infty d\tau \Delta^<(\tau, \boldsymbol{x}) = -\frac{i}{2}\int_{-\infty}^\infty d\tau \tilde{\Delta}(\tau, \boldsymbol{x}) = \frac{i}{2}\Delta_A(0, \boldsymbol{x}) + \frac{1}{2}\Delta^<(0, \boldsymbol{x}), \tag{116}$$

Now we recall that

$$\Delta^R(0) = \Delta^A(0) = -V(\boldsymbol{r}), \\ \Delta^<(0) = -W(\boldsymbol{r}), \qquad \Delta'^<(0) = -\Delta'^>(0) = \frac{\beta}{2}W(\boldsymbol{r}). \tag{117}$$

We have therefore

$$\int_0^\infty d\tau \Delta^>(\tau; \boldsymbol{x})) = \frac{i}{2}V(\boldsymbol{x}) - \frac{1}{2}W(\boldsymbol{x}), \\ \int_0^\infty d\tau\, \Delta^<(\tau; \boldsymbol{x}) = -\frac{i}{2}V(\boldsymbol{x}) - \frac{1}{2}W(\boldsymbol{x}), \\ \int_0^\infty d\tau\, \tau\, \Delta^>(\tau; \boldsymbol{x})) = -\frac{i}{2}\Delta'^>(0) = \frac{i}{4T}W(\boldsymbol{x}), \\ \int_0^\infty d\tau\, \tau\, \Delta^<(\tau; \boldsymbol{x}) = \frac{i}{2}\Delta'^<(0) = -\frac{i}{4T}W(\boldsymbol{x}). \tag{118}$$

Another way to obtain these results uses the Fourier transform. Recall that

$$\Delta(\tau, \boldsymbol{x}) = \int_\omega e^{-i\omega\tau}\Delta(\omega, \boldsymbol{x}). \tag{119}$$

Now

$$\int_0^\infty d\tau e^{-i\omega\tau} = \lim_{\epsilon\to 0^+}\int_0^\infty d\tau e^{-i\omega\tau}e^{-\epsilon\tau} = \frac{1}{i\omega + \epsilon}. \tag{120}$$

Therefore, we have

$$\int_0^\infty d\tau \Delta^>(\tau, \boldsymbol{x}) = \int_\omega \frac{\Delta^>(\omega, \boldsymbol{x})}{i\omega + \epsilon} \\ = -iP\int_\omega \frac{\Delta^>(\omega, \boldsymbol{x})}{\omega} + \frac{1}{2}\Delta^>(\omega = 0, \boldsymbol{x}) \\ = = \frac{i}{2}V(\boldsymbol{x}) - \frac{1}{2}W(\boldsymbol{x}). \tag{121}$$

This provides other expressions of V and W in terms of the Fourier transform of the correlator $\Delta^>$:

$$V(\boldsymbol{x}) = -2P\!\int \frac{\mathrm{d}\omega}{2\pi} \frac{\Delta^>(\omega, \boldsymbol{x})}{\omega} = -\Delta_R(\omega = 0, \boldsymbol{x}), \tag{122}$$

$$W(\boldsymbol{x}) = -\Delta^>(\omega = 0, \boldsymbol{x}) = -\Delta^<(\omega = 0, \boldsymbol{x}). \tag{123}$$

[1] A. Beraudo, J. P. Blaizot, P. Faccioli, and G. Garberoglio. A path integral for heavy-quarks in a hot plasma. *Nucl. Phys.*, A846:104–142, 2010.

[2] Esteban A Calzetta and Bei-Lok B Hu. *Nonequilibrium quantum field theory.* Cambridge University Press, 2009.

[3] Richard Phillips Feynman and FL Vernon Jr. The theory of a general quantum system interacting with a linear dissipative system. *Annals of physics*, 281(1-2):547–607, 2000.

[4] Hagen Kleinert. *Path integrals in quantum mechanics, statistics, polymer physics, and financial markets.* World Scientific, 2009.

[5] Note that we use the notation $\int_{\mathcal{C}}$ to denote either a path integrals where the paths are defined on the contour, as in $\int_{\mathcal{C}} D\boldsymbol{Q}$, or an ordinary integral, as in $\int_{\mathcal{C}} \mathrm{d}t^{\mathcal{C}}$ where the time variable $t^{\mathcal{C}}$ lives on the contour.

[6] We follow closely here the approximation scheme developed in Ref. [1]