Projected ensemble in a system with conserved charges with local support

arxiv : 2501.01823

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Outline of this talk

1. Quantum state ensemble - Density matrix and beyond
2. Distribution of states in Hilbert space , Projected Ensemble
3. L- bit Hamiltonain
4. Probability distribution of output bitstrings.
5. Projected ensemble with l-bit Hamiltonian
6. Numerical evidence
7. Conclusion

Ensemble of quantum states

$$\{p(\psi), |\psi\rangle\}$$

$$|\psi_1\rangle\ |\psi_2\rangle\ |\psi_3\rangle \cdots |\psi_2\rangle\ \cdots |\psi_n\rangle$$

Observe $\hat{O}$

Expectation value $\langle \hat{O} \rangle = Tr(\rho^{(1)} \hat{O})$

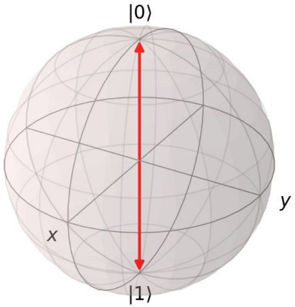
Possible to measure experimentally and depends only on $\rho^{(1)}$

Ensemble of quantum states

$\rho^{(1)}$ doesn't uniquely describe a quantum state ensemble.

$$\rho^{(1)} = \sum_i p(\psi_i)|\psi_i\rangle\langle\psi_i|$$ Density matrix

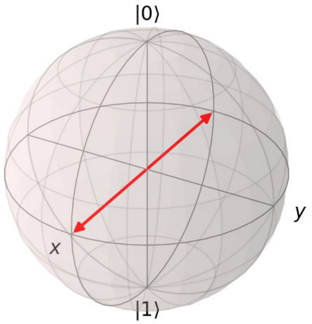
$\frac{1}{2}, |0\rangle$
$\frac{1}{2}, |1\rangle$



$$\frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\langle\sigma^x\rangle = Tr(\rho^{(1)}\sigma^x) = 0$$

$\frac{1}{2}, |+\rangle$
$\frac{1}{2}, |-\rangle$



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Can't be distinguished by just $\langle\hat{O}\rangle$

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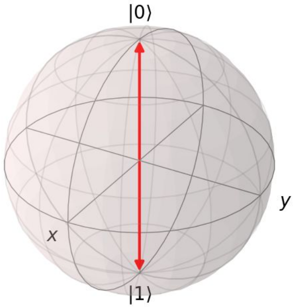
Density matrix

Distribution of expectation values for the states in the ensemble.

$$\frac{1}{2}, |0\rangle \qquad \frac{1}{2}, |1\rangle$$



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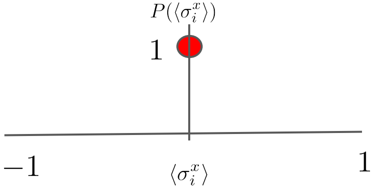

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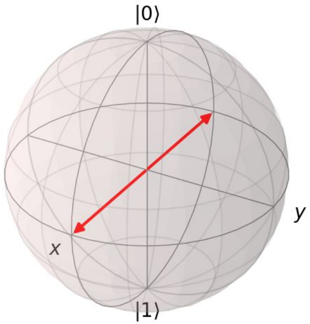

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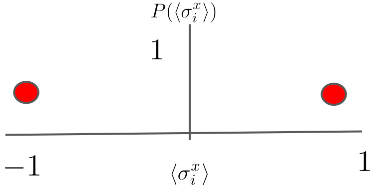

Can't be distinguished by just $\langle\hat{O}\rangle$

Ensemble of quantum states

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Density matrix

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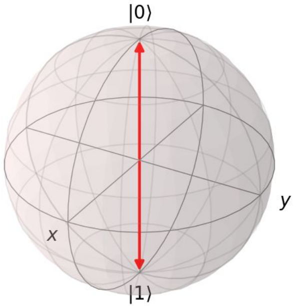
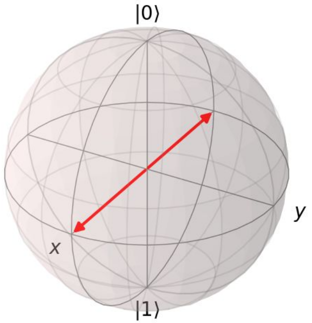


$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \sigma^x \rangle = Tr(\rho^{(1)}\sigma^x) = 0$$

$$\frac{1}{2}, |+\rangle \qquad \frac{1}{2}, |-\rangle$$



$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \sigma^x \rangle = Tr(\rho^{(1)}\sigma^x) = 0$$

Can distinguish between two ensembles

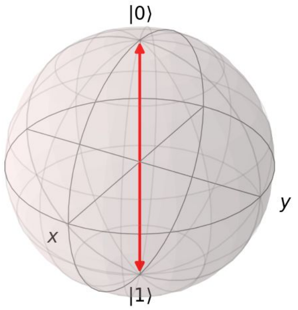
Can't be distinguished by just $\langle \hat{O} \rangle$

Ensemble of quantum states

$\rho^{(1)}$ doesn't uniquely describe a quantum state ensemble.

$$\rho^{(1)} = \sum_i p(\psi_i)|\psi_i\rangle\langle\psi_i| \qquad \rho^{(2)} = \sum_i p(\psi_i)(|\psi_i\rangle\langle\psi_i|)^{\otimes 2}$$

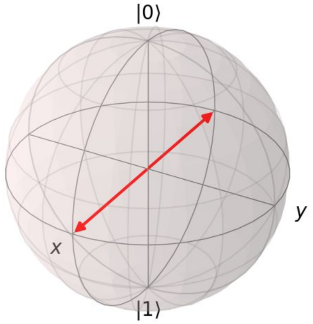
$\frac{1}{2}, |0\rangle$
$\frac{1}{2}, |1\rangle$



$$\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

Captures 2nd moment of the distribution of observable expectation values in the ensemble.

$\frac{1}{2}, |+\rangle$
$\frac{1}{2}, |-\rangle$



$$\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \frac{1}{4}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}$$

Can't be distinguished by just $\langle\hat{O}\rangle$

Ensemble of quantum states : Higher moments

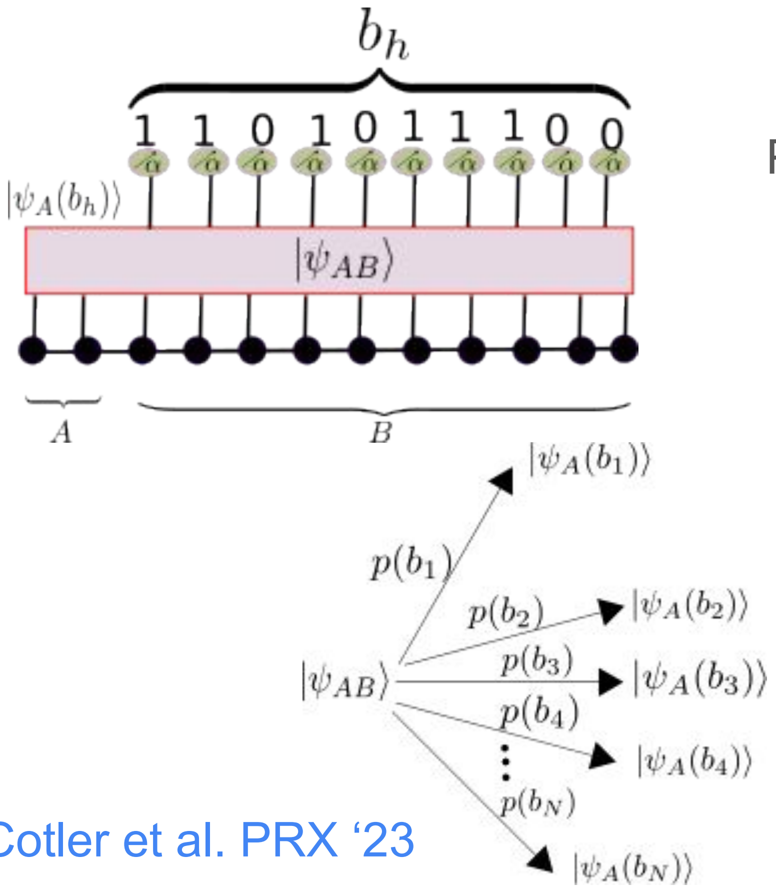
- The higher moments of an ensemble of pure quantum states contain information that is not accessible from density matrix.

  e.g. $\langle \hat{O}^{\otimes 2} \rangle = Tr(\rho^{(2)} \hat{O}^{\otimes 2}) = \sum_i p(\psi_i) \langle \psi_i | \hat{O} | \psi_i \rangle^2$

- The conventional study on fate of isolated closed quantum system has focused on the quantities at the level of density matrix.
- Do the higher moments (and the ensemble) approach to any universal ensemble? → Not well-posed in the context of thermalization of a pure quantum state. (no unique ensemble corresponding to the density matrix)

  Can we associate an ensemble to the reduced density matrix? -> Projected ensemble

Generate a quantum state ensemble : Projected ensemble



Pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

Post measurement state on the composite system

$$|\psi^{b_h}\rangle = \frac{1}{\sqrt{p(b_h)}} \Pi_{b_h} \psi_{AB} = \frac{1}{\sqrt{p(b_h)}} |\psi_A(b_h)\rangle \otimes |b_h\rangle$$

Projected ensemble is constructed as,

$$\mathcal{E}_{\mathrm{PE}} = \{p(b_h), |\psi_A(b_h)\rangle\}$$

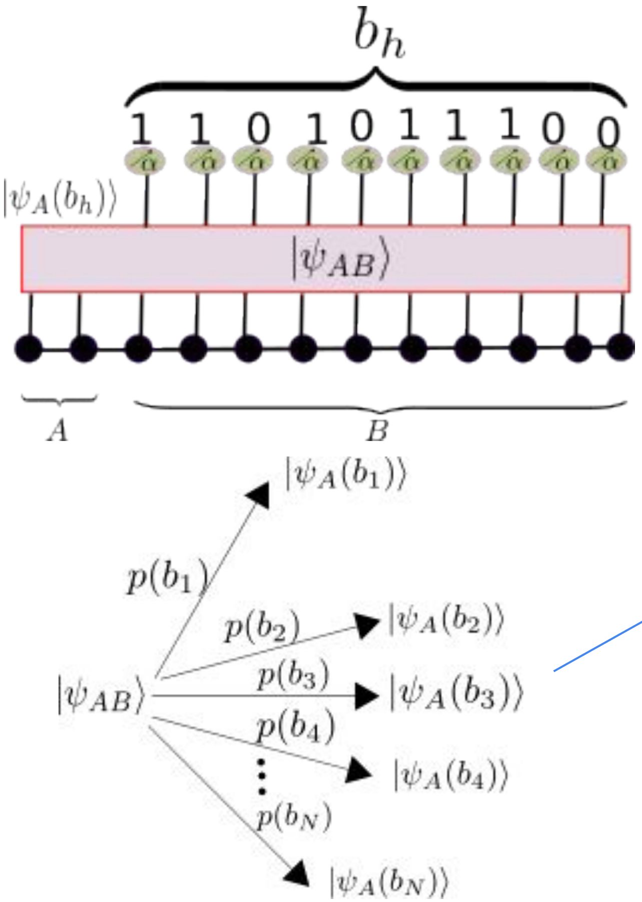
Allows access to any moment of observables by sampling.

$$p(b_h) = \langle \psi_{AB} | \Pi_{b_h} | \psi_{AB} \rangle$$

$$\Pi_{b_h} = \mathbb{I}_A \otimes |b_h\rangle\langle b_h|$$

*Cotler et al. PRX '23

Generate a quantum state ensemble : Projected ensemble



$$\mathcal{E}_{\mathrm{PE}} = \{p(b_h), |\psi_A(b_h)\rangle\}$$

States in A labelled by measurement outcome in B.

Get many copies of pure states in A labelled by measurement outcome in B.

Calculate the expectation value on these pure states.

Look at the statistics of these expectation values. -> Accessible in modern quantum simulators.

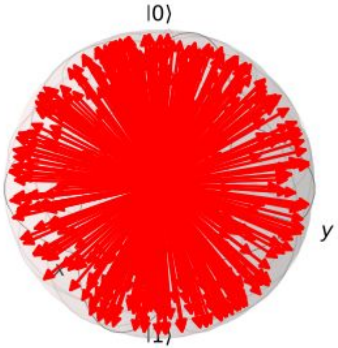
Generate a quantum state ensemble : Projected ensemble

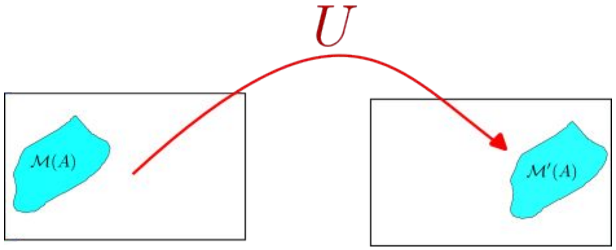
- Projected ensemble -> A protocol to generate a qunatum state esnemble given a pure state.
- ETH describes the local expectation values of the observable -> predicts the reduced density matrix of a subsystem. -> Gibbs ensemble
- Projected ensemble is a natural protocol to associate an ensemble of quantum states with the density matrix.
- Does Projected ensemble has any universal feature ?

Chaotic system with effective infinite temperature

Haar ensemble : Unitary invariant measure on Hilbert space.

- Produces identity matrix as the first moment (density matrix) always.
- Represents an effective infinite temperature state.
- Maximum entropy ensemble in Hilbert space.





Ensemble of quantum states :Scrooge ensemble

What happens when the density matrix is known and not identity ?

Different possible ensembles

$\{p_1(\psi), \|\psi\rangle\}$ $\rho^{(1)}$	$\{p_2(\psi), \|\psi\rangle\}$ $\rho^{(1)}$	$\{p_3(\psi), \|\psi\rangle\}$ $\rho^{(1)}$	$\cdots$	$\{p_n(\psi), \|\psi\rangle\}$ $\rho^{(1)}$

Hilbert space "Entropy" E_1 E_2 E_3 E_n

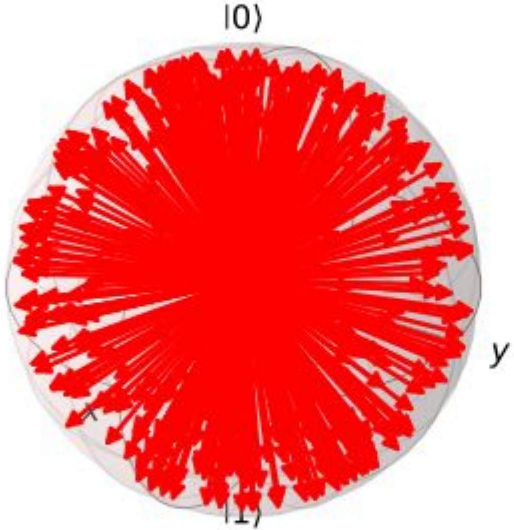
The ensemble with minimum accessible information (maximal entropy) consistent with the density matrix.

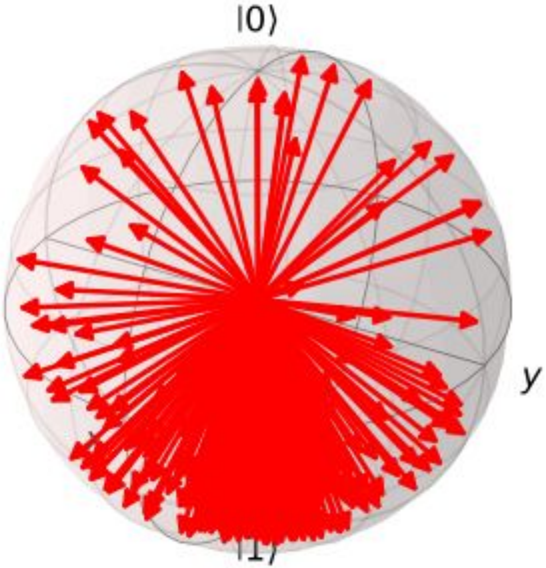
$$\mathcal{E}_{\text{Scrooge}}(\rho) = \left\{ D\langle\psi|\rho|\psi\rangle, \frac{\sqrt{\rho}|\psi\rangle}{\sqrt{\langle\psi|\rho|\psi\rangle}} \right\}_{\psi\in\mathcal{E}_{\text{Haar}}}$$

Jozsa, R., Robb, D., & Wootters, W. K. (1994). Lower bound for accessible information in quantum mechanics. Physical Review A, 49(2), 668.
Goldstein, S., Lebowitz, J. L., Tumulka, R., & Zanghi, N. (2006). On the distribution of the wave function for systems in thermal equilibrium. *Journal of statistical physics*, *125*, 1193-1221.

Ensemble of quantum states :Scrooge vs Haar ensemble

What happens when the density matrix is known and not identity ?





$$\langle S_z \rangle = -0.95$$

Effect of measurement basis on asymptotic ensemble

Hamiltonian with Global U(1) symmetry

Charge revealing measurement

- The global charge $Q = Q_A + Q_B$ is conserved under dynamics.
- Measurement outcome b_h ,reveals Q_B
- The possible conditional states in A are constrained.
- Approaches to a convex mixture of multiple Scrooge ensembles.

Charge non-revealing measurement

- Measurement reveals no information regarding charges.
- Conditional states in A are unconstrained (apart from global symmetry)
- Approaches to a single Scrooge ensemble

Hamiltonian with 1-local charge : Measurement never reveals any information about states in A.

Mark, D., Surace, F., Elben, A., Shaw, A., Choi, J., Refael, G., Endres, M., & Choi, S. (2024). Maximum Entropy Principle in Deep Thermalization and in Hilbert-Space Ergodicity. *Phys. Rev. X, 14, 041051.*

ℓ -bit model

$$H_{\ell\text{-bit}} = \sum_i J_i \sigma_i^z + \sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z + \sum_{i<j<k} J_{ijk}\sigma_i^z\sigma_j^z\sigma_k^z$$

Strongly disordered interacting spin chain

where

$$J_i = u_i, \;\; J_{ij} = u_{ij} e^{-(j-i)/\xi}, \;\; J_{ijk} = u_{ijk} e^{-(k-i)/\xi}$$

The model has strictly 1-local conserved charges

$$\{\sigma_i^z | 1 \leq i \leq L\}$$

Not a classical Hamiltonian.
Entanglement grows under dynamics.

Initial state

$$|\psi_0\rangle = \bigotimes_{i=1}^{L} \left(\cos\frac{\theta_i}{2} |\uparrow\rangle_i + e^{\imath\phi_i} \sin\frac{\theta_i}{2} |\downarrow\rangle_i \right)$$

$$\rho_{A,\infty} \to \bigotimes_{i=1}^{L_A} \rho_i^\infty \quad \text{with } \rho_i^\infty = \text{diag}[\cos^2\frac{\theta_i}{2}, \sin^2\frac{\theta_i}{2}]$$

Measurement basis

$$|0\rangle \equiv \cos\frac{\alpha}{2}|\uparrow\rangle + \sin\frac{\alpha}{2}|\downarrow\rangle$$

$$|1\rangle \equiv \sin\frac{\alpha}{2}|\uparrow\rangle - \cos\frac{\alpha}{2}|\downarrow\rangle$$

Time- evolution of moments of PE with ℓ-bit Hamiltonian

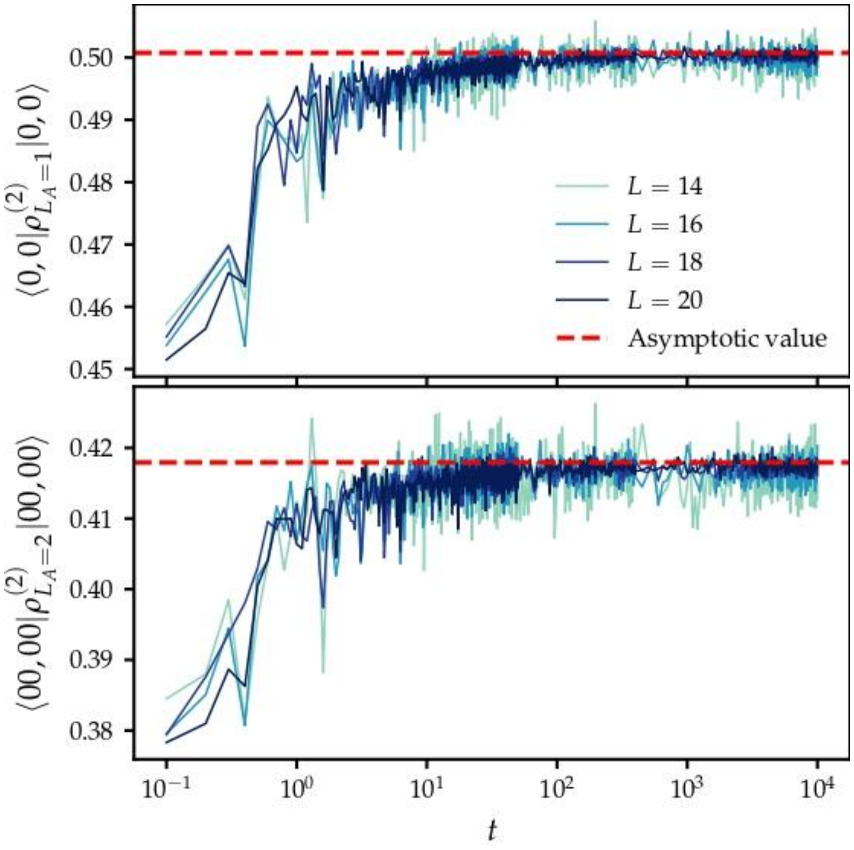
Asymptotic value is time-independent at large L limit.

Consider time-averaged quantities

Construct temporal ensemble as,

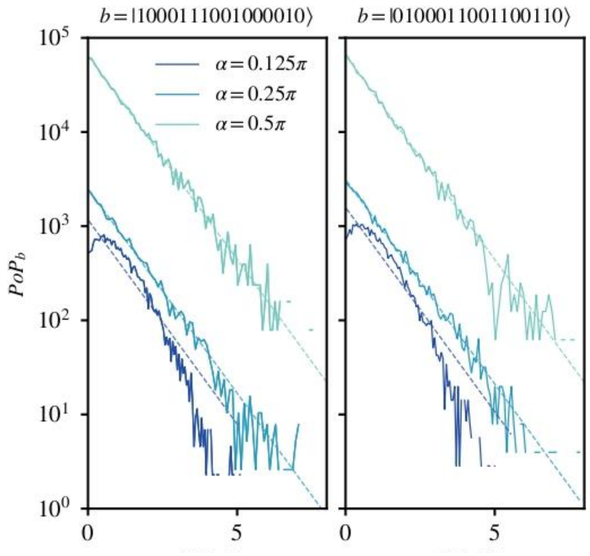
$$\mathcal{E}_{\text{temp}} = \left\{ \frac{\delta t}{\tau_2 - \tau_1}, e^{-\imath H t} \left| \psi_0 \right\rangle \right\}_{t \in [\tau_1, \tau_2]}$$

Time average implemented through averaging over the temporal ensemble.



Distribution of probabilities of bitstrings in temporal ensemble

$$p(b) = |\langle\psi|b\rangle|^2$$



- Follows PT distribution for all bitstring at measurement angles away from 0 with different bitstrings having different mean for PT dist.

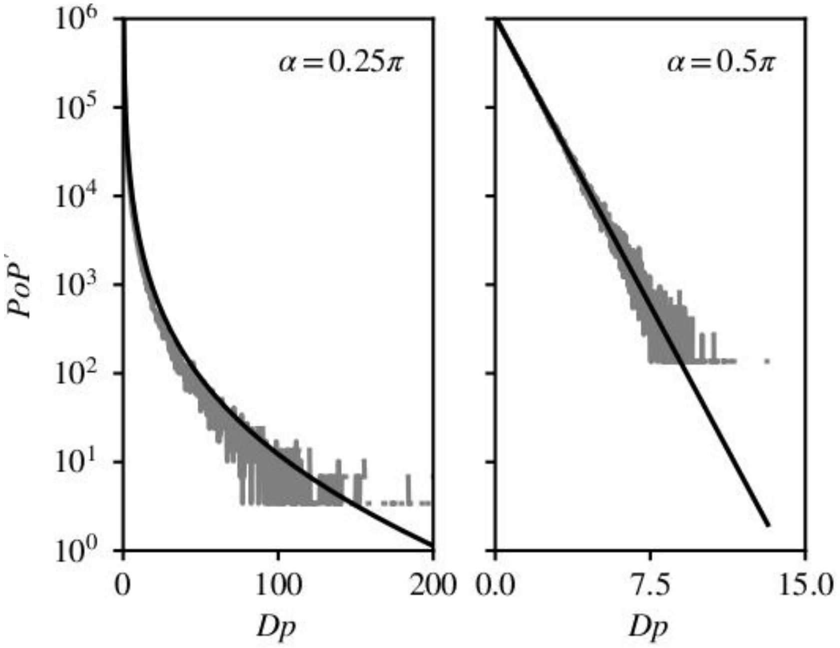
$$PT(p) = \frac{1}{\mu} e^{-p/\mu}$$

$$\mu_b(\alpha, \psi_0) = \sum_{z_\nu} |\langle\psi_0|z_\nu\rangle|^2 |\langle b|z_\nu\rangle|^2 ,$$

Porter, C. E., & Thomas, R. G. (1956). Fluctuations of nuclear reaction widths. *Physical Review*, *104*(2), 483.

Boixo, S., Isakov, S. V., Smelyanskiy, V. N., Babbush, R., Ding, N., Jiang, Z., ... & Neven, H. (2018). Characterizing quantum supremacy in near-term devices. Nature Physics, 14(6), 595-600.

Distribution of probabilities of bitstrings in a single late-time state



At measurement angle $\alpha = \pi/2$

All the bitstrings follow PT with same mean -> A single PT distribution with mean $1/2^L$

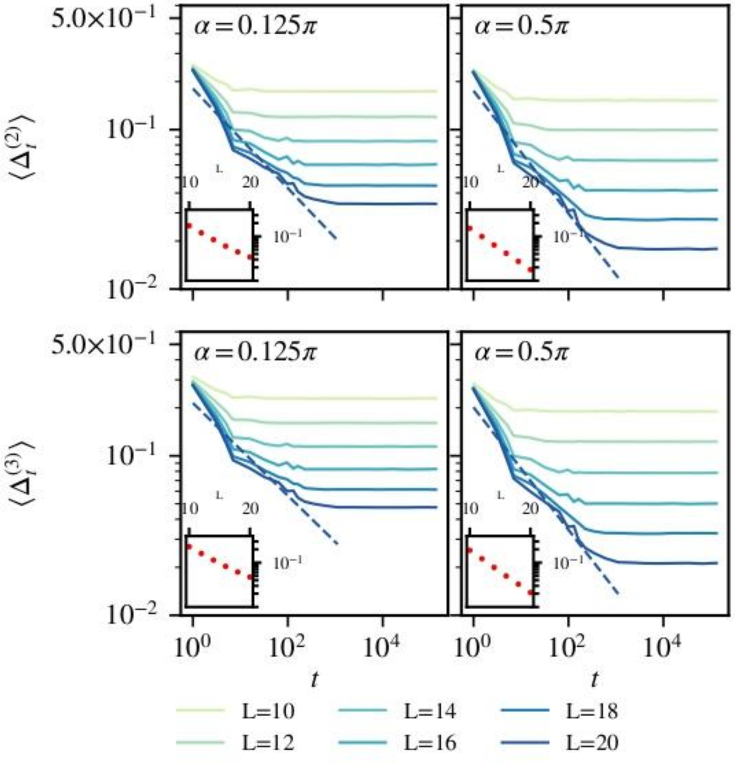
At any other measurement angle :

All the bitstrings follow PT with different mean -> sum of PT distributions in single state

Porter, C. E., & Thomas, R. G. (1956). Fluctuations of nuclear reaction widths. *Physical Review*, *104*(2), 483.

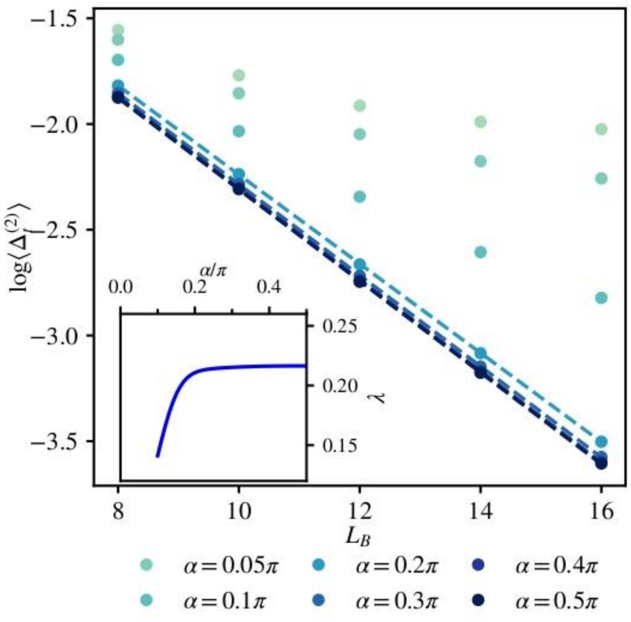
Boixo, S., Isakov, S. V., Smelyanskiy, V. N., Babbush, R., Ding, N., Jiang, Z., ... & Neven, H. (2018). Characterizing quantum supremacy in near-term devices. Nature Physics, 14(6), 595-600.

Numerical evidence



- Decay of distance between instantaneous PE from Scrooge ensemble with time.
- The distance saturates to an asymptotic value depending on system size.

Numerical evidence



Decay of distance with bath size

- PE approaches Scrooge ensemble with increasing L_B
- Holds true for any measurement angle away from 0

Conclusion

- For l-bit model, the late-time PE is described by an universal ensemble - Scrooge ensemble for any measurement angle except 0 (distinct from the case with global charges).
- Can be associated with emergence of PT distribution in temporal ensemble.
- The PT distribution emerges in temporal ensemble under l-bit Hamiltonian even though the dynamics is far from chaotic.
- Additional results regarding the angle and system size dependence of the approach to Scrooge ensemble, semi-analytical proof , convergence to random phase ensemble and studies in a Floquet MBL model are described in the preprint.

arxiv : 2501.01823

Elements of Projected ensemble

$$\varrho^{(k)}_{Z_A} = \sum_b \frac{\prod_{i=1}^k p(z_A^{(i)}) p_t(b|z_A^{(i)})}{[\sum_{z^A \in \mathcal{Z}^A} p(z_A) p_t(b|z_A)]^{k-1}} .$$

1. Replace $p_t(b|z_A^{(i)})$ with PT dist having appropriate mean
2. $p_t(b|z_A^{(i)})$ & $p_t(b|z_A^{(j)})$ are uncorrelated.

$$\varrho^{(k)}_{Z_A} = \sum_b \int \left(\prod_{z_A} \frac{dp(b|z_A)}{\mu(b)} e^{-\frac{p(b|z_A)}{\mu(b)}} \right) \times \quad \frac{\prod_{i=1}^k p(z_A^{(i)}) p_t(b|z_A^{(i)})}{[\sum_{z^A \in \mathcal{Z}^A} p(z_A) p_t(b|z_A)]^{k-1}}$$

The expression evaluates to the corresponding expression for Scrooge ensemble.

Elements of Projected ensemble

σ_z measurement ?

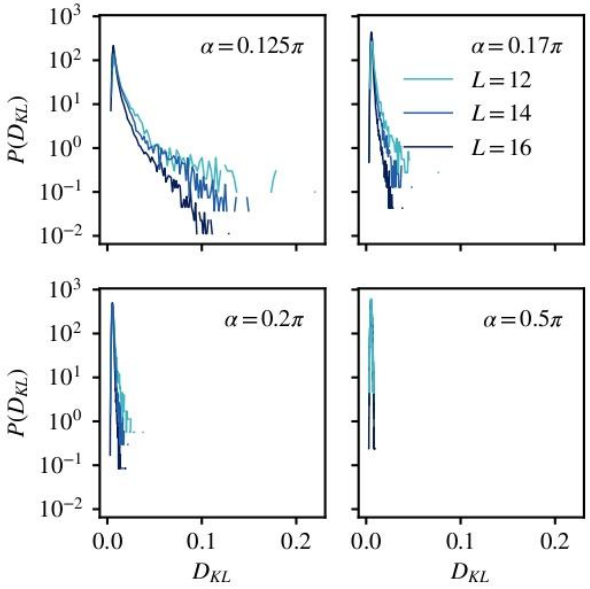
$$\varrho^{(k)}_{Z_A} = \sum_b \frac{\prod_{i=1}^k p(z_A^{(i)}) p_t(b|z_A^{(i)})}{[\sum_{z^A \in \mathcal{Z}^A} p(z_A) p_t(b|z_A)]^{k-1}} .$$

1. Replace $p_t(b|z_A^{(i)})$ with PT dist having appropriate mean ✗
2. $p_t(b|z_A^{(i)})$ & $p_t(b|z_A^{(j)})$ are uncorrelated. ✗

Both the assumptions maximally violated

Distribution of probabilities of bitstrings in temporal ensemble

$$p(b) = |\langle \psi | b \rangle|^2$$



A long tail in KL- div distribution for angles close to 0.

$$PT(p) = \frac{1}{\mu} e^{-p/\mu}$$

→ Signature of uniformity.

$$\mu_b(\alpha, \psi_0) = \sum_{z_\nu} |\langle \psi_0 | z_\nu \rangle|^2 |\langle b | z_\nu \rangle|^2 \,,$$

Porter, C. E., & Thomas, R. G. (1956). Fluctuations of nuclear reaction widths. *Physical Review*, *104*(2), 483.

Boixo, S., Isakov, S. V., Smelyanskiy, V. N., Babbush, R., Ding, N., Jiang, Z., ... & Neven, H. (2018). Characterizing quantum supremacy in near-term devices. Nature Physics, 14(6), 595-600.