Problems for Complex numbers and Analytic Functions

Problem Sheet 1 (31.07.2026)

1. Complex numbers and Polynomials

1.1. **Introduction.** The aim of this module of Math Circles is to give an elementary introduction to Modular Forms, fundamental objects in modern mathematics. Modular forms play a very important role in diverse areas such as Number Theory, Complex Geometry, and String Theory from Theoretical Physics.

Modular forms occupy the central stage in Langlands Program, a vast and influential network of mathematical conjectures proposed by Robert Langlands that link number theory, representation theory, and algebraic geometry. Certain formulae conceived by Ramanujan can be Famous German Number Theorist Martin Eichler called them the fifth fundamental operation.

Our strategy is to introduce complex numbers and analytic functions in Problem Sheet 1. We then introduce modular forms as analytic functions on the upper half space, satisfying certain transformation properties in the subsequent problem sheets. We discuss theta functions, Ramanujan's discriminant modular form, and end our module with a strategy for computing $r(n,4)$, number of ways in which the integer n can be represented as a sum of 4 squares.

1.2. **Complex numbers.** Let $\mathbb{R}$ denote the set of real numbers, and set $i := \sqrt{-1}$. Let

$$\mathbb{C} := \big\{x + iy \big|\, x, y \in \mathbb{R}\big\}$$

denote the set of complex numbers.

For any $z = x + iy \in \mathbb{C}$, x is called the real-part of z, and y is called the imaginary part of z. Let Re(z) and Im(z) denote the real and imaginary parts of z.

We can identify the set of complex numbers with the set of 2-tuples

$$\mathbb{R}^2 := \big\{(x, y) \big|\, x, y \in \mathbb{R}\big\},$$

via the following bijective map

$$f : \mathbb{C} \longrightarrow \mathbb{R}^2,\ x + iy \mapsto (x, y). \tag{1}$$

From the above bijection, we identify $\mathbb{C}$ with the 2-dimensional real plane, where the traditional X-axis represents the real part of a complex number, and the traditional Y-axis represents the imaginary part of complex number.

For example the point $z = 2 + i3 \in \mathbb{C}$ represents the coordinate $(2, 3)$ on the real plane $\mathbb{R}^2$.

Binary Operations. For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we define

$$\textbf{Addition} : z_1 + z_2 := x_1 + x_2 + i(y_1 + y_2),\ \text{i.e.,}$$
$$\mathrm{Re}(z_1 + z_2) := x_1 + iy_1,\ \text{and}\,\mathrm{Im}(z_1 + z_2) := y_1 + y_2; \tag{2}$$

$$\textbf{Multiplication} : z_1 \cdot z_2 := (x_1 + iy_1) \times (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1),$$
$$\text{i.e., } \mathrm{Re}(z_1 \cdot z_2) := x_1x_2 - y_1y_2,\ \text{and}\,\mathrm{Im}(z_1 \cdot z_2) := x_1y_2 + x_2y_1. \tag{3}$$

Absolute value and distance function. For $z = x+iy \in \mathbb{C}$, the absolute value of z is defined as

$$|z| := \sqrt{x^2 + y^2}.$$

From the identification of $\mathbb{C}$ with the real-plane $\mathbb{R}^2$ (as in (1)), $|z|$ represents is nothing but the distance of the point (x, y) from the origin $(0, 0)$.

From the above formula, the Euclidean distance between the points $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2 \in \mathbb{C}$ is given by the following formula

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{4}$$

Problems

(1) Find all the complex numbers z such that the coordinates represented by the complex numbers $z, z_1 := 2+i$, $z_2 := 2-i$ represent the vertices of an equilateral triangle on the real plane $\mathbb{R}^2$.

(2) Prove the triangular inequalities

$$|z_1 + z_2| \le |z_1| + |z_2|;$$
$$\big||z_1| - |z_2|\big| \le |z_1 - z_2|.$$

(3) Find all the complex numbers, which satisfy the equations:

$$|z - 1 + 2i| = |z + 1 - i|;$$
$$\big||z - 1| - |z + 1|\big| = 1.$$

1.3. **Euler's formula.** As discussed in the previous section, any $z = x + iy \in \mathbb{C}$ can be represented by a point (x, y) in the real plane $\mathbb{R}^2$. We now describe a famous formula from one of the great masters, Leonhard Euler, which will make the identification of $\mathbb{C}$ with the real-plane $\mathbb{R}^2$ (as in (1)), geometrically and intuitively concrete.

For any given $z \in \mathbb{C}$, define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The above series is called a power series, and it is possible to show that the above series converges for all $z \in \mathbb{C}$ and is never zero.

For any $x \in \mathbb{R}$, Euler's formula is

$$e^{ix} = \cos(x) + i\sin(x). \tag{5}$$

We now describe a proof for equation (5). Consider the function

$$f(x) := \frac{\cos(x) + i\sin(x)}{e^{ix}}.$$

It is easy to see that

$$\frac{df(x)}{dx} = \frac{e^{ix}\big(-\sin(x) + i\cos(x)\big) - ie^{ix}\big(\cos(x) + i\sin(x)\big)}{e^{2ix}} =$$
$$\frac{-ie^{ix}\big(\cos(x) + i\sin(x)\big) + ie^{ix}\big(\cos(x) + i\sin(x)\big)}{e^{2ix}} = 0 \implies f \text{ is a constant function on } \mathbb{R}.$$

Furthermore

$$f(0) = 1 \implies \frac{\cos(x) + i\sin(x)}{e^{ix}} = 1 \implies e^{ix} = \cos(x) + i\sin(x).$$

Hence, any $z = x + iy \in \mathbb{C}$ can be expressed as

$$z = re^{i\theta}, \text{ where } r = |z| = \sqrt{x^2 + y^2}, \text{ and } \theta := \tan^{-1}(\mathrm{Im}(z)/\mathrm{Re}(z)) = \tan^{-1}(y/x).$$

Problems

(1) For any $r > 0$, show that

$$\left|r^i\right| \leq 1.$$

(2) Compute the limit

$$\lim_{n\to\infty} i^{4n}.$$

(3) For any $z = x + iy \in \mathbb{C}$, define

$$\cosh(z) := \frac{e^z + e^{-z}}{2}, \ \sinh(z) := \frac{e^z - e^{-z}}{2}.$$

Compute $\cosh(i)$.

(4) For any $z = x + iy \in \mathbb{C}$, define

$$\cos(z) := \frac{e^z + e^{-z}}{2}, \ \sin(z) := \frac{e^z - e^{-z}}{2i}, \ \tan(z) := \frac{\sin(z)}{\cos(z)}.$$

Find all the solutions of the equation

$$\tan(z) = z.$$

1.4. **Analytic functions.** We will restrict ourselves to discussing analytic functions on $\mathbb{H} \subset \mathbb{C}$, the hyperbolic upper half-plane, which is defined as

$$\mathbb{H} := \left\{z = x + iy \in \mathbb{C} \middle| \, y > 0\right\}.$$

A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called an analytic function, if the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Cauchy-Riemann Equations. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be an analytic function. For any $z = x + iy$, we can write

$$f(z) = u(x, y) + iv(x, y), \text{ where } u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

are two differentiable functions.

Problems

(1) With notation as above, show that

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}$$
$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}.$$

The above equations are known as the famous Cauchy-Riemann equations.

(2) For any $z = x + iy \in \mathbb{C}$, let $\bar{z} = x - iy$ denote the complex conjugate of z. Using the chain rule, we can express derivatives with respect to x and y in terms of z and $\bar{z}$:

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}};$$
$$\frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial y}\frac{\partial}{\partial \bar{z}} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}.$$

Solving this system for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, we derive

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ an analytic function as above. Using Cauchy-Riemann equations, prove that

$$\frac{\partial f(z)}{\partial \bar{z}} = 0.$$

(3) For any $z \in \mathbb{H}$, set $q(z) := e^{2\pi i z}$. Consider the function

$$\theta_3(q(z)) := \sum_{n=-\infty}^{\infty} (q(z))^{n^2}.$$

Show that the above series converges, i. e., $|\theta_3(z)| < \infty$. Furthermore, prove that the theta function $\theta_3(z)$ is an analytic function.

(4) With notation as above, consider the analytic function

$$\theta_4(q(z)) := \sum_{n=-\infty}^{\infty} (-1)^n (q(z))^{n^2}.$$

From (3), we can easily see that θ_4 is an analytic function satisfying

$$\theta_4(q(z)) = \theta_3(-q(z)).$$

Furthermore, observe that θ_3 counts the number of integer solutions to $n^2 = m$, θ_4 counts the difference between number of even n solutions and the number of odd n solutions, i.e.,

$$\text{coefficient of } (qz)^m = \#\{n \,\text{even} \,|\, n^2 = m\} - \#\{n \,\text{odd} \,|\, n^2 = m\}$$

(5) Finally, for any $x \in \mathbb{R}$, set

$$u(x;t) := \theta_3(x; 4\pi i t) := \sum_{n=-\infty}^{n=\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}.$$

Show that the above series converges, and that is satisfies the heat equation

$$\frac{\partial u(x;t)}{\partial t} = \frac{1}{4\pi^2}\frac{\partial^2 u(x;t)}{\partial x^2}.$$