Number systems, sequences, series, exponential series

Problem sheet 1 (07.06.2024)

1. Number systems

1.1. **Natural numbers.** We start our discussion for this exploration by recalling that natural numbers usually refer to the following set of numbers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Sometimes we also consider 0 as a natural number, but in this exploration, we will start with 1.

1.2. **Problems.** Let X and Y be two sets, we say a function $f : X \to Y$ is *one to one* if $f(x) = f(y) \implies x = y$ and *onto* if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. If f is both one to one and onto, we say that f is a *bijection.*

(1) Can you find a bijection from the set $X = \{2, 7, 8\}$ to the set $Y = \{blue, green, yellow\}$?

(2) Can you find a bijection from the set $X = \{1, 2, 3\}$ and $\{5, 6\}$?

(3) If there is a bijection from a set X to a set Y, then can you find a bijection from Y to X.

(4) If there is a bijection from a set X to a set Y, and another bijection from the set Y to a set Z, then can you find a bijection from the set X to the set Y?

(5) Can you find a bijection from $\mathbb{N}$ to $\{3, 4, 5, \ldots\}$?

(6) Can you find a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N} = \{(n, m) : n \in \mathbb{N}, m \in \mathbb{N}\}$?

$*$(7) Let Y be the set of all functions from $\mathbb{N} \to \{0, 1\}$. Can you find a bijection from $\mathbb{N}$ to Y?

1.3. **Integers.** We note that addition and multiplication is of two natural numbers is a natural number and hence addition is a well defined concept within the realm of natural numbers. However subtraction or division does not always make sense as the result of such operation may not be natural number.

If you add 0 to the set of natural numbers, then the quest for well defined notion of subtraction boils down to finding the *additive inverse* of a number m, i.e. we can think of the the operation $n - m$ to be $n + (-m)$ with $-m$ being the additive inverse of m. In a rather naive way, finding the additive inverse of a natural number m would be equivalent to finding a root of the following equation,

$$x + m = 0 \tag{1}$$

Our attempt to solve the above equation, leads us to the world of *integers*,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Of course we can check that 1 always has a root in $\mathbb{Z}$ for any choice of $m \in \mathbb{Z}$.

1.4. **Rational numbers.** A similar but slightly modified argument about finding multiplicative inverses and making sense of division leads us to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\} \tag{2}$$

1.5. **Problems.** Note that the definition 2 is not correct as it is inconsistent with our standard knowledge of rational numbers.

(1) Can you describe what is the issue with this definition?

(2) Can you fix this and come up with a better definition for rational numbers?

1.6. **Real numbers.** After rational numbers, *real numbers* are often considered which like rational numbers, is well behaved with respect to addition, subtraction, multiplication and division. However this is a strictly bigger set of numbers than the rational numbers and those real numbers which are not rational are referred to as *irrational numbers*.

We will not attempt to give a description of the properties which lead to consideration of real numbers (interested readers can look into the property called Dedekind's completeness), but we remind the readers that every real number has a (possibly non terminating) decimal representation.

Finally we can find a rational number as close as we want to a given real number. This property is known as the *density* of rational numbers in reals. An easy way to understand this property would be as follows: For any two real numbers a and b satisfying $a < b$, we have $\mathbb{Q} \cap (a, b) \neq \emptyset$.

1.7. **Problems.**

(1) Find a rational number whose decimal representation is given by 0.33

(2) Find a rational number whose decimal representation is given by $0.333333333\ldots$

(3) Find a rational number whose decimal representation is given by $3.311111111\ldots$

(4) Find a rational number whose decimal representation is given by $0.9999999999\ldots$

(5) Find a rational number which does not have a unique decimal representation.

(6) Find the decimal representation of $11/9$.

(7) Find a rational number x whose decimal representation is given by $0.142857142857\ldots\ldots$, and additionally can you observe any pattern when computing $2x, 3x, 4x, \ldots$?

(8) Suppose the non terminating decimal expansion of a real number eventually results in a set of digits being repeated. What can you say about such numbers?

(9) Show that the root of the equation $x^2 - 2 = 0$, denoted by $\sqrt{2}$ can not be a rational number.

(10) How to makes sense of an irrational exponent? For examples $3^2 = 3 \times 3$, or $3^{\frac{3}{2}} = \sqrt{3 \times 3 \times 3}$, but how to make sense of $3^{\sqrt{2}}$?

$*$(11) Can an irrational to the power of an irrational be a rational? [1]

$*$(12) For positive integers m and n the decimal representation for the fraction m/n begins 0.711 followed by other digits. Find the least possible value for n.

In the next exploration we will review a bit about complex numbers.

2. Sequences

2.1. **Basic definitions.** Recall that in mathematics a *sequence* refers to an enumerated list of objects, allowing repetition. The order in which these objects appears in the sequence matters.

We denote a sequence using expressions like

$$(a_n)_{n \in \mathbb{N}}$$

[1] Hint: Consider $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

so that

$$\begin{aligned}
a_1 &= \text{1st element of the sequence,}\\
a_2 &= \text{2nd element of the sequence,}\\
a_3 &= \text{3rd element of the sequence,}\\
&\dots\\
a_{n-1} &= \text{n-1 th element of the sequence,}\\
a_n &= \text{n th element of the sequence,}\\
a_{n+1} &= \text{n+1 th element of the sequence,}\\
&\dots
\end{aligned}$$

Expanded out the sequence $(a_n)_{n\in\mathbb{N}}$ can be written as $(a_1, a_2, a_3, \ldots)$. Some examples of sequences are $(2, 3, 5, 7, 11, 13, \ldots)$, $(0.1, 0.11, 0.111, 0.1111, \ldots)$ and $(0.9, 0.99, 0.999, 0.9999, \ldots)$.

Note that here the indexing set is an 'one-sided' infinite set, and we could as well use 'two-sided' infinite sets like the integers $\mathbb{Z}$ or even finite sets. In the later case a sequence is just an n-tuple.

2.2. **Problems.**

(1) It is often common to describe a sequence using a recursive relation. In each of the following cases, write down the first ten terms of the sequence.

(a) The Fibonacci sequence

$$a_n = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2\\ a_{n-1} + a_{n-2} & \text{if } n > 2 \end{cases}$$

(b) Recamán's sequence

$$a_n = \begin{cases} 1 & \text{if } n = 0\\ a_{n-1} - n & \text{if } a_{n-1} - n > 0 \text{ and not already in the sequence}\\ a_{n-1} + n & \text{otherwise.} \end{cases}$$

(2) Show that any (one sided infinite) sequence of real numbers can be described by a function $f : \mathbb{N} \to \mathbb{R}$ and conversely any such function can be thought of as a sequence. What about finite sequences and two sided infinite sequences?

2.3. **Convergence.** We say that the sequence $(a_n)_{n\in\mathbb{N}}$ *converges* to the real number a if as n becomes larger and larger the terms a_n comes closer and closer to a. In this case we write

$$\lim_{n\to\infty} a_n = a.$$

Otherwise we say that the sequence *diverges*.

More formally we say that $(a_n)_{n\in\mathbb{N}}$ converges to a if for every $\varepsilon > 0$ we can find a large enough $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$.

2.4. **Problems.**

(1) Determine if the following sequence converges:

(a)

$$a_n = \begin{cases} 2 & \text{if } n = 1\\ a_{n-1} + \frac{1}{2} & \text{if } n \geq 2 \end{cases}$$

(b)

$$a_n = \begin{cases} 2 & \text{if } n = 1 \\ \frac{a_{n-1}}{2} & \text{if } n \geq 2 \end{cases}$$

(2) Conduct the following experiment: Let

$$f(x) = 1 + \frac{1}{x}$$

For any given x, construct a sequence as follows: $a_1 = x, a_2 = f(x), a_3 = f(f(x)), a_4 = f(f(f(x))), a_5 = f(f(f(f(x)))), \ldots$. For different values of x, compute the first 5 terms. Do you see any pattern emerging?

3. Series

Given any (infinite) sequence of number $(a_n)_{n \in \mathbb{N}}$, the term *series* refers to the sum of the terms in the sequence. We denote this series by

$$a_1 + a_2 + a_3 + \ldots \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

3.1. **Summing the series.** Since the series need us to sum infinitely many terms from the sequence we have to carefully understand what exactly is meant when we say we want to find the sum of the series.

To understand this, we define the *partial sums* of the series by summing the first n terms,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

We say that the series *converges* or the series is *summable* if the following limit exists

$$S = \lim_{n \to \infty} S_n$$

The limit S in this case is called the sum of the series. We say that the series is *divergent* if it is not summable.

We say that the series $\sum a_n$ is *absolutely summable* if $\sum |a_n|$ is summable.

3.2. **Problems.**

(1) A *geometric series* is a series whose terms are obtained by multiplying the previous term with a fixed number, i.e. the n th term $a_n = ar^n$.

   (a) Find a formula for the n th partial sum S_n.

   (b) For what values of a and r is the series summable?

   (c) Determine a formula for the sum of the series when it is summable.

   (d) An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. Find the common ratio of the original series.

(2) Find

$$\sum_{j=1}^{n} \frac{1}{j(j+1)}$$

and determine if the series $\sum 1/(n(n+1))$ is summable.

(3) Suppose the terms a_n of a series $\sum a_n$ satisfy $a_n > 0.03$ when $n > 389$. Determine if the series converges.

3.3. **Test for convergence.** There are some tests that are used to quickly determine if a series is convergent. We will not go into the details but rather present a few of them briefly here:

- Comparison test: If $\sum b_n$ is absolutely convergent and $|a_n| \leq |b_n|$, then $\sum b_n$ is also absolutely convergent.
- Ratio test: If there exists a constant $C < 1$ such that $|a_{n+1}/a_n| < C$ for all sufficiently large n, then $\sum a_n$ converges absolutely.
- Root test: If there exists a constant $C < 1$ such that $\sqrt[n]{|a_n|} < C$ for all sufficiently large n, then $\sum a_n$ converges absolutely.

3.4. **Problems.**

(1) Determine if the harmonic series $\sum \frac{1}{n}$ is summable.

(2) Determine if the series $\sum 2^n/n^9$ is summable.

3.5. **The exponential series.** For a real number x, we define the exponential of x as

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

And in particular we use the letter e to denote $\exp(1)$. e turns out to be an irrational number and an explicit computation shows that $e = 2.71828\ldots$.

3.6. **Problems.**

(1) Explicitly write down the first 10 terms of the exponential series.

(2) Show that the series $\sum 1/n!$ is summable.

(3) For what values of x the exponential is defined, or in other words, when is the series summable?

$*$(4) Suppose $a_n > 0$ and $\sum a_n$ diverges. What can you say about the convergence of the following series?

$$\sum \frac{a_n}{1 + na_n}$$