ORDER THEORY

MATHS CIRCLE INDIA–MODULE 20

Countability

A *list* is a collection of elements $a_0, a_1, a_2, \cdots$ that may or may not end. Natural numbers are usually listed as $0, 1, 2, \cdots$. Similarly, we can list integers as $0, 1, -1, 2, -2, \cdots$.

- A1 Suppose you are given a list $L_1, L_2, \cdots$ of lists. Show that there is a single list that contains all the elements of all lists.
- A2 Show that there is a list of pairs (a, b) of natural numbers.
- A3 Is it possible to list all rational numbers?
- HW Is it possible to list all real numbers?

Linear Orderings

Suppose there is a set L of candidates and you rank all those candidates according to the following rules:

1. If x is ranked lower than y and y is ranked lower than z, then x is ranked lower than z;
2. If x and y are distinct candidates then either x is ranked lower than y or y is ranked lower than x, but not both;
3. No element is ranked lower than itself.

We say that a set L equipped with such a ranking is a *linear ordering*. The name "linear" is justified because the ranking could be used to imagine candidates as points of a line together with their relative positioning. If x is ranked lower than y, then we write $x < y$ and position x to the left side of y.

- B1 Verify that the standard ordering on each of the following sets is a linear ordering:
  - (a) the set of natural numbers;
  - (b) the set of integers;
  - (c) the set of rational numbers.
- B2 Can you think of infinitely many rankings on the set of natural numbers?

Consider the following rankings on the set of natural numbers:
$L_1 : 1 < 2 < 3 < \cdots < 0$
$L_2 : 0 < 2 < 3 < \cdots < 1$
$L_3 : \cdots < 2 < 1 < 0$
Then the "shape" of the linear orderings L_1 and L_2 is the same but is different from L_3. In fact, one can get L_2 from L_1 just by swapping 0 with 1. Two linear orderings like this are called *isomorphic*. Any relabelling of elements produces an isomorphic copy.

- B3 Can you find infinitely many rankings on the set of natural numbers such that no two of them are isomorphic to each other?
- B4 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of integers?
- B5 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of rationals?

The standard ordering of natural numbers has a smallest element, namely 0. However, the standard orderings of integers or rationals do not have smallest or largest elements. Also notice that there is a "gap" between two consecutive natural numbers or integers in their standard orderings.

- B5 Show that the standard ordering of rationals does not have any gaps.
- B6 Suppose you can list all elements of a set L, and have chosen and fixed a ranking of elements of L. Show that for each element x of L, there is a rational $\bar{x}$ such that whenever x is ranked lower than y in L then $\bar{x} < \bar{y}$ in rationals.
- B7 A famous result due to Cantor states that if L_1 and L_2 are any rankings of natural numbers without smallest element, largest element or gaps, then they are isomorphic.

  Prove Cantor's result using a modification of the technique used for B6!