Math Circles

Modular Forms

Problem sheet 2 (14.08.2026)

1. Ramanujan's Discriminant Modular Form

1.1. **Basic Definitions.** We start working towards modular forms through a very famous example, namely the Discriminant modular form, also known as the Ramanujan's modular forms.

Let

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

denote the upper half plane. and let $q = e^{2\pi i z}$.

The Ramanujan discriminant modular form is defined by the infinite product

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where the coefficients $\{\tau(n)\}_{n \geq 1}$ are given by the famous Ramanujan tau function

$$\begin{aligned} \tau :& \mathbb{N} \longrightarrow \mathbb{Z} \\ & n \mapsto \tau(n). \end{aligned}$$

Our goal is to prove that $\Delta(z)$ is a cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$. That is, we must show:

(1) $\Delta(z)$ is an analytic function.

(2) For every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, it satisfies the following transformation property

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z). \tag{1}$$

(3) $\Delta(i\infty) = 0$.

1.2. **Generators of $\mathrm{SL}_2(\mathbb{Z})$.** Our first goal is to prove that the set of matrices

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\}.$$

is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

i.e., every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be written as

$$\gamma = S \circ T^{\beta_1} \circ \cdots \circ S \circ T^{\beta_n},$$

and we denote this by

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

Problems.

(i) For any integer $n \in \mathbb{Z}$, show that

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

(ii) Find the inverse of S.

(iii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, let $c \neq 0$ with $a = q_1 c + r$ and $0 < r < |c|$, show that

$$\gamma_1 = ST^{-q_1}\gamma = \begin{pmatrix} -c & -d \\ r & b - q_1 d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

(iv) Repeat the above procedure to arrive at a

$$\gamma_k = S \circ T^{-q_k} \circ \cdots S \circ T^{-q_1} \circ \gamma = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

for some $\alpha \in \mathbb{Z}$. Thus, we have proved that

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

(v) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, show that the map

$$\begin{aligned} \gamma : &\mathbb{H} \longrightarrow \mathbb{H} \\ z &\mapsto \gamma z = \frac{az+b}{cz+d} \end{aligned}$$

is well-defined. For any $z, z' \in \mathbb{H}$, the hyperbolic distance function is given by the formula

$$\sinh^2(d_{\mathrm{hyp}}(z,z')/2) = \frac{|z-z'|^2}{4\mathrm{Im}(z)\mathrm{Im}(z')}.$$

Then, for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ show that

$$\sinh^2(d_{\mathrm{hyp}}(z,z')/2) = \sinh^2(d_{\mathrm{hyp}}(\gamma z, \gamma z')/2),$$

i.e., the map $\gamma : \mathbb{H} \longrightarrow \mathbb{H}$ is an isometry.

2. Transformation property of Δ

In this section we prove the transformation property of Δ, namely equation (1).

2.1. Dedekind Eta Function.

2.1. Dedekind Eta Function. The key to proving the transformation property is the Dedekind eta function, which is given by the following equation

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

It is immediate from the definitions that

$$\Delta(z) = \eta(z)^{24}.$$

Thus, if we can prove that $\eta(z)$ transforms with a half-integer weight (up to a root of unity), then raising to the 24th power will yield the desired transformation law for Δ.

Problems.

(i) Show that
$$\eta(Tz) = \zeta_{24}\eta(z) \implies \Delta(Tz) = \Delta(z),$$
where $\zeta_{24} = e^{2\pi i/24}$.

(ii) Show that
$$\log \eta(z) = \frac{2\pi i z}{24} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{e^{2\pi i m n z}}{m} =$$
$$\frac{2\pi i z}{24} - \sum_{N=1}^{\infty} \frac{1}{N} \sum_{n|N} n\, e^{2\pi i N z} = \frac{2\pi i z}{24} - \sum_{N=1}^{\infty} \frac{\sigma(N)}{N} e^{2\pi i N z},$$
where $\sigma(N) = \sum_{d|N} d$ is the divisor sum function.

2.2. **Jacobi Theta Function and Poisson Summation Formula.** For a sufficiently well-behaved function $f : \mathbb{R} \to \mathbb{C}$, the Poisson summation formula states:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \widehat{f}(k), \tag{4}$$

where $\widehat{f}$ is the Fourier transform defined by
$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x}\, dx.$$

For any $z \in \mathbb{H}$, define
$$\vartheta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}.$$
When $z = it$ with $t > 0$, this becomes
$$\vartheta(it) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}.$$

Problems.

(i) Using Poisson summation formula for the function $f(x) = e^{-\pi t x^2}$, show that
$$\vartheta(it) = \frac{1}{\sqrt{t}} \vartheta(i/t). \tag{2}$$
By analytic continuation, we can extend this from the imaginary axis to the entire upper half-plane:
$$\vartheta(-1/z) = \sqrt{-iz}\, \vartheta(z).$$

(ii) For $z \in \mathbb{H}$ and $q = e^{2\pi i z}$, the Jacobi triple product formula is defined as
$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + z q^{2n-1})(1 + z^{-1} q^{2n-1}).$$
Using which show that
$$\vartheta_4(z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}})^2 = \frac{\prod_{n=1}^{\infty}(1 - q^{n/2})^2}{\prod_{n=1}^{\infty}(1 - q^n)}.$$

(iii) Tracing definitions, show that
$$\vartheta(z) = \frac{\eta\left(\frac{z+1}{2}\right)^2}{\eta(z+1)}, \tag{3}$$
which is known as the **theta transformation law**.

(iv) Using (2) and (3), show that

$$\sqrt{-iz}\,\frac{\eta\left(\frac{z+1}{2}\right)^2}{\eta(z+1)} = \frac{\eta\left(\frac{z-1}{2z}\right)^2}{\eta(-1/z+1)}.$$

(v) Finally prove that

$$\eta(-1/z) = \sqrt{-iz}\,\eta(z),$$

which implies that for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\Delta(\gamma z) = \Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z).$$

Hence, we can conclude that Δ is a cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$.