Lectures on PBH and SIGW

Gianmassimo Tasinato
e-mail: g.tasinato2208 at gmail.com

1 Plan

- INTRO: A novel source of GW from the early universe (fraction of second after big bang)

  Ingredients: cosmic inflation (bit different variety wrt what you're accustomed to) and good control of dynamics of cosmo fluctuations

  Main realization so far: a byproduct of the production of PBH : hence we start with lightening intro to PBH

- More material:

  - Reviews on PBH:

    Sasaki et al arxiv.org/abs/1801.05235
    Green-Kavanagh arxiv.org/abs/2007.10722 ,
    Ozsoy et al arxiv.org/abs/2301.03600

  - Lectures on PBH: Byrnes and Cole arxiv.org/abs/2112.05716,
  - Reviews on SIGW: arxiv.org/abs/2109.01398 , arxiv.org/abs/2307.06964 (second one is lecture)

2 Motivations for PBH

- Dark matter exists, but so far dit not find a BSM corresponding to it. What about if its made of PBH? a very economical possibility, only based on GR (and special initial conditions)
- Some of LVK events might be associated with BH of no astro origin (too small mass, smaller than Chandrasekhar limit 1.4 solar mass)
- What about SMBH? difficult to produce via astro channels, maybe they're produced in primordial epochs?

3 How to form PBH?

- How do PBH form? Collapse of primordial overdensities $\delta\rho/\bar{\rho}$ in the early universe. We do not talk about stars, just cosmological fluctuations.
- Suppose that in early universe, during RD, distribution of energy density is homogeneous with background value $\bar{\rho}(t)$ independent from time, plus inhomogeneities $\delta\rho/\bar{\rho}$ of different sizes. Homogeneous and isotropic space-time described by FLRW metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \tag{3.1}$$

  Size of observable universe controlled by horizon scale $1/H(t)$. For us, convenient to work with comoving horizon $1/(a(t)\,H(t))$ since we can compare its size with comoving scales.

  - During RD, the comoving horizon size increases with time. $a(t) \sim t^{1/2}$, $H \sim t^{-1}$, $1/(aH) \sim t^{1/2} \sim a$.
  - During inflation, instead, the comoving horizon reduces its size.

  Suppose at a certain point comoving horizon becomes as large as the typical comoving wavelength of a primordial inhomogeneity.

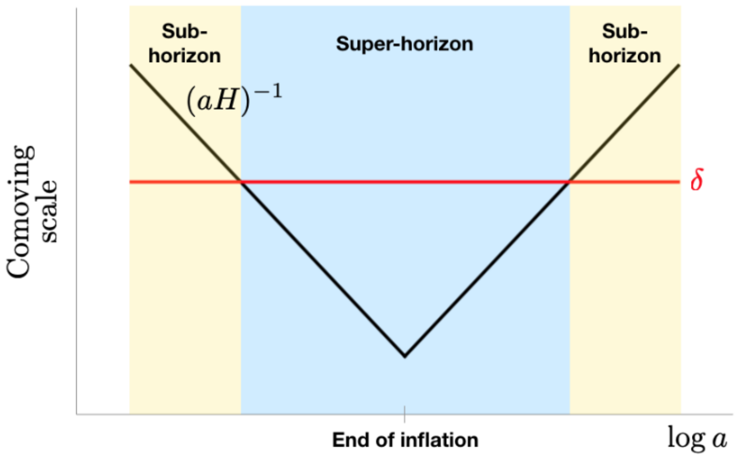


Figure 1: Behaviour of comoving horizon during cosmic history. From Byrnes-Cole

- At this stage, the primordial inhomogeneity enters in causal contact with observed universe: if its size is large enough to contrast RD pressure, it starts to collapse, and form a PBH.

  Roughly, the PBH mass is comparable to the total mass of the energy density within the universe horizon at that time:

$$M_{\rm PBH}\,=\,M_{\rm Hor}\,=\,\rho\,\mathcal{V} \tag{3.2}$$

- What's the threshold δ_c for formation? First estimated by Carr around 50 years ago, using Jeans-type instability arguments for fluids in expanding universe. Result is simple: it depends on the speed square c_s^2 of density fluctuations in RD, corresponding to velocity of pressure wave travels between different regions through the RD medium:

$$\delta_c\,=\,c_s^2 \tag{3.3}$$

In RD, $c_s = 1/\sqrt{3}$ hence $\delta_c\,=\,1/3$. More refined estimates using numerical simulations give

$$\delta_c\,\simeq\,0.45 \qquad \text{more refined value} \tag{3.4}$$

Importantly, notice that $\delta_c \sim \mathcal{O}(0.1)$ hence quite a **large value**!
We need an early universe mechanism able to:

i) Produce inhomogeneities with wavelengths larger than comoving horizon, that will then re-enter the horizon during RD (inflation can do it)

ii) The size of these fluctuations must be large just at the specific scale (ie specific wavelength) we're interested to to produce PBH (inflation can do it, with quite some efforts)

Question: How many PBH we need at formation, to give sizeable amount of DM today?

- Two important quantities

1. $f_{\rm PBH}$: fraction of PBH vs DM **today**

$$f_{\rm PBH}\,=\,\frac{\rho_{\rm PBH}}{\rho_{\rm DM}}\Big|_0 \tag{3.5}$$

2. β the fraction of PBH versus total energy density at time of formation

- hence if $f_{\rm PBH}\,=\,1$ then all DM is PBH. Recall that $\rho_{\rm RD}\,\sim\,1/a^4$, while $\rho_{\rm DM}\,\sim\,1/a^3$, hence PBH relative fraction against total energy density

$\rho_{\rm PBH}/\rho_{\rm tot}$ increases from their formation during RD up to matter-radiation equality.

Hence during RD

$$\left(\frac{\rho_{PBH}}{\rho_{tot}}\right)(t) = \frac{a(t)}{a_{\rm form}}\,\beta \tag{3.6}$$

Then such fraction freezes from $a_{\rm eq}$ onwards (ignoring DE).

- We can then relate $f_{\rm PBH}$ with β as

$$f_{\rm PBH} = \frac{\rho_{\rm PBH}}{\rho_{\rm DM}}\Big|_0 = \frac{\rho_{\rm PBH}}{\rho_{\rm tot}}\Big|_{\rm eq} = \frac{a_{\rm eq}}{a_{\rm form}}\,\beta \tag{3.7}$$

Hence if $a_{\rm eq}/a_{\rm form}$ is very large, we only need a **very small** β:
very few PBH at time of formation can lead to totality of DM today.

- Let's put some numbers, to recollect formulas so far. We assume to work within RD where $a \propto t^{1/2}$, $\rho \propto a^{-4}$, $H \propto \rho^{1/2}$. At formation

$$M_{\rm PBH} = M_{\rm Hor} = \rho\,\mathcal{V} = \frac{4\pi\,\rho}{3}\,H^{-3} \propto \rho^{-1/2} \propto a^2 \propto t_{\rm form} \tag{3.8}$$

Hence during RD mass of PBH linearly depends on time $t_{\rm form}$ when it forms:

$$M_{\rm PBH} = \left(\frac{a_{\rm form}}{a_{\rm eq}}\right)^2 M_{\rm eq} = \left(\frac{a_{\rm form}}{a_{\rm eq}}\right)^2 10^{16}\,M_\odot \tag{3.9}$$

PBH formed at equality are very massive (we do not consider them).

- Recall that $M_\odot = 2\times 10^{33}$ g. Converting to time:

$$M_{\rm PBH} \simeq 10^{15}{\rm g}\left(\frac{t_{\rm form}}{10^{-23}\,{\rm s}}\right) \tag{3.10}$$

where the time pivot value is chosen to identify minimal mass to avoid Hawking evaporation (mass of a small mountain). Smaller mass PBHs, produced at earlier times, are evaporated by today. For example, if we wish to produce a solar-mass PBH, we get $t_{\rm form} \simeq 10^{-5}$ s, and $a_{\rm form}/a_{\rm eq} = 10^{-8}$. Hence we only need

$$\beta = 10^{-8}$$

to produce a population of solar-mass PBH that constitutes DM. Full DM as PBH with these masses is excluded. But they can be a fraction of DM, and contribute to LVK events.

Try yourself the computation for $M_{\rm PBH} \sim 10^{17}$ g: asteroid-size PBH.

- We can also estimate the characteristic wavenumber k (= 1/wavelength) corresponding to PBH of a given mass. Recall $M_{\rm PBH} \simeq M_{\rm Hor}$. During RD, $M_{\rm Hor} \propto a^2$. Then, the comoving scale k of horizon re-entry in RD

$$k \,=\, aH \,\propto\, t^{1/2}\, t^{-1} \,\propto\, a^{-1} \qquad \Rightarrow \qquad M_{\rm Hor} \sim a^2 \sim k^{-2} \tag{3.11}$$

Putting numbers,

$$M_{\rm PBH} \,=\, 10^{13}\, M_{\odot} \left(\frac{\rm Mpc^{-1}}{k} \right)^2 \tag{3.12}$$

for $M_{\rm PBH} \simeq M_{\odot}$, we get $k \simeq 10^7$ Mpc^{-1}. A pretty small scale wrt CMB $k_{\rm CMB} \simeq 10^{-2}$ Mpc^{-1}.

- This's good news because

  1. Such small scales are unconstrained by CMB bounds. Theories are free to speculate about new physics happening at those scales.
  2. If detected, PBH tell us about early universe physics that cant be probed otherwise.

4 Inflation and PBH

- Use **inflation** to produce **PBH**. Inflation is a short period of quasi-exponential expansion

$$a(t) \,\sim\, e^{H_I t} \qquad \text{with} \quad H_I \text{ nearly constant} \tag{4.1}$$

Start from of QM at microscopic scales, and exponential expansion drives quantum effects at astronomical scales (larger than the size of universe horizon).

- Simplest way to get inflation: slow-roll inflation driven by single scalar field. Its EOM

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4.2}$$

with dot derivative along time, prime derivative along field.

- Inflation requires slow-roll parameter ϵ to be small

$$\epsilon \,\equiv\, -\frac{\dot{H}}{H^2} \,=\, \frac{\dot{\phi}^2}{2M_{\rm Pl}^2 H^2} \,\ll\, 1 \tag{4.3}$$

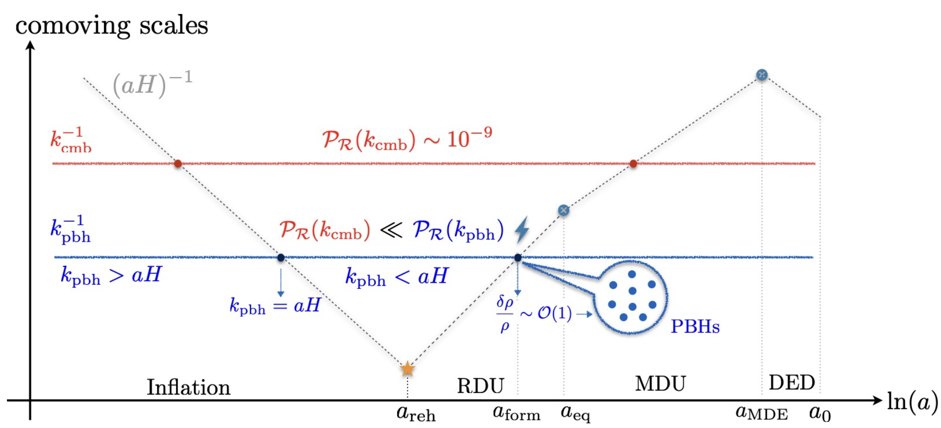


Figure 2: Pictorial representation of PBH production from inflation. From arxiv.org/abs/2301.03600

Moreover, for lasting sufficiently long, also a second slow-roll parameter η should be small

$$\eta = \frac{\dot{\epsilon}}{\epsilon\, H} = 2\epsilon + \frac{2\,\ddot{\phi}}{H\,\dot{\phi}} \ll 1 \tag{4.4}$$

but this last condition can be avoided for a short period of time. Precisely this case is what we'll be interested to.

- Standard slow-roll requires ϵ and η small everytime during inflation. Since η small, $\ddot{\phi}$ small. We can then simplify EOM for scalar:

$$3H\dot{\phi} \simeq -V' \qquad \Rightarrow \qquad \dot{\phi} = -\frac{V'}{3H} \tag{4.5}$$

Since $\dot{\phi}$ enters in ϵ, we want this small: the potential is flat, and the scalar is slowly rolling along the potential profile, with its motion is slowed down by friction. In this case, $|\dot{\phi}|$ is **nearly constant** during all inflation, since by hypothesis $\ddot{\phi}$ is small.

4.1 Dynamics of curvature perturbation

- We now have to work with cosmological perturbation theory: a very interesting subject, mathematically challenging but well developed, that allows us to put together theory with observations in exquisite details. Hence, although rather technical, its worth learning!

- The density contrast $\delta\rho/\rho$ can be expressed in terms of **curvature perturbation** $\mathcal{R}(t, \vec{x})$. This variable can be defined during different epochs (inflation, RD) in terms of fluctuations of energy density relevant at that stage.

- Lets start from inflation

$$\mathcal{R}(t,\vec{x}) = \frac{H\,\delta\phi(t,\vec{x})}{\dot{\phi}(t)} = \frac{\delta\phi(t,\vec{x})}{\sqrt{2\epsilon}\,M_{\rm Pl}} \tag{4.6}$$

during slow-roll.

- Observations are sensitive to correlators among curvature fluctuations, typically expressed in Fourier space. The 2-point function for scalar fluctuations evaluated at horizon exit during inflation, introducing the notion of **power spectrum**

$$\langle \delta\phi_{\vec{k}}(\tau)\delta\phi_{\vec{q}}(\tau)\rangle = \delta(\vec{k}+\vec{q})\,|\delta\phi_{\vec{k}}(\tau)|^2 \quad \Rightarrow \quad \mathcal{P}_{\delta\phi} = \frac{k^3}{2\pi^2}\,|\delta\phi_{\vec{k}}(\tau)|^2$$

can be computed using techniques of QFT in curved space-time

$$\mathcal{P}_{\delta\phi} = \left(\frac{H_I}{2\pi}\right)^2 \quad \Rightarrow \quad \mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2M_{\rm Pl}^2\epsilon}\left(\frac{H_I}{2\pi}\right)^2 \tag{4.7}$$

where these quantities are evaluated at horizon exit during inflation, $k = aH$. **Roughly then, $\mathcal{P}_{\mathcal{R}}$ controls the size of curvature fluctuation**.

- After inflation ends, we can relate **density contrast in RD** $\delta\rho/\bar{\rho} = \delta$ with $\mathcal{R}$ computed during inflation:

$$|\delta| = \frac{4}{9}\left(\frac{k}{aH}\right)^2 |\mathcal{R}| \tag{4.8}$$

Hence to increase the size of $|\delta|$ at certain small scales, we need to increase size of $\mathcal{P}_{\mathcal{R}}$ at those scales. A possible way is to have a short epoch during which ϵ rapidly reduces its size – so to boost the curvature power spectrum.

4.2 Inflection point and USR

- But, for our PBH-production purposes, we wish to consider the possibility that $|\dot{\phi}|$ has a rapid decrease, and $|\eta|$ becomes large for a short amount of time. This boosts the parameter ϵ, helping to increase the size of density contrast/primordial fluctuations at certain scales, so to produce PBH. Among many, lets discuss one realization of a phenomenon called **Ultra Slow Roll** inflation:

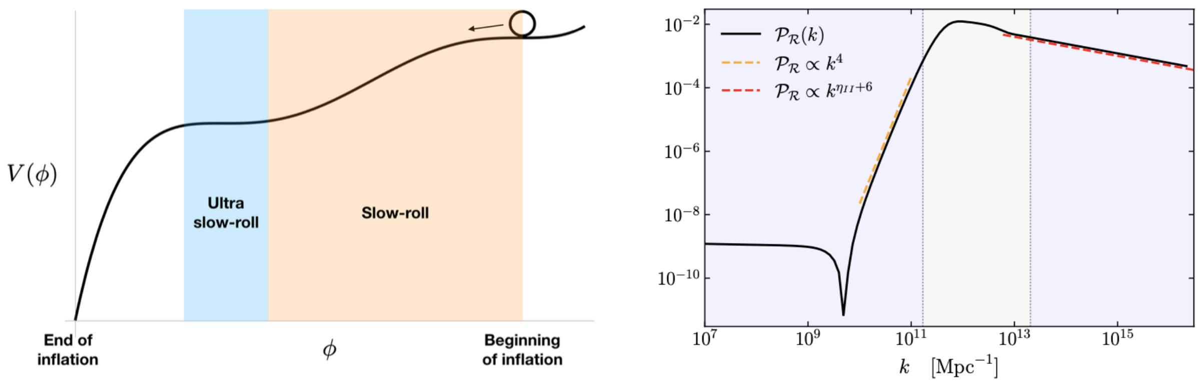


Figure 3: **Left:** Representation of a potential with inflection point. From Byrnes-Cole. **Right:** Plot of the curvature spectrum.

The potential has inflection point region where $V' = 0$. Then scalar EOM is

$$\ddot{\phi} + 3H\dot{\phi} = 0 \quad \Rightarrow \quad \frac{d \ln \dot{\phi}}{dt} = -\frac{d \ln a^3}{dt} \quad \Rightarrow \quad \dot{\phi} \simeq a^{-3} \tag{4.9}$$

The scalar speed is rapidly decreasing in size, reducing rapidly the value of ϵ:

$$\epsilon \simeq a^{-6}$$

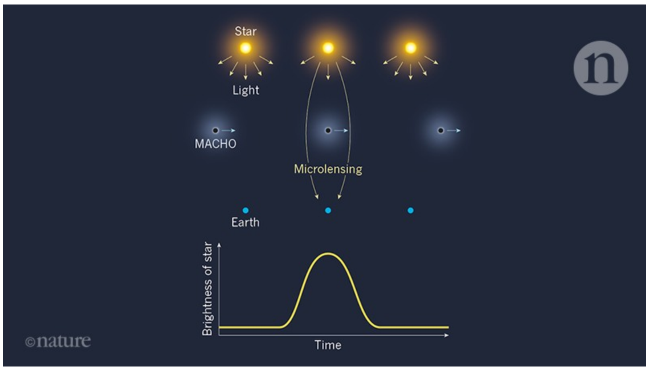
The η parameter is

$$\eta = 2\epsilon + \frac{2}{\dot{\phi}\, H}\left(-3H\dot{\phi}\right) \simeq -6 \tag{4.10}$$

5 Constraints on PBH: present and future

Constraints on presence of PBH can be phrased in terms of $f_{\rm PBH}$ (or β) vs $M_{\rm PBH}$. Interesting to identify mass ranges where $f_{\rm PBH} = 1$ and PBH are totality of DM. But also other mass ranges can be phenomenologically interesting.

- **PBH evaporation** BH temperature is inversely proportional to its mass: $T_{\rm BH} \propto 1/M_{\rm BH}$. Too small PBH are very hot: even if not yet disappeared, their radiation might interfere with observations (CMB) etc. This sets constraints on small-mass PBH.
- **Microlensing** Accurately monitor a number of distant stars, and check whether their observed luminosity changes in time passing in front of an object.



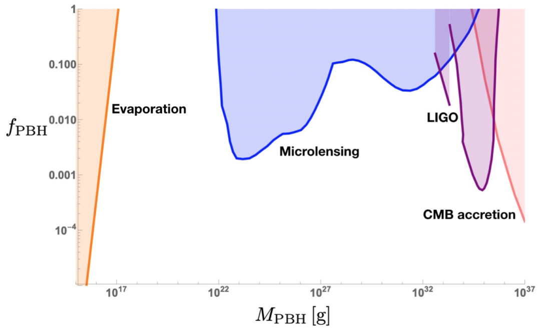


Figure 4: **Left:** Microlensing phenomenon. **Right:** Summary of constraints. From arxiv.org/abs/2112.05716

E.g. Subaru telescope ruled out $f_{\rm PBH} = 1$ for the mass ranges $10^{-12}\,M_\odot \leq M_{\rm PBH} \leq 10^{-6} M_\odot$. Other experiments constrain other mass ranges.

- **LVK constraints** Current GW detections of astro sources (mostly if not all) do not favour $f_{\rm PBH} = 1$ in the mass range $10^0\,M_\odot \leq M_{\rm PBH} \leq 10^2 M_\odot$.
- Interestingly, summing up the constraints, there's an allowed mass range for asteroid size PBH, $10^{-16}\,M_\odot \leq M_{\rm PBH} \leq 10^{-12} M_\odot$, or $10^{17}\,{\rm g} \leq M_{\rm PBH} \leq 10^{22}$ g.
- A further interesting avenue to probe PBH is discussed in the next section:

6 Scalar-induced GW

- This's a timely topic. Possible observational signatures now or in the foreseeable future with GW experiments.
- **Idea**: the starting point is the primordial stochastic background of GW produced during inflation. They are produced by quantum mechanical effects during the early universe.

  At linearized order, the spin-2 tensor modes from inflation h_{ij} (primordial GW) follow an evolution eq

$$h''_{ij}(\tau, k) + \frac{2a'(\tau)}{a(\tau)}\, h'_{ij}(\tau, k) + k^2\, h_{ij}(\tau, k) \;=\; 0 \tag{6.1}$$

  Inflation naturally produces a SGWB. In vanilla inflation, its size is however too small to be directly detected with GW experiments. It might be detected through CMB B-mode observations.

- However, since scalar perturbations acquire large amplitudes for a short-range of scales in PBH-forming models, it makes sense to push perturbation theory at second order.
  Then scalars source tensors at **quadratic order in fluctuations**:

$$h''_{ij}(\tau, k) + \frac{2a'(\tau)}{a(\tau)} h'_{ij}(\tau, k) + k^2 h_{ij}(\tau, k) \;=\; S_{ij}(\tau, k) \tag{6.2}$$

- A bit more details. Decompose metric in scalar and tensor fluctuations as

$$ds^2 \;=\; a^2(\tau)\left[-e^{2\Phi}\, d\tau^2 + e^{-2\Psi}\left(\delta_{ij} + h_{ij}\right)\, dx^i dx^j\right] \tag{6.3}$$

  In absence of anisotropic stress: $\Phi = \Psi$: this is propto curvature fluctuations:

$$|\Phi| \sim \frac{4}{9}\,|\mathcal{R}| \tag{6.4}$$

  Call $\mathcal{H} = a'/a$. The source reads (in RD)

$$S^r_s \;=\; -2\Phi\;(\partial^r\partial_s\,\Phi) + \partial^r\left(\Phi + \frac{\Phi'}{\mathcal{H}}\right)\partial_s\left(\Phi + \frac{\Phi'}{\mathcal{H}}\right) \tag{6.5}$$

  If we amplify curvature fluctuations $\Rightarrow$ amplify primordial spectrum.

- This fact enhances the inflationary tensor spectrum, and the energy density in GW. We can understand this fact in few steps:
  - Pass to Fourier space, $h_{\vec{k}}(\tau)$ being the GW Fourier mode. Rescale $a(\tau)h_{\vec{k}}(\tau) = v_{\vec{k}}(\tau)$. Evolution eq gets rid of term linear on time derivs:

$$v''_{\vec{k}}(\tau) + \left(k^2 - \frac{a''(\tau)}{a(\tau)}\right) v_{\vec{k}}(\tau) = a(\tau) S_{\vec{k}}(\tau) \tag{6.6}$$

  - Formal solution expressed in terms of Green function method

$$h_k(\tau) \;=\; \frac{1}{a(\tau)}\int^\tau d\tilde{\tau}\, g_k(\tau, \tilde{\tau})\, a(\tilde{\tau})\, S_k(\tilde{\tau}) \tag{6.7}$$

    Where Green function:

$$\partial^2_{\tilde{\tau}} g_k(\tau, \tilde{\tau}) + \left(k^2 - \frac{a''}{a}\right) g_k(\tau, \tilde{\tau}) \;=\; \delta(\tau - \tilde{\tau})$$

  - You can compute tensor power spectrum $\mathcal{P}_h = (k^3/(2\pi^2))\,\langle h^2_k\rangle$.

– Then the GW energy density for log-scale at subhorizon:

$$\rho_{\rm GW}(\tau) \;=\; \frac{M_{Pl}^2}{16a^2}\,\langle\overline{\partial_k h_{ij}\partial_k h_{ij}}\rangle \quad\Rightarrow\quad \rho_{\rm GW}(\tau) = \int d\ln k\,\rho_{\rm GW}(\tau,k) \tag{6.8}$$

and the density in GW compared with critical density:

$$\Omega_{\rm GW} \;=\; \frac{\rho_{\rm GW}}{\rho_{\rm crit}} \;=\; \frac{1}{24}\left(\frac{k}{aH}\right)^2\overline{\mathcal{P}_h} \tag{6.9}$$

- The primordial stochastic background produced during inflation gets amplified for a small range of scales. Computations using perturbation theory give, for PBH produced during RD,

$$\Omega_{\rm GW}(k) \;=\; \int_0^\infty dv \int_{|1-v|}^{|1+v|} du\,\mathcal{T}_{\rm RD}(u,v)\,\mathcal{P}_{\mathcal{R}}(uk)\,\mathcal{P}_{\mathcal{R}}(vk) \tag{6.10}$$

These convolution integrals are typical when considering effects of second-order fluctuations.

- **Case study:** for

$$\mathcal{P}_{\mathcal{R}} \;=\; A_s\,\delta\left(\ln(k/k_\star)\right) \tag{6.11}$$

one gets ($\tilde{k} = k/k_\star$)

$$\Omega_{\rm GW}(\tilde{k}) = \frac{3A_{\mathcal{R}}^2}{64}\left(\frac{4-\tilde{k}^2}{4}\right)^2\tilde{k}^2\left(3\tilde{k}-2\right)^2$$
$$\times\left(\pi^2\left(3\tilde{k}^2-2\right)^2\Theta(2\sqrt{3}-3\tilde{k}) + \left(4+\left(3\tilde{k}^2-2\right)\ln\left|1-\frac{4}{3\tilde{k}^2}\right|\right)^2\right)\Theta(2-\tilde{k}) \tag{6.12}$$

The first gentle peak is at $f = k_\star/(\sqrt{3}\,\pi)$. Then there is a resonance at small scales, leading to a rich spectrum profile: easy to distinguish wrt astrophysical signals!

- In general Hence the production of PBH leads to enhancement of SGWB from inflation at characteristic scales, related with PBH properties. In fact, converting to frequencies $k = 2\pi f$, and expressing in Hz, one gets

$$f_{\rm peak}^{\rm GW} \;=\; 1.2\times 10^8\,{\rm Hz}\left(\frac{M_{\rm PBH}}{1\,{\rm g}}\right)^{-1/2} \tag{6.13}$$

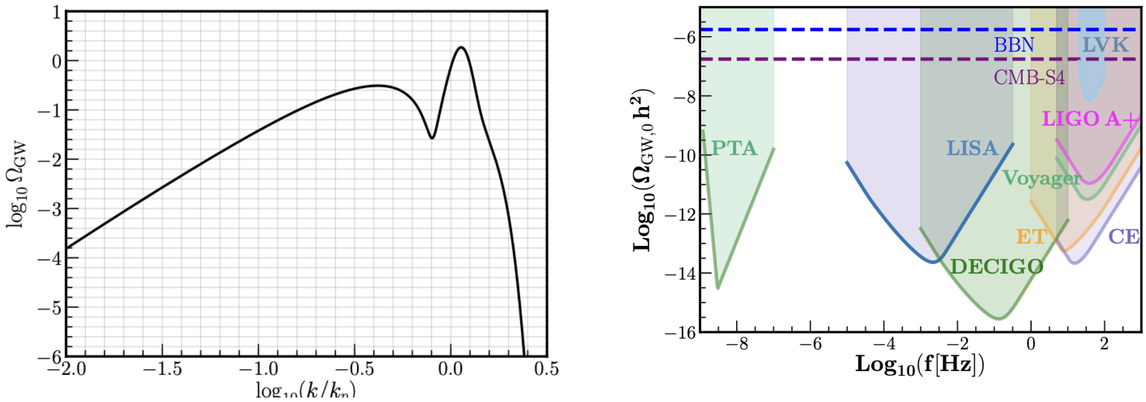


Figure 5: **Left**: GW spectrum from a monochromatic delta-like scalar source. **Right**: sensitivity curves for different experiments. From Domenech.

- Examples
  - Solar mass PBH $M_{\rm PBH} = 10^{34}$ g ($\simeq 5M_\odot$)
    $\Rightarrow f^{\rm GW}_{\rm peak} = 10^8 \times 10^{-17}$ Hz , PTA frequency band.
  - Asteroid mass PBH $M_{\rm PBH} = 10^{22}$ g
    $\Rightarrow f^{\rm GW}_{\rm peak} = 10^8 \times 10^{-11}$ Hz , LISA frequency band.

7 Open questions

i) Better investigate dependence of induced GW on non-linearities (as primordial nonG) as well as non-standard cosmological histories.
   - $\Rightarrow$ Excellent opportunities to probe **very early** universe at scales that can not be probed otherwise
   - $\Rightarrow$ Is there any **further cosmo info** we can squeeze out from the spectrum, if detected?

ii) Are there observational **'smoking gun' signals** of SIGW? Besides frequency profile, look for anisotropies etc...

iii) Are there more **convenient computational strategies** to obtain induced spectra? So far convolution nested integrals are numerically challenging – above for the effects of rapid oscillations in the integrand functions.
   - ∗ Are there analytical tricks to do them?
   - ∗ Are there fast and reliable methods to extract info on the original source from the measured spectrum? Inverse problem of reconstruction. Machine learning?